EMS Tracts in Mathematics 16

EMS Tracts in Mathematics

This series includes advanced texts and monographs covering all fields in pure and applied mathematics. *Tracts* will give a reliable introduction and reference to special fields of current research. The books in the series will in most cases be authored monographs, although edited volumes may be published if appropriate. They are addressed to graduate students seeking access to research topics as well as to the experts in the field working at the frontier of research.

1 Panagiota Daskalopoulos and Carlos E. Kenig, *Degenerate Diffusions*
2 Karl H. Hofmann and Sidney A. Morris, *The Lie Theory of Connected Pro-Lie Groups*
3 Ralf Meyer, *Local and Analytic Cyclic Homology*
4 Gohar Harutyunyan and B.-Wolfgang Schulze, *Elliptic Mixed, Transmission and Singular Crack Problems*
5 Gennadiy Feldman, *Functional Equations and Characterization Problems on Locally Compact Abelian Groups*
6 Erich Novak and Henryk Woźniakowski, *Tractability of Multivariate Problems. Volume I: Linear Information*
7 Hans Triebel, *Function Spaces and Wavelets on Domains*
8 Sergio Albeverio et al., *The Statistical Mechanics of Quantum Lattice Systems*
9 Gebhard Böckle and Richard Pink, *Cohomological Theory of Crystals over Function Fields*
10 Vladimir Turaev, *Homotopy Quantum Field Theory*
11 Hans Triebel, *Bases in Function Spaces, Sampling, Discrepancy, Numerical Integration*
12 Erich Novak and Henryk Woźniakowski, *Tractability of Multivariate Problems. Volume II: Standard Information for Functionals*
13 Laurent Bessières et al., *Geometrisation of 3-Manifolds*
14 Steffen Börm, *Efficient Numerical Methods for Non-local Operators. $\mathcal{H}^2$-Matrix Compression, Algorithms and Analysis*
15 Ronald Brown, Philip J. Higgins and Rafael Sivera, *Nonabelian Algebraic Topology. Filtered Spaces, Crossed Complexes, Cubical Homotopy Groupoids*

Marek Jarnicki
Peter Pflug

Separately Analytic Functions

European Mathematical Society

Authors:

Marek Jarnicki
Jagiellonian University
Institute of Mathematics
Łojasiewicza 6
30-348 Kraków
Poland

E-mail: Marek.Jarnicki@im.uj.edu.pl

Peter Pflug
Carl von Ossietzky Universität Oldenburg
Institut für Mathematik
Postfach 2503
26111 Oldenburg
Germany

E-mail: Peter.Pflug@uni-oldenburg.de

2010 Mathematical Subject Classification: 32-02, 32D15, 32A10, 32A17, 32D05, 32D10, 32D26, 32U15

Key words: Separately holomorphic/meromorphic functions, Riemann domains, *N*-fold crosses, generalized crosses, relative extremal functions, holomorphic extension with singularities

ISBN 978-3-03719-098-2

The Swiss National Library lists this publication in The Swiss Book, the Swiss national bibliography, and the detailed bibliographic data are available on the Internet at http://www.helveticat.ch.

Contact address:

European Mathematical Society Publishing House
Seminar for Applied Mathematics
ETH-Zentrum FLI C4
CH-8092 Zürich
Switzerland

Phone: +41 (0)44 632 34 36
Email: info@ems-ph.org
Homepage: www.ems-ph.org

Typeset using the authors' TEX files: I. Zimmermann, Freiburg
Printing and binding: Druckhaus Thomas Müntzer GmbH, Bad Langensalza, Germany
∞ Printed on acid free paper
9 8 7 6 5 4 3 2 1

Preface

As a result of the incorrect statement by A. Cauchy in his book *Cours d'Analyse* (1821) that any separately continuous function on $\mathbb{R} \times \mathbb{R}$ is in fact continuous, a discussion was generated as to whether certain separate regularity properties for functions in several variables are in fact regularity properties with respect to all variables simultaneously. While in the case of continuous functions this kind of implication fails to hold, the situation of holomorphic functions is much better:

Any separately holomorphic function in several complex variables is automatically holomorphic as a function of all variables.

This deep result was proved by Friedrich Hartogs in 1906. Since that time, generalizations of this result in various directions have been intensively pursued. Now, more than a hundred years after the seminal work by Hartogs, we felt that the time had come to gather some of the main streams of these developments into one single source book. Since we have also been working in that area, we reached the conclusion that it might even be our duty to complete such a project. The book may thus be understood as a kind of hundredth anniversary of separate holomorphicity.

The book is divided into two parts. The first one, which is more elementary, deals with separately holomorphic functions "without singularities", while in the second part the situation of existing singularities is discussed. For more details, the reader is asked to consult the Introduction.

Holomorphic extension problems are important questions in several complex variables (SCV). Therefore, the book should be interesting for anyone seeking further knowledge on this type of phenomena, in particular, for students (if they have attended a course on SCV). We are confident that a part of this book will serve as a basis for seminars on several complex variables and that other parts will lead to further research.

We do not claim that the text is easy (especially in Part II). However, to help the reader as much as possible, much of the necessary prerequisite knowledge has been collected in Chapter 3 (most of it without proofs but with hints on where to find them). Moreover, most later sections begin with $\boxed{>}$ § … showing which part of the previous sections may be helpful in understanding the current one. As an orientation for the reader, each chapter starts with a short summary of the topics one may find in it.

There are a few theorems for which no proofs are given; they are denoted by Theorem*. Moreover, there are many points in the proofs that we have marked EXERCISE. By this we mean that the reader is encouraged to write out the argument in more detail than we have done. At the end of the book a list of general symbols with short explanations is found; in addition, each chapter has its own list of symbols introduced in it. Finally, from time to time we pose open problems (marked by $\boxed{?}$ … $\boxed{?}$) that to the best of our knowledge have not yet been solved. We encourage the reader to try to solve them.

We want to point out that we, the authors, are responsible for all mistakes that may be still in the text. It would be much appreciated by us if the readers could inform us about any mistakes they have found, using our email addresses:

Marek.Jarnicki@im.uj.edu.pl, Peter.Pflug@uni-oldenburg.de.

Finally, it is our great pleasure to thank the following institutions for their support during the writing of this book:

Jagiellonian University in Kraków,
Carl von Ossietzky Universität Oldenburg,
Polish Ministry of Science and Higher Education – grant N N201 361436,
Deutsche Forschungsgemeinschaft – grant 436POL113/103/0-2.

Without this help it would have been impossible to finish this project.

We are deeply indebted to Dr. M. Karbe – Publishing Director of the EMS Publishing House – for his constant help and to Dr. I. Zimmermann for taking care of the text.

Kraków–Oldenburg, June 2011

Marek Jarnicki
Peter Pflug

Contents

Introduction

Let us begin with the following (elementary) problem.

(S-$\mathcal{C}$) We are given two domains $D \subset \mathbb{R}^p$, $G \subset \mathbb{R}^q$ and a function

$$f : D \times G \to \mathbb{R}$$

that is *separately continuous* on $D \times G$, i.e.,

- $f(a, \cdot)$ is continuous on G for arbitrary $a \in D$,
- $f(\cdot, b)$ is continuous on D for arbitrary $b \in G$.

We ask whether the above conditions imply that f is continuous on $D \times G$.

It is well known that the answer is negative. However, recall that the answer was not known for instance to A. Cauchy, who in 1821 in his *Cours d'Analyse* claimed that f must be continuous.

> **I.$^{\text{er}}$ Théorème.** *Si les variables $x, y, z, \dots$ ont pour limites respectives les quantités fixes et déterminées $X, Y, Z, \dots$, et que la fonction $f(x, y, z, \dots)$ soit continue par rapport à chacune des variables $x, y, z, \dots$ dans le voisinage du systéme des valeurs particuliéres $x = X$, $y = Y, z = Z, \dots$, $f(x, y, z, \dots)$ aura pour limite* $f(X, Y, Z, \dots)$.
> A. Cauchy [Cau 1821], p. 39

See also [Pio 1985-86], [Pio 1996], [Pio 2000].

According to C. J. Thomae (cf. [Tho 1870], p. 13, [Tho 1873], p. 15, see also [Rose 1955]), the first counterexample had been discovered by E. Heine:

$$f(x, y) := \begin{cases} \sin(4 \arctan \frac{x}{y}) & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$$

A simpler counterexample is the function

$$f(x, y) := \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0), \end{cases} \tag{$*$}$$

which was already known to G. Peano in 1884 (cf. [Gen 1884], p. 173).

Since the answer is in general negative, one can ask how big is the set $\mathcal{S}_{\mathcal{C}}(f)$ of discontinuity points $(a, b) \in D \times G$ of a separately continuous function f. A partial answer was first given in 1899 by R. Baire ([Bai 1899], see also [Rud 1981]), who proved that every separately continuous function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is of the *first Baire class*, i.e. there exists a sequence $(f_k)_{k=1}^{\infty} \subset \mathcal{C}(\mathbb{R}^2, \mathbb{R})$ such that $f_k \to f$ pointwise on $\mathbb{R}^2$. Consequently, if $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is separately continuous, then f is Borel measurable.

> Several years ago I used to pose this question to randomly selected analysts. The typical answer was something like this: "Hmm – well – probably not – why should it be ?" The only group that did a little better were the probabilists. And there was just one person who said: "Let's see, yes, it is – and it is of Baire class 1 – and …". He knew.
>
> W. Rudin [Rud 1981]

Moreover, $\mathcal{S}_{\mathcal{C}}(f)$ must be of the *first Baire category*, i.e. $\mathcal{S}_{\mathcal{C}}(f) \subset \bigcup_{k=1}^{\infty} F_k$, where $\operatorname{int} \bar{F}_k = \varnothing, k \in \mathbb{N}$. Baire also proved that if $f : [0,1] \times [0,1] \to \mathbb{R}$ is separately continuous, then $\mathcal{S}_{\mathcal{C}}(f)$ is an $\mathcal{F}_\sigma$-set (i.e. a countable union of closed sets) whose projections are of the first Baire category. Conversely, if $S \subset [0,1] \times [0,1]$ is an $\mathcal{F}_\sigma$-set whose projections are of the first Baire category, then there exists a separately continuous function $f : [0,1] \times [0,1] \to \mathbb{R}$ with $\mathcal{S}_{\mathcal{C}}(f) = S$ (cf. [Ker 1943], [Mas-Mik 2000]). Moreover, if $S \subset [0,1] \times [0,1]$ is an $\mathcal{F}_\sigma$-set whose projections are nowhere dense, then there exists a separately $\mathcal{C}^\infty$ function $f : [0,1] \times [0,1] \to \mathbb{R}$ with $\mathcal{S}_{\mathcal{C}}(f) = S$ (cf. [Ker 1943]). Summarizing, the singularity sets $\mathcal{S}_{\mathcal{C}}(f)$ are small in the topological sense. However, G. P. Tolstov ([Tol 1949]) showed that for any $\varepsilon \in (0,1)$ there exists a separately $\mathcal{C}^\infty$ function $f : [0,1] \times [0,1] \to \mathbb{R}$ such that the measure of $\mathcal{S}_{\mathcal{C}}(f)$ is larger than ε.

It is natural to ask whether the above results may be generalized to the case of separately continuous functions $f : \mathbb{R}^n \to \mathbb{R}$, $n \geq 3$, i.e. those functions f for which $f(x_1, \dots, x_{j-1}, \cdot, x_{j+1}, \dots, x_n) \in \mathcal{C}(\mathbb{R})$ for arbitrary $(x_1, \dots, x_n) \in \mathbb{R}^n$ and $j \in \{1, \dots, n\}$. H. Lebesgue proved ([Leb 1905]) that every such a function is of the $(n-1)$ *Baire class*, i.e. there exists a sequence $(f_k)_{k=1}^{\infty}$ of functions of the $(n-2)$ Baire class such that $f_k \to f$ pointwise on $\mathbb{R}^n$. In particular, every separately continuous function $f : \mathbb{R}^n \to \mathbb{R}$ is Borel measurable. Moreover, H. Lebesgue proved that the above result is exact, i.e. for $n \geq 3$ there exists a separately continuous function $f : \mathbb{R}^n \to \mathbb{R}$ that is not of the $(n-2)$ Baire class.

It is clear that one may formulate similar problems substituting the class $\mathcal{C}$ of continuous functions by other classes $\mathcal{F}$, e.g.:

- $\mathcal{F} = \mathcal{C}^k$ = the class of $\mathcal{C}^k$*-functions*, $k \in \mathbb{N} \cup \{\infty, \omega\}$, where $\mathcal{C}^\omega$ means the class of *real analytic functions*,
- $\mathcal{F} = \mathcal{H}$ = the class of *harmonic functions*,
- $\mathcal{F} = \mathcal{SH}$ = the class of *subharmonic functions* (in this case we allow that $f : D \times G \to [-\infty, +\infty)$).

Thus our more general problem is the following one.

(S-$\mathcal{F}$) We are given two domains $D \subset \mathbb{R}^p$, $G \subset \mathbb{R}^q$ and a function

$$f : D \times G \to \mathbb{R}$$

that is *separately of class* $\mathcal{F}$ on $D \times G$, i.e.,

- $f(a, \cdot) \in \mathcal{F}(G)$ for arbitrary $a \in D$,
- $f(\cdot, b) \in \mathcal{F}(D)$ for arbitrary $b \in G$.

We ask whether $f \in \mathcal{F}(D \times G)$.

Moreover, in the case where the answer is negative, one may study the set

$$\mathcal{S}_{\mathcal{F}}(f) := \{(a, b) \in D \times G : f \notin \mathcal{F}(U) \text{ for every neighborhood } U \text{ of } (a, b)\}.$$

Observe that the Peano function (∗) is separately real analytic. Consequently, our problem has a negative solution for $\mathcal{F} = \mathcal{C}^k$ with arbitrary $k \in \mathbb{N} \cup \{\infty, \omega\}$ and, therefore, one may be interested in the structure of $\mathcal{S}_{\mathcal{C}^k}(f)$. The structure of $\mathcal{S}_{\mathcal{C}^\omega}(f)$ was completely characterized in [StR 1990], [Sic 1990], and [Bło 1992] (cf. Theorem 5.8.2). In particular, in contrast to Tolstov's result, the set $\mathcal{S}_{\mathcal{C}^\omega}(f)$ must be of zero measure.

Surprisingly, in the case of harmonic functions the answer is positive – *every separately harmonic function is harmonic*, cf. [Lel 1961] (Theorem 5.6.5).

In the case of separately subharmonic functions the answer is once again negative, cf. § 5.8.2.

Analogous problems may be formulated in the case where $D \subset \mathbb{C}^p$, $G \subset \mathbb{C}^q$ are domains and $f : D \times G \to \mathbb{C}$ is a function that is separately of class $\mathcal{F}$ with:

- $\mathcal{F} = \mathcal{O} =$ the class of all *holomorphic functions*,
- $\mathcal{F} = \mathcal{M} =$ the class of all *meromorphic functions*.

In the case of holomorphic functions the answer is positive – *every separately holomorphic function is holomorphic* (Theorem 1.1.7) – this is the famous *Hartogs theorem* (cf. [Har 1906]). In the sequel we will be mostly concentrated on the holomorphic case. We would like to point out that investigations of separately holomorphic functions began in 1899 ([Osg 1899]), that is almost at the same time as Baire's first results on separately continuous functions ([Bai 1899]).

Since the answer to the main question (S-$\mathcal{O}$) is positive, we may consider the following strengthened problem.

(S-$\mathcal{O}_{\mathrm{H}}$) We are given two domains $D \subset \mathbb{C}^p$, $G \subset \mathbb{C}^q$, a non-empty *test set* $B \subset G$, and a function $f : D \times G \to \mathbb{C}$ such that:

- $f(a, \cdot) \in \mathcal{O}(G)$ for every $a \in D$,
- $f(\cdot, b) \in \mathcal{O}(D)$ for every $b \in B$ (only in B).

We ask whether $f \in \mathcal{O}(D \times G)$.

The problem has a long history that began with M. Hukuhara [Huk 1942] (Theorem 1.4.2). (We have to mention that there was a misunderstanding related to the year of publication of [Huk 1942]. Many papers (including ours) name 1930, but the reader may verify (pp. 281–283) that, in fact, it was 1942.) The problem has been continued in [Ter 1967], [Ter 1972], see Theorems 4.2.2 and 4.2.5; compare also the survey article

[Pfl 2003]. Terada was the first to use the pluripotential theory – the newest tool at that time. Roughly speaking, the final result says that the answer is positive iff the set B is not *pluripolar* (i.e. B is not thin from the point of view of the pluricomplex potential theory, cf. Definition 2.3.19).

The problem (S-$\mathcal{O}_{\mathrm{H}}$) leads to the following general question.

(S-$\mathcal{O}_{\mathrm{C}}$) We are given two domains $D \subset \mathbb{C}^p$, $G \subset \mathbb{C}^q$, two non-empty *test sets* $A \subset D$, $B \subset G$. We ask whether there exists an open neighborhood $\widehat{\boldsymbol{X}} \subset D \times G$ of the *cross* $\boldsymbol{X} := (A \times G) \cup (D \times B)$ such that every *separately holomorphic function* $f : (A \times G) \cup (D \times B) \to \mathbb{C}$, i.e.,

- $f(a, \cdot) \in \mathcal{O}(G)$ for every $a \in A$,
- $f(\cdot, b) \in \mathcal{O}(D)$ for every $b \in B$,

extends holomorphically to $\widehat{\boldsymbol{X}}$.

Note that (S-$\mathcal{O}_{\mathrm{H}}$) is just the case where $A = D$ (and, consequently, $\boldsymbol{X} = \widehat{\boldsymbol{X}} = D \times G$).

Investigations of (S-$\mathcal{O}_{\mathrm{C}}$) began with [Ber 1912] and have been continued for instance in [Cam-Sto 1966], [Sic 1968], [Sic 1969a], [Sic 1969b], [Akh-Ron 1973], [Zah 1976], [Sic 1981a], [Shi 1989], [NTV-Sic 1991], [NTV-Zer 1991], [NTV-Zer 1995], [NTV 1997], [Ale-Zer 2001], [Zer 2002]. It turned out (Theorem 5.4.1) that if the sets A, B are *regular* (i.e. every point of A (resp. B) is a density point of A (resp. B) in the sense of the pluricomplex potential theory, cf. Definition 3.2.8), then such a neighborhood $\widehat{\boldsymbol{X}}$ exists.

Similar questions as above may be formulated for a *boundary cross*. To be more precise:

(S-$\mathcal{O}_{\mathrm{B}}$) We are given two domains $D \subset \mathbb{C}^p$, $G \subset \mathbb{C}^q$ and two non-empty sets $A \subset \partial D$, $B \subset \partial G$. We ask whether there exists an open subset $\widehat{\boldsymbol{X}}$ of $D \times G$ with $\boldsymbol{X} \subset \overline{\widehat{\boldsymbol{X}}}$ such that every function $f : \big(A \times (G \cup B)\big) \cup \big((D \cup A) \times B\big) \to \mathbb{C}$ for which

- $f(a, \cdot) \in \mathcal{O}(G)$ for every $a \in A$,
- $f(\cdot, b) \in \mathcal{O}(D)$ for every $b \in B$,
- $f(a, b) = \lim_{D \ni z \to a} f(z, b) = \lim_{G \ni w \to b} f(a, w)$, $(a, b) \in A \times B$ (the limits are taken in a certain sense, e.g. non-tangential),

extends to an $\hat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}})$ and

$$f(a, b) = \lim_{\widehat{\boldsymbol{X}} \ni (z, w) \to (a, b)} \hat{f}(z, w), \quad (a, b) \in \boldsymbol{X},$$

cf. e.g. [Dru 1980], [Gon 1985], [Pfl-NVA 2004], [Pfl-NVA 2007], see Chapter 8.

So far our separately holomorphic functions $f : \boldsymbol{X} \to \mathbb{C}$ had no singularities on $\boldsymbol{X}$. The fundamental paper by E. M. Chirka and A. Sadullaev ([Chi-Sad 1987]) and next some applications to mathematical tomography ([Ökt 1998], [Ökt 1999]) showed that the following problem seems to be important.

(S-$\mathcal{O}_S$) Suppose that $\widehat{X}$ is a solution of (S-$\mathcal{O}_C$). We are given a relatively closed "thin" set (in a certain sense, e.g. pluripolar) $M \subset X := (A \times G) \cup (D \times B)$. We ask whether there exists a "thin" relatively closed set $\widehat{M} \subset \widehat{X}$ such that every *separately holomorphic* function $f : X \setminus M \to \mathbb{C}$, i.e.,

- $f(a, \cdot)$ is holomorphic in $\{w \in G : (a, w) \notin M\}$ for every $a \in A$,
- $f(\cdot, b)$ is holomorphic in $\{z \in D : (z, b) \notin M\}$ for every $b \in B$,

extends holomorphically to $\widehat{X} \setminus \widehat{M}$.

The problem (S-$\mathcal{O}_S$) has been studied for example in [Sic 2001], [Jar-Pfl 2001], [Jar-Pfl 2003a], [Jar-Pfl 2003b], [Jar-Pfl 2003c], [Jar-Pfl 2007], [Jar-Pfl 2010a], [Jar-Pfl 2010b], [Jar-Pfl 2011], see Chapter 10. Observe that the case where $M = \varnothing$ reduces to (S-$\mathcal{O}_C$).

Analogous problems may also be stated for separately meromorphic functions, but their solutions are essentially more difficult. For example, the Hartogs problem corresponds to the following question for separately meromorphic functions.

(S-$\mathcal{M}$) We are given two domains $D \subset \mathbb{C}^p$, $G \subset \mathbb{C}^q$, a "thin" (in a certain sense) relatively closed set $S \subset D \times G$, and a function $f : D \times G \setminus S \to \mathbb{C}$ that is *separately meromorphic* on $D \times G$, i.e.,

- $f(a, \cdot)$ extends meromorphically to G for "almost all" (in a certain sense) $a \in D$,
- $f(\cdot, b)$ extends meromorphically to D for "almost all" $b \in B$.

We ask under what assumptions on S the function f extends meromorphically to $D \times G$.

The problem has been studied for instance in [Kaz 1976], [Kaz 1978], [Kaz 1984], [Shi 1989], [Shi 1991], [Jar-Pfl 2003c], [Pfl-NVA 2003], see Chapter 11.

All the above problems may be formulated also for more general objects than crosses and in the category of Riemann domains over $\mathbb{C}^n$ and/or complex manifolds.

Notice that, instead of complex-valued functions $f : X \to \mathbb{C}$, we may discuss mappings $f : X \to Z$ with values in a complex manifold or even a complex space Z. We will not go in this direction. Let us only mention that in this general case results may be essentially different than for complex-valued functions. For example ([Bar 1975]), let $f : \mathbb{C} \times \mathbb{C} \to \overline{\mathbb{C}}$ be given by

$$f(z_1, z_2) := \begin{cases} \frac{(z_1+z_2)^2}{z_1-z_2} & \text{if } z_1 \neq z_2, \\ \infty & \text{if } z_1 = z_2 \neq 0, \\ 0 & \text{if } z_1 = z_2 = 0. \end{cases}$$

Then f is separately holomorphic but nevertheless, it is not continuous at the origin.

On the other hand we have the following positive result (cf. [Gau-Zer 2009]). Let $\mathbb{P}^m$ be an m-dimensional complex projective space, let $D \subset \mathbb{C}^n$ be a domain, and let $f : D \to \mathbb{P}^m$ be such that for any $(a_1, \dots, a_n) \in D$ and $j, k \in \{1, \dots, n\}$, $j < k$, the function

$$(z_j, z_k) \mapsto f(a_1, \dots, a_{j-1}, z_j, a_{j+1}, \dots, a_{k-1}, z_k, a_{k+1}, \dots, a_n)$$

is holomorphic in an open neighborhood of (a_j, a_k). Then f is holomorphic on D.

See also [Shi 1990], [Shi 1991], [LMH-NVK 2005] for characterizations of those complex spaces Z for which Hartogs' theorem on separately holomorphic mappings $f : D \to Z$ hold.

Part I

Cross theorems without singularities

Chapter 1
Classical results I

Summary. The chapter has an introductory character and collects those fundamental results related to separately holomorphic functions which can be proved by using only "classical" methods of complex analysis, without pluripotential theory. The main result is the famous Hartogs theorem (Theorem 1.1.7) which says that every separately holomorphic function is in fact jointly holomorphic. The key tool used in the proof is the Hartogs lemma (Lemma 1.1.6). We present two (rather non-standard) independent proofs of this lemma based on Leja's polynomial lemma (Lemma 1.1.8) and Koseki's lemma (Lemma 1.1.9). Subsection 1.1.4 contains counterexamples to certain "weakened" versions of the Hartogs lemma. Observe that, in contrast to the holomorphic case, separately continuous functions need not be jointly continuous. This leads to certain intermediate "separately continuous-holomorphic" problems. One of them is presented in § 1.2. As an application of the Hartogs lemma, in § 1.3 we present a characterization of separately pluriharmonic functions. Finally, in § 1.4, we begin our discussion of the so-called Hukuhara problem related to those functions that are holomorphic in all "vertical" directions but only in certain "horizontal" ones. The complete answer needs pluripotential theory and will only be possible in § 4.2.

1.1 Osgood and Hartogs theorems

Definition 1.1.1. For an open set $\varnothing \neq \Omega \subset \mathbb{C}^n$ let $\mathcal{O}(\Omega)$ be the space of all functions holomorphic on Ω.

We say that a function $f\colon \Omega \to \mathbb{C}$ is *separately holomorphic* on Ω (we write $f \in \mathcal{O}_s(\Omega)$) if for any $a = (a_1, \dots, a_n) \in \Omega$ and $j \in \{1, \dots, n\}$, the function $\zeta \mapsto f(a_1, \dots, a_{j-1}, \zeta, a_{j+1}, \dots, a_n)$ is holomorphic in an open neighborhood of a_j (as a function of one complex variable).

For $f\colon \Omega \to \mathbb{C}$, let $\mathcal{S}_{\mathcal{O}}(f)$ denote the set of all points $a \in \Omega$ such that $f \notin \mathcal{O}(U)$ for every open neighborhood $U \subset \Omega$ of a.

Clearly, $\mathcal{O}(\Omega) \subset \mathcal{O}_s(\Omega)$ and $\mathcal{S}_{\mathcal{O}}(f)$ is closed in Ω. The natural problem is whether $\mathcal{O}_s(\Omega) = \mathcal{O}(\Omega)$, i.e. whether every separately holomorphic function is jointly holomorphic. In other words, we ask whether $\mathcal{S}_{\mathcal{O}}(f) = \varnothing$ for arbitrary $f \in \mathcal{O}_s(\Omega)$.

At the end of the 19$^{\text{th}}$ century, due to the Cauchy integral representation, the following equivalence was well known.

Theorem 1.1.2. *Let $\Omega \subset \mathbb{C}^n$ be open and let $f\colon \Omega \to \mathbb{C}$. Then the following conditions are equivalent:*

(i) *f is complex differentiable at any point of Ω;*

(ii) *$f \in \mathcal{O}(\Omega)$;*

(iii) $f \in \mathcal{O}_s(\Omega) \cap \mathcal{C}(\Omega)$.

Proof. EXERCISE. □

Thus $\mathcal{O}_s(\Omega) \cap \mathcal{C}(\Omega) = \mathcal{O}(\Omega)$. The first result dealing with separately holomorphic functions without the continuity assumption was the following one.

Theorem 1.1.3 (Osgood). (a) [Osg 1899] *If $f \in \mathcal{O}_s(\Omega)$ is locally bounded, then f is continuous. Consequently, by Theorem 1.1.2, $\mathcal{O}_s(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega) = \mathcal{O}(\Omega)$.*

(b) [Osg 1900] *Suppose that $n = p + q$ and $f : \Omega \to \mathbb{C}$ is such that for every $(a, b) \in \Omega \subset \mathbb{C}^p \times \mathbb{C}^q$ the functions $z \mapsto f(z, b)$ and $w \mapsto f(a, w)$ are holomorphic in neighborhoods of a and b, respectively (e.g., $n = 2$, $p = q = 1$, $f \in \mathcal{O}_s(\Omega)$). Then the set $\mathcal{S}_{\mathcal{O}}(f)$ is nowhere dense in Ω.*

Define (cf. Symbols, p. 285)

$$\|z\|_\infty := \max\{|z_1|, \dots, |z_n|\}, \quad z = (z_1, \dots, z_n) \in \mathbb{C}^n,$$
$$\mathbb{P}(a, r) = \mathbb{P}_n(a, r) := \{z \in \mathbb{C}^n : \|z - a\|_\infty < r\}, \quad a \in \mathbb{C}^n,\ 0 < r \le +\infty,$$
$$\mathbb{P}(r) = \mathbb{P}_n(r) := \mathbb{P}_n(0, r),$$
$$\mathbb{D}(a, r) := \mathbb{P}_1(a, r), \quad \mathbb{D}(r) := \mathbb{P}_1(r), \quad \mathbb{D} := \mathbb{D}(1), \quad \mathbb{T} = \partial\mathbb{D}.$$

Proof. (a) Nowadays standard proofs of (a) are based on the Schwarz lemma. If $|f(z)| \le C$ for $z \in \mathbb{P}(a, r) \subset \Omega$, then

$$\begin{aligned}|f(z) - f(a)| &\le |f(z_1, z_2, z_3, \dots, z_n) - f(a_1, z_2, z_3, \dots, z_n)| \\ &\quad + |f(a_1, z_2, z_3, \dots, z_n) - f(a_1, a_2, z_3, \dots, z_n)| \\ &\quad + \dots + |f(a_1, \dots, a_{n-1}, z_n) - f(a_1, \dots, a_{n-1}, a_n)| \\ &\le \frac{2C}{r}(|z_1 - a_1| + |z_2 - a_2| + \dots + |z_n - a_n|),\end{aligned}$$

and consequently f is continuous at a.

(b) follows from a Baire argument. Let $\mathbb{P}_p(a, r) \times \mathbb{P}_q(b, r) \subset\subset \Omega$ be arbitrary. Define

$$A_k := \{z \in \mathbb{P}_p(a, r) : \forall_{w \in \mathbb{P}_q(b,r)} : |f(z, w)| \le k\}, \quad k \in \mathbb{N}.$$

Then A_k is closed in $\mathbb{P}_p(a, r)$ and $\mathbb{P}_p(a, r) = \bigcup_{k=1}^\infty A_k$. Hence, by Baire's theorem, there exists a k_0 such that A_{k_0} has a non-empty interior. Thus f is bounded on a non-empty open set $U = \mathbb{P}_p(c, \delta) \times \mathbb{P}_q(b, r) \subset \mathbb{P}_p(a, r) \times \mathbb{P}_q(b, r)$. By (a), $f \in \mathcal{O}(U)$. Hence $U \subset \Omega \setminus \mathcal{S}_{\mathcal{O}}(f)$. □

Remark 1.1.4. Another proof of Theorem 1.1.3 (a), based on Montel's theorem, may be done by induction on n.

Indeed, suppose the result is true for $n - 1$ and let $f \in \mathcal{O}_s(\Omega)$ be locally bounded. We only need to show that f is continuous.

Take a polydisc $\mathbb{P}(a,r) \subset\subset \Omega$. Write $z = (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C}$. Take a sequence $\mathbb{P}(a,r) \ni a^k \to a$ such that $f(a^k) \to \alpha \in \mathbb{C}$. We want to show that $\alpha = f(a)$. Since f is locally bounded in Ω, the sequence $(f((a^k)', \cdot))_{k=1}^{\infty} \subset \mathcal{O}(\mathbb{D}(a_n, r))$ is locally bounded. By Montel's theorem, there exists a subsequence $(k_s)_{s=1}^{\infty}$ such that $f((a^{k_s})', \cdot) \to g$ locally uniformly in $\mathbb{D}(a_n, r)$ with $g \in \mathcal{O}(\mathbb{D}(a_n, r))$.

By the inductive assumption, $f(\cdot, z_n) \in \mathcal{O}(\mathbb{P}_{n-1}(a', r))$ for all $z_n \in \mathbb{D}(a_n, r)$. In particular, $f(\cdot, z_n) \in \mathcal{C}(\mathbb{P}_{n-1}(a', r))$ for all $z_n \in \mathbb{D}(a_n, r)$. Thus $g = f(a', \cdot)$ and, finally, $\alpha = \lim_{s\to+\infty} f((a^{k_s})', a_n^{k_s})) = g(a_n) = f(a)$.

Remark 1.1.5. Observe that the assumption in Theorem 1.1.3 (a) may be weakened, namely:

If $f \in \mathcal{O}_s(\Omega)$ is locally integrable, then f is continuous. Consequently, by Theorem 1.1.2, $\mathcal{O}_s(\Omega) \cap L^1_{\mathrm{loc}}(\Omega) = \mathcal{O}(\Omega)$.

Indeed, take a polydisc $\mathbb{P}(a,r) \subset\subset \Omega$. Then the one-dimensional Cauchy integral formula gives

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\partial\mathbb{D}(a_1,r)} \frac{f(\zeta_1, z_2, \dots, z_n)}{\zeta_1 - z_1} d\zeta_1 \\ &= \dots = \frac{1}{2\pi i} \int_{\partial\mathbb{D}(a_1,r)} \Big(\dots \Big(\frac{1}{2\pi i} \int_{\partial\mathbb{D}(a_n,r)} \frac{f(\zeta_1, \dots, \zeta_n)}{\zeta_n - z_n} d\zeta_n \Big) \dots \Big) \frac{1}{\zeta_1 - z_1} d\zeta_1, \\ & \qquad z = (z_1, \dots, z_n) \in \mathbb{P}(a,r). \end{aligned}$$

Since f is integrable on $\partial_0 \mathbb{P}(a,r) := \partial\mathbb{D}(a_1,r) \times \dots \times \partial\mathbb{D}(a_n,r)$, we may apply Fubini's theorem, which gives

$$f(z) = \Big(\frac{1}{2\pi i}\Big)^n \int_{\partial_0\mathbb{P}(a,r)} \frac{f(\zeta_1, \dots, \zeta_n)}{(\zeta_1 - z_1)\cdots(\zeta_n - z_n)} d\zeta_1 \cdots d\zeta_n, \qquad z = (z_1, \dots, z_n) \in \mathbb{P}(a,r),$$

and consequently, f is continuous on $\mathbb{P}(a,r)$.

W. Osgood also observed that the proof of Theorem 1.1.3 (b) gives more. Namely, it shows that in order to prove that $\mathcal{O}_s(\Omega) = \mathcal{O}(\Omega)$ for an arbitrary open set $\Omega \subset \mathbb{C}^n$, it suffices to check the following lemma.

Lemma 1.1.6. *Let $f : \mathbb{D}(r) \times \mathbb{P}_m(r) \to \mathbb{C}$ be such that*

- *$f(a, \cdot) \in \mathcal{O}(\mathbb{P}_m(r))$ for every $a \in \mathbb{D}(r)$,*
- *$f \in \mathcal{O}(\mathbb{D}(r) \times \mathbb{P}_m(\delta))$ for some $0 < \delta < r$.*

Then $f \in \mathcal{O}(\mathbb{D}(r) \times \mathbb{P}_m(r))$.

Proof that Lemma 1.1.6 implies that $\mathcal{O}_s(\Omega) = \mathcal{O}(\Omega)$. We use induction on n. For $n = 1$ the theorem is trivial. Suppose that $\mathcal{O}_s(\Omega) = \mathcal{O}(\Omega)$ for an arbitrary open set $\Omega \subset \mathbb{C}^{n-1}$.

Fix $\Omega \subset \mathbb{C}^n = \mathbb{C} \times \mathbb{C}^{n-1}$ and $f \in \mathcal{O}_s(\Omega)$. It is sufficient to show that f is holomorphic in a neighborhood of an arbitrary point $(z_0, w_0) \in \Omega$. Let $\mathbb{P}_n((z_0, w_0), 2r) \subset \Omega$ and let

$$A_k := \{w \in \mathbb{P}_{n-1}(w_0, r) : \forall_{z\in\mathbb{D}(z_0,r)} : |f(z,w)| \le k\}, \quad k \in \mathbb{N}.$$

Since $f(z,\cdot) \in \mathcal{C}(\mathbb{P}_{n-1}(w_0, 2r))$ for arbitrary $z \in \mathbb{D}(z_0, 2r)$ (by the inductive assumption), the sets A_k are closed in $\mathbb{P}_{n-1}(w_0, r)$. Moreover, we have $\bigcup_{k\in\mathbb{N}} A_k = \mathbb{P}_{n-1}(w_0, r)$. Using Baire's property we conclude that $\operatorname{int} A_{k_0} \ne \varnothing$ for some k_0. Let $\mathbb{P}_{n-1}(\xi_0, \delta) \subset A_{k_0}$. In particular, by Theorem 1.1.3 (a), $f \in \mathcal{O}(\mathbb{D}(z_0, r) \times \mathbb{P}_{n-1}(\xi_0, \delta))$. Now we apply Lemma 1.1.6 (with $m := n - 1$) to the function

$$\mathbb{D}(r) \times \mathbb{P}_{n-1}(r) \ni (z, w) \mapsto f(z_0 + z, \xi_0 + w),$$

and we conclude that $f \in \mathcal{O}(\mathbb{P}_n((z_0, \xi_0), r))$. It remains to observe that $(z_0, w_0) \in \mathbb{P}_n((z_0, \xi_0), r)$. □

Finally, F. Hartogs, based on the above remark by Osgood, finished the discussion on the equality $\mathcal{O}_s(\Omega) = \mathcal{O}(\Omega)$ and proved in [Har 1906] the following theorem.

Theorem 1.1.7 (Hartogs' theorem). *Lemma 1.1.6 is true. Consequently, $\mathcal{O}_s(\Omega) = \mathcal{O}(\Omega)$ for an arbitrary open set $\Omega \subset \mathbb{C}^n$.*

In his proof Hartogs used for the first time methods from potential theory in complex analysis. Nowadays there exist various proofs of Lemma 1.1.6. We present below two of them:

- Leja's proof [Lej 1950], based on *Leja's polynomial lemma* (Lemma 1.1.8),
- Koseki's proof [Kos 1966], based on an elementary version of the Hartogs lemma for special subharmonic functions (Lemma 1.1.9).

We like to point out that both proofs are based on classical complex analysis and are independent of the Hartogs lemma for plurisubharmonic functions (cf. Proposition 2.3.13).

1.1.1 Leja's proof

Let $\mathcal{P}(\mathbb{C}^n)$ denote the space of all complex polynomials of n complex variables. For $A \subset \mathbb{C}$, let $\operatorname{diam} A := \sup\{|z - w| : z, w \in A\}$ be the *diameter* of A.

Lemma 1.1.8 (Leja's polynomial lemma, cf. [Lej 1933a], [Lej 1933b]). *Let $K \subset \mathbb{C}$ be a compact set such that*

$$\inf\{\operatorname{diam} S : S \text{ is a connected component of } K\} > 0$$

(*e.g. K is a compact connected set having more than one point* (***continuum***)). *Let $\mathcal{F} \subset \mathcal{P}(\mathbb{C})$ be such that*

$$\forall_{z \in K} : \sup_{p \in \mathcal{F}} |p(z)| < +\infty,$$

i.e. $\mathcal{F}$ is pointwise bounded on K. Then

$$\forall_{a \in K}\ \forall_{\omega > 1}\ \exists_{M = M(K,a,\omega,\mathcal{F}) > 0}\ \exists_{\eta = \eta(K,a,\omega) > 0}\ \forall_{p \in \mathcal{F}} : \sup_{z \in \mathbb{D}(a,\eta)} |p(z)| \le M\omega^{\deg p},$$

or equivalently (EXERCISE)*,*

$$\forall_{\omega > 1}\ \exists_{M = M(K,\omega,\mathcal{F}) > 0}\ \exists_{\substack{\Omega = \Omega(K,\omega) \text{ open} \\ K \subset \Omega}}\ \forall_{p \in \mathcal{F}} : \sup_{z \in \Omega} |p(z)| \le M\omega^{\deg p}.$$

Notice that η and Ω are independent of $\mathcal{F}$.

Proof. Let $r > 0$ be such that $\operatorname{diam} S \ge 2r$ for every connected component S of K.

Step 1^0: Let $A \subset [0, r]$ be a closed set with $m := \mathcal{L}^1(A) > 0$. Then for every $d \in \mathbb{N}$ there exist $t_0, \dots, t_d \in A$, $0 \le t_0 < \dots < t_d \le r$, such that

$$t_k - t_j \ge \frac{k^2 - j^2}{d^2} m, \quad j, k = 0, \dots, d,\ j < k.$$

Proof of Step 1^0. Let $t_0 = \min A$, $s_0 := t_0 + \frac{1^2}{d^2} m$. Then $A_1 := A \setminus [0, s_0)$. Observe that A_1 is closed and non-empty (if $A \subset [t_0, s_0)$, then $m = \mathcal{L}^1(A) < s_0 - t_0 = \frac{1^2}{d^2} m \le m$ – a contradiction).

Put $t_1 := \min A_1$, $s_1 := t_1 + \frac{2^2 - 1^2}{d^2} m$. Then $t_1 - t_0 \ge s_0 - t_0 = \frac{1^2 - 0^2}{d^2} m$. Let $A_2 := A \setminus [0, s_1)$; A_2 is again closed and non-empty (if $A \subset [t_0, s_0) \cup [t_1, s_1)$), then $m < s_0 - t_0 + s_1 - t_1 = \frac{2^2}{d^2} m \le m$ – a contradiction). Let $t_2 := \min A_2$. Then $t_2 - t_1 \ge s_1 - t_1 = \frac{2^2 - 1^2}{d^2} m$.

We continue and we get $t_0, \dots, t_{d-1} := \min A_{d-1}$, $s_{d-1} := t_{d-1} + \frac{d^2 - (d-1)^2}{d^2} m$, $A_d := A \setminus [0, s_{d-1})$. Suppose that $A_d = \varnothing$. Then $A \subset [t_0, s_0) \cup \dots \cup [t_{d-1}, s_{d-1})$ and hence $m < s_0 - t_0 + \dots + s_{d-1} - t_{d-1} = \frac{d^2}{d^2} m = m$ – a contradiction. Thus $t_d := \min A_d$ is well defined and $t_d - t_{d-1} \ge s_{d-1} - t_{d-1} = \frac{d^2 - (d-1)^2}{d^2} m$.

Step 2^0: For any $B \subset K$ and $a \in K$, let

$$\pi_a(B) := \{t \in [0, r] : B \cap \partial\mathbb{D}(a, t) \ne \varnothing\}.$$

Observe that if B is closed, then so is $\pi_a(B)$ (EXERCISE).

Step 3^0: Let

$$I(\alpha) := \exp\Big(\int_0^1 \log \frac{\alpha^2 + x^2}{x^2} dx\Big), \quad \alpha \ge 0.$$

Observe that $\log I(\alpha) = \log(1+\alpha^2) + 2\alpha\arctan(1/\alpha)$ (EXERCISE). In particular, $\log I(\alpha) \le (\pi+\alpha)\alpha$.

From now on p is an arbitrary polynomial from the family $\mathcal{F}$ and $d := \deg p$.

Step 4^0: For any closed set $B \subset K$, $a \in K$, and $\eta > 0$ we have

$$\|p\|_{\overline{\mathbb{D}}(a,\eta)} \le \|p\|_B [I(\alpha)]^d,$$

where

$$\|p\|_C := \sup_C |p|, \quad A := \pi_a(B), \quad m := \mathcal{L}^1(A), \quad \alpha := \sqrt{\frac{\eta+r-m}{m}}.$$

In particular, if $B = K$, then $A = [0,r]$ and, consequently,

$$\|p\|_{K^{(\eta)}} \le \|p\|_K [I(\sqrt{\eta/r})]^d,$$

where $K^{(\eta)} := \bigcup_{a\in K} \overline{\mathbb{P}}_n(a,\eta)$; notice that $K^{(\eta)}$ is also compact (EXERCISE).

Proof of Step 4^0. The case where $m = 0$ is obvious. Assume that $m > 0$. Let $t_0, \dots, t_d$ be as in Step 1^0. Take arbitrary $z_k \in B \cap \partial\mathbb{D}(a, t_k)$, $k = 0, \dots, d$. Let $T_d := \{z_0, \dots, z_d\}$,

$$L^{(k)}(z, T_d) := \prod_{\substack{j=0\\ j\ne k}}^{d} \frac{z - z_j}{z_k - z_j}, \quad k = 0, \dots, d,$$

be the *Lagrange interpolation polynomials with nodes at* T_d. If $z \in \overline{\mathbb{D}}(a,\eta)$, then $|z - z_j| \le |z-a| + |z_j - a| \le \eta + t_j$. Moreover, since $r - t_j \ge t_d - t_j \ge \frac{d^2-j^2}{d^2}m$, we get $t_j \le r - m + \frac{j^2}{d^2}m$, and, consequently, $|z - z_j| \le (\alpha^2 + \frac{j^2}{d^2})m$, $j = 0, \dots, d$. On the other hand, $|z_k - z_j| \ge |t_k - t_j| \ge \frac{|k^2-j^2|}{d^2}m$. Thus, if $z \in \overline{\mathbb{D}}(a,\eta)$, then

$$|L^{(k)}(z, T_d)| \le \prod_{\substack{j=0\\ j\ne k}}^{d} \frac{\alpha^2 + \frac{j^2}{d^2}}{\frac{|k^2-j^2|}{d^2}} \overset{(*)}{\le} 2\prod_{j=1}^{d} \frac{\alpha^2 + \frac{j^2}{d^2}}{\frac{j^2}{d^2}} \overset{(**)}{\le} 2[I(\alpha)]^d,$$

where $(*)$ follows from the inequality $\prod_{\substack{j=0\\ j\ne k}}^{d} |j^2 - k^2| = \frac{1}{2}(d+k)!(d-k)! \ge \frac{1}{2}(d!)^2$ (EXERCISE), and $(**)$ follows from the fact that the function $[0,1] \ni x \mapsto \log\frac{\alpha^2+x^2}{x^2}$ is decreasing and, therefore (EXERCISE),

$$\frac{1}{d}\sum_{j=1}^{d} \log \frac{\alpha^2 + \frac{j^2}{d^2}}{\frac{j^2}{d^2}} \le \int_0^1 \log\frac{\alpha^2+x^2}{x^2}dx = \log I(\alpha).$$

Observe that $p^s(z) = \sum_{k=0}^{sd} p^s(z_k)L^{(k)}(z, T_{sd})$, $z \in \mathbb{C}$. Hence, for $z \in \overline{\mathbb{D}}(a, \eta)$, we have

$$|p(z)|^s = \Big|\sum_{k=0}^{sd} p^s(z_k)L^{(k)}(z, T_{sd})\Big| \le \|p\|_B^s(sd + 1)2[I(\alpha)]^{sd},$$

which implies that

$$|p(z)| \le \|p\|_B \sqrt[s]{2(sd + 1)}[I(\alpha)]^d.$$

It remains to let $s \to +\infty$.

Step 5^0: Let $(K_s)_{s=1}^{\infty}$ be a sequence of compact subsets of K such that $K_s \subset K_{s+1}$, $K = \bigcup_{s=1}^{\infty} K_s$. Then for every $\eta > 0$ there exists a sequence $(m_s(\eta))_{s=1}^{\infty} \subset [0, r]$ with $m_s(\eta) \to r$ such that

$$\|p\|_{K^{(\eta)}} \le \|p\|_K[I(\sqrt{\eta/r})I(\alpha_s)]^d,$$

where

$$\alpha_s := \sqrt{\frac{\eta + r - m_s(\eta)}{m_s(\eta)}}, \quad s \in \mathbb{N}.$$

Proof of Step 5^0. Take $a_1, \dots, a_N \in K$ such that $K \subset \bigcup_{k=1}^{N} \overline{\mathbb{D}}(a_k, \eta) =: L$. Let $A_{k,s} := \pi_{a_k}(K_s)$. Then $A_{k,s} \nearrow [0, r]$ when $s \nearrow +\infty$, which implies that $\mathcal{L}^1(A_{k,s}) \nearrow r$ when $s \nearrow +\infty$, $k = 1, \dots, N$. Put $m_s(\eta) := \min_{k=1,\dots,N} \mathcal{L}^1(A_{k,s})$. It is clear that $m_s(\eta) \to r$. Fix an $s \in \mathbb{N}$. Then, by Step 4^0 (with $B := K_s$), we get

$$\|p\|_{\overline{\mathbb{D}}(a_k,\eta)} \le \|p\|_{K_s}[I(\alpha_{k,s})]^d \le \|p\|_{K_s}[I(\alpha_s)]^d,$$

where

$$\alpha_{k,s} := \sqrt{\frac{\eta + r - \mathcal{L}^1(A_{k,s})}{\mathcal{L}^1(A_{k,s})}}.$$

Hence,

$$\|p\|_L \le \|p\|_{K_s}[I(\alpha_s)]^d,$$

and, finally, by Step 4^0, we get the required inequality.

Step 6^0: Let

$$K_s := \{z \in K : \forall_{p\in\mathcal{F}} : |p(z)| \le s\}, \quad s \in \mathbb{N}.$$

Observe that K_s is compact, $K_s \subset K_{s+1}$, and $K = \bigcup_{s=1}^{\infty} K_s$ (because $\mathcal{F}$ is pointwise bounded on K). Let $F_{s,d} = \{z_{s,0}, \dots, z_{s,d}\}$ be the *d-th system of Fekete points for* K_s, i.e. $F_{s,d}$ realizes the maximum of the continuous function

$$K_s^{d+1} \ni (z_0, \dots, z_d) \mapsto \prod_{0\le j<k\le d} |z_j - z_k|.$$

Observe that $|L^{(k)}(z, F_{s,d})| \le 1$ for $z \in K_s$. Hence, by Step 5^0, we get

$$|p(z)| = \Big|\sum_{k=0}^{d} p(z_{s,k}) L^{(k)}(z, F_{s,d})\Big| \le s(d+1)[I(\sqrt{\eta/r})I(\alpha_s)]^d,$$

where $z \in K^{(\eta)}$, $s \in \mathbb{N}$.

Step 7^0: We move to the main proof. Fix an $\omega > 1$. Let $d_0 = d_0(\omega) \in \mathbb{N}$ be such that $\sqrt[d]{d+1} \le \sqrt[3]{\omega}$ for $d \ge d_0$. Let $\eta = \eta(r,\omega) > 0$ be so small that $I(\sqrt{\eta/r}) < \sqrt[3]{\omega}$. Finally, let $s_0 = s_0(r,\omega) \in \mathbb{N}$ be such that $I(\alpha_s) \le \sqrt[3]{\omega}$ for $s \ge s_0$. In view of Step 5^0, if $d \ge d_0$, then

$$\|p\|_{K^{(\eta)}} \le s(d+1)[I(\sqrt{\eta/r})I(\alpha_s)]^d \le s\omega^d.$$

It remains to find an estimate for $d < d_0$. Let $S := \{z_0, \dots, z_{d_0}\} \subset K$ be an arbitrary set of $d_0 + 1$ pairwise distinct points. Put

$$M_0 := \max_{k=0,\dots,d_0} \sup_{p \in \mathcal{F}} |p(z_k)| < +\infty.$$

Then, for $d \le d_0$ we get

$$|p(z)| = \Big|\sum_{k=0}^{d_0} f(z_k) L^{(k)}(z, S)\Big| \le M_0 \widetilde{M} \le M_0 \widetilde{M} \omega^d, \quad z \in K^{(\eta)},$$

where $\widetilde{M} := \max_{z \in K^{(\eta)}} |L^{(k)}(z, S)|$. It remains to put $M := \max\{s_0, M_0 \widetilde{M}\}$. □

Proof of Lemma 1.1.6 via Leja's polynomial lemma (*cf.* [Lej 1950]). Let $n := 1 + m$. Observe that it is sufficient to show that $f \in \mathcal{O}(\mathbb{P}_n(r'))$ for arbitrary $0 < r' < r$. Thus we may assume that $|f| \le C < +\infty$ in $\mathbb{D}(r) \times \mathbb{P}_m(\delta)$ and $f(z, \cdot)$ is bounded for any $z \in \mathbb{P}_m(r)$. We have

$$f(z, w) = \sum_{\alpha \in \mathbb{Z}_+^m} f_\alpha(z) w^\alpha, \quad z \in \mathbb{D}(r),\ w \in \mathbb{P}_m(r),$$

where

$$f_\alpha(z) = \frac{1}{\alpha!}(D^\alpha f(z, \cdot))(0) = \frac{1}{\alpha!}(D^{(0,\alpha)} f)(z, 0), \quad z \in \mathbb{D}(r),\ \alpha \in \mathbb{Z}_+^m.$$

The last equality follows from the fact that $f \in \mathcal{O}(\mathbb{D}(r) \times \mathbb{P}_m(\delta))$. In particular, $f_\alpha \in \mathcal{O}(\mathbb{D}(r))$ for arbitrary α. Moreover, by the Cauchy inequalities, we obtain

$$|f_\alpha(z)| \le C/\delta^{|\alpha|}, \quad z \in \mathbb{D}(r),\ \alpha \in \mathbb{Z}_+^m. \tag{1.1.1}$$

Applying once more the Cauchy inequalities (for the function $f(z, \cdot)$), we have

$$|f_\alpha(z)| \le \frac{\|f(z, \cdot)\|_{\mathbb{P}_m(r)}}{r^{|\alpha|}}, \quad z \in \mathbb{D}(r),\ \alpha \in \mathbb{Z}_+^m. \tag{1.1.2}$$

Consequently,

$$\limsup_{|\alpha|\to+\infty} |f_\alpha(z)|^{1/|\alpha|} \le 1/r, \quad z \in \mathbb{D}(r).$$

Our aim is to show that the series $\sum_{\alpha\in\mathbb{Z}_+^m} f_\alpha(z)w^\alpha$ converges locally normally in $\mathbb{D}(r) \times \mathbb{P}_m(r)$.

Take an arbitrary $\theta \in (0,1)$ and let $\omega > 1$ be such that $\theta_0 := \omega^2\theta < 1$. Fix a point $a \in \mathbb{D}(r)$ and $0 < \rho < r - |a|$. Let $0 < \rho_0 < \rho$ be so small that $r\rho_0 \le \omega\delta\rho$. Write

$$f_\alpha(z) = \sum_{k=0}^{\infty} f_{\alpha,k}(z-a)^k, \quad z \in \mathbb{D}(a, r-|a|),$$

$$p_\alpha(z) := \sum_{k=0}^{|\alpha|} f_{\alpha,k}(z-a)^k, \quad \mathcal{F} := \{(r/\omega)^{|\alpha|} p_\alpha : \alpha \in \mathbb{Z}_+^m\}.$$

In view of (1.1.1), the Cauchy inequalities imply that

$$|f_{\alpha,k}| \le \frac{C}{\delta^{|\alpha|}\rho^k}.$$

Consequently, in view of (1.1.2), if $z \in \overline{\mathbb{D}}(a, \rho_0)$, then

$$\begin{aligned}
|p_\alpha(z)| &\le |f_\alpha(z)| + \sum_{k=|\alpha|+1}^{\infty} |f_{\alpha,k}(z-a)^k| \le C(z)\Big(\frac{\omega}{r}\Big)^{|\alpha|} + \frac{C}{\delta^{|\alpha|}}\Big(\frac{\rho_0}{\rho}\Big)^{|\alpha|+1}\frac{1}{1-\frac{\rho_0}{\rho}} \\
&\le C(z)\Big(\frac{\omega}{r}\Big)^{|\alpha|} + \frac{C}{1-\frac{\rho_0}{\rho}}\frac{\rho_0}{\rho}\Big(\frac{\rho_0}{\delta\rho}\Big)^{|\alpha|} \le C(z)\Big(\frac{\omega}{r}\Big)^{|\alpha|} + C_1\Big(\frac{\omega}{r}\Big)^{|\alpha|}.
\end{aligned}$$

Hence, the family $\mathcal{F}$ is pointwise bounded on $\overline{\mathbb{D}}(a, \rho_0)$. By Leja's polynomial lemma there exist $0 < \eta \le \rho_0$ and $M > 0$ such that

$$\Big(\frac{r}{\omega}\Big)^{|\alpha|}|p_\alpha(z)| \le M\omega^{|\alpha|}, \quad z \in \mathbb{D}(a,\eta),\ \alpha \in \mathbb{Z}_+^m.$$

Finally, for $(z,w) \in \mathbb{D}(a,\eta) \times \mathbb{P}_m(\theta r)$ we get

$$\begin{aligned}
|f_\alpha(z)w^\alpha| &\le \Big(|p_\alpha(z)| + C_1\Big(\frac{\omega}{r}\Big)^{|\alpha|}\Big)(\theta r)^{|\alpha|} \\
&\le M(\omega^2\theta)^{|\alpha|} + C_1(\omega\theta)^{|\alpha|} \le (M + C_1)\theta_0^{|\alpha|},
\end{aligned}$$

which implies that the series $\sum_{\alpha\in\mathbb{Z}_+^m} f_\alpha(z)w^\alpha$ converges normally in $\mathbb{D}(a,\eta)\times\mathbb{P}_m(\theta r)$. □

1.1.2 Koseki's proof

The main ingredient of Koseki's proof is the following lemma (cf. Proposition 2.3.13).

Lemma 1.1.9 (Koseki's lemma, cf. [Kos 1966]). *Let $\Omega \subset \mathbb{C}$ be open, $\varphi_\nu \in \mathcal{O}(\Omega)$, $p_\nu > 0$, $\nu \geq 1$. Assume that the sequence $(|\varphi_\nu|^{p_\nu})_{\nu=1}^{\infty}$ is locally uniformly bounded in Ω and*

$$\limsup_{\nu\to+\infty} |\varphi_\nu(z)|^{p_\nu} \leq c, \quad z \in \Omega.$$

Then for any $K \subset\subset \Omega$ and $\varepsilon > 0$ there exists a ν_0 such that

$$|\varphi_\nu(z)|^{p_\nu} \leq c + \varepsilon, \quad z \in K, \ \nu \geq \nu_0.$$

Proof. The result is local – it is sufficient to show that for any $\varepsilon > 0$ and $a \in \Omega$ there exist a disc $\mathbb{D}(a,\eta) \subset \Omega$ and ν_0 such that

$$|\varphi_\nu(z)|^{p_\nu} \leq c + \varepsilon, \quad z \in \mathbb{D}(a,\eta), \ \nu \geq \nu_0.$$

We may assume that $\Omega = \mathbb{D}(2)$, $a = 0$. Let $C > 0$ be such that $|\varphi_\nu|^{p_\nu} \leq C$ in $\mathbb{D}$ for arbitrary ν. We may also assume that $\varphi_\nu \not\equiv 0$, $\nu \geq 1$. Write $\varphi_\nu = B_\nu \psi_\nu$ in $\mathbb{D}$, where B_ν is a finite Blaschke product and ψ_ν has no zeros in $\mathbb{D}$. Let $\chi_\nu \in \mathcal{O}(\mathbb{D})$ be a branch of $\psi_\nu^{p_\nu}$ in $\mathbb{D}$. Given arbitrary $\zeta \in \partial\mathbb{D}$, we have

$$\limsup_{\mathbb{D}\ni z\to\zeta} |\chi_\nu(z)| = \limsup_{\mathbb{D}\ni z\to\zeta} |\psi_\nu(z)|^{p_\nu} = \limsup_{\mathbb{D}\ni z\to\zeta} |\varphi_\nu(z)|^{p_\nu} \leq C,$$

and so $|\chi_\nu| \leq C$ in $\mathbb{D}$, $\nu \in \mathbb{N}$. In particular, the family $(\chi_\nu)_{\nu=1}^{\infty}$ is equicontinuous in $\mathbb{D}$. Fix an $\varepsilon > 0$ and let $0 < \eta < 1$ be such that $|\chi_\nu(z) - \chi_\nu(0)| \leq \varepsilon/2$ for $z \in \overline{\mathbb{D}}(\eta)$ and $\nu \geq 1$. Then

$$|\varphi_\nu(z)|^{p_\nu} \leq |\psi_\nu(z)|^{p_\nu} = |\chi_\nu(z)| \leq \varepsilon/2 + |\chi_\nu(0)|, \quad z \in \overline{\mathbb{D}}(\eta), \ \nu \geq 1.$$

It remains to estimate $\chi_\nu(0)$. Observe that

$$|\chi_\nu(0)| \leq \frac{1}{2\pi}\int_0^{2\pi} |\varphi_\nu(e^{i\theta})|^{p_\nu}\, d\theta, \quad \nu \geq 1.$$

Let

$$A_k := \{\theta \in [0,2\pi] : |\varphi_\nu(e^{i\theta})|^{p_\nu} \leq c + \varepsilon/4, \ \nu \geq k\}.$$

The sets A_k are closed, $A_k \subset A_{k+1}$, and $\bigcup_{k\in\mathbb{N}} A_k = [0,2\pi]$. For $\nu \geq k$ we have

$$\begin{aligned}|\chi_\nu(0)| &\leq \frac{1}{2\pi}\Big(\int_{A_k} |\varphi_\nu(e^{i\theta})|^{p_\nu} d\theta + \int_{[0,2\pi]\setminus A_k} |\varphi_\nu(e^{i\theta})|^{p_\nu} d\theta\Big)\\ &\leq \frac{1}{2\pi}\big((c+\varepsilon/4)\mathcal{L}^1(A_k) + C(2\pi - \mathcal{L}^1(A_k))\big) \to c + \varepsilon/4,\end{aligned}$$

where $\mathcal{L}^1$ denotes the Lebesgue measure in $\mathbb{R}$. Hence $|\chi_\nu(0)| \leq c + \varepsilon/2$ for $\nu \gg 1$. $\square$

Proof of Lemma 1.1.6 via Koseki's lemma. We begin as in the proof based on Leja's polynomial lemma:

$$f(z,w) = \sum_{\alpha\in\mathbb{Z}_+^m} f_\alpha(z)w^\alpha, \quad z\in\mathbb{D}(r),\ w\in\mathbb{P}_m(r),$$

where

$$f_\alpha\in\mathcal{O}(\mathbb{D}(r)), \quad |f_\alpha(z)|\le C/\delta^{|\alpha|}, \quad z\in\mathbb{D}(r),\ \alpha\in\mathbb{Z}_+^m,$$
$$\limsup_{|\alpha|\to+\infty} |f_\alpha(z)|^{1/|\alpha|}\le 1/r, \quad z\in\mathbb{D}(r).$$

Write $\mathbb{Z}_+^m = \{\alpha_1,\alpha_2,\dots\}$ so that $|\alpha_\nu|\le|\alpha_{\nu+1}|$, $\nu = 1,2,\dots$. Let $\Omega := \mathbb{D}(r)$, $\varphi_\nu := f_{\alpha_\nu}$, $p_\nu := 1/|\alpha_\nu|$. Fix a $\theta\in(0,1)$ and let $\varepsilon>0$ be such that $(1+r\varepsilon)\theta<1$. Applying Lemma 1.1.9 to $K := \overline{\mathbb{D}}(\theta r)$, we obtain $|\varphi_\nu(z)|^{p_\nu}\le 1/r+\varepsilon$ for $z\in\overline{\mathbb{D}}(\theta r)$ and $\nu\ge\nu_0$. This means that

$$|f_\alpha(z)|\le(1/r+\varepsilon)^{|\alpha|}, \quad z\in\mathbb{D}(\theta r),\ |\alpha|\gg 1.$$

Hence

$$|f_\alpha(z)w^\alpha|\le((1+r\varepsilon)\theta)^{|\alpha|}, \quad z\in\mathbb{D}(\theta r),\ w\in\mathbb{P}_m(\theta r),\ |\alpha|\gg 1.$$

Consequently, the series $\sum_{\alpha\in\mathbb{Z}_+^m} f_\alpha(z)w^\alpha$ is convergent normally in $\mathbb{P}_n(\theta r)$, which implies that $f\in\mathcal{O}(\mathbb{P}_n(\theta r))$. Since θ was arbitrary, we conclude that $f\in\mathcal{O}(\mathbb{P}_n(r))$. □

1.1.3 Applications

Theorem 1.1.7 permits us to generalize Lemma 1.1.6 to the following useful form.

Proposition 1.1.10. (a) *Let $D\subset\mathbb{C}^p$, $G\subset\mathbb{C}^q$ be domains and let $\varnothing\ne B\subset G$ be open. Let $f: D\times G\to\mathbb{C}$ be such that*

- *$f(a,\cdot)\in\mathcal{O}(G)$, $a\in D$,*
- *$f(\cdot,b)\in\mathcal{O}(D)$, $b\in B$.*

Then $f\in\mathcal{O}(D\times G)$.

(b) *Let $D\subset\mathbb{C}^p$, $\Omega\subset D\times\mathbb{C}^q$ be domains such that for every $a\in D$ the fiber $\Omega_{(a,\cdot)} := \{w\in\mathbb{C}^q : (a,w)\in\Omega\}$ is connected. Let $U\subset\Omega$ be an open set such that for every $a\in D$ the fiber $U_{(a,\cdot)}$ is non-empty. Let $f:\Omega\to\mathbb{C}$ be such that*

- *$f(a,\cdot)\in\mathcal{O}(\Omega_{(a,\cdot)})$, $a\in D$,*
- *$f\in\mathcal{O}(U)$.*

Then $f\in\mathcal{O}(\Omega)$.

Proof. (a) Observe that Hartogs' Theorem 1.1.7 implies that $f \in \mathcal{O}(D \times B)$.

First, consider the case where $D = \mathbb{D}^p$, $G = \mathbb{D}^q$, and $B = \mathbb{P}_q(\delta)$. We apply induction on p. The case $p = 1$ reduces to Lemma 1.1.6. Assume that $p \geq 2$.

By virtue of Hartogs' theorem, it suffices to prove that $f \in \mathcal{O}_s(\mathbb{D}^{p+q})$. In fact, we only need to check that $f(\cdot, w) \in \mathcal{O}_s(\mathbb{D}^p)$ for every $w \in \mathbb{D}^q$. Fix $a \in \mathbb{D}^p$ and $j \in \{1, \dots, p\}$. Define

$$g(\zeta, w) := f(a_1, \dots, a_{j-1}, \zeta, a_{j+1}, \dots, a_p, w), \quad (\zeta, w) \in \mathbb{D} \times \mathbb{D}^q.$$

Then g satisfies all the assumptions of the lemma with $p = 1$. Consequently, $g \in \mathcal{O}(\mathbb{D} \times \mathbb{D}^q)$, which shows that f is holomorphic as a function of z_j.

In the general case let Ω denote the set of all $(a, b) \in D \times G$ such that $f \in \mathcal{O}(U)$ for an open neighborhood $U \subset D \times G$ of (a, b). It is clear that Ω is open and non-empty. The first part of the proof shows that if $(a, b) \in \Omega$, then $\{a\} \times G \subset \Omega$ (Exercise). Thus $\Omega = D \times G$.

(b) Exercise. □

Exercise 1.1.11. (a) Find a counterexample showing that Proposition 1.1.10 (b) may be not true if the fibers $\Omega_{(a,\cdot)}$ are disconnected for some $a \in D$.

(b) Prove that Proposition 1.1.10 (b) remains true for arbitrary open $\Omega \subset D \times G$ if for all $a \in D$ each connected component of $\Omega_{(a,\cdot)}$ intersects $U_{(a,\cdot)}$.

As a first elementary application of Hartogs' Theorem 1.1.7 we present the following remark on separately polynomial functions; more developed results will be presented in § 5.3.

Corollary 1.1.12. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be **separately polynomial**, i.e. for any $(a_1, \dots, a_n) \in \mathbb{C}^n$ and $j \in \{1, \dots, n\}$, the function*

$$\mathbb{C} \ni z_j \mapsto f(a_1, \dots, a_{j-1}, z_j, a_{j+1}, \dots, a_n)$$

is a polynomial. Then $f \in \mathcal{P}(\mathbb{C}^n)$.

Proof. We use induction on n. The case $n = 1$ is trivial. Suppose that the result is true for $n-1$. By Hartogs' theorem f must be holomorphic. Write f in the form of the Hartogs series $f(z', z_n) = \sum_{s=0}^{\infty} f_s(z') z_n^s$, $(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C}$, where $f_s \in \mathcal{O}(\mathbb{C}^{n-1})$, $s \in \mathbb{Z}_+$. Put $A_k := \{z' \in \mathbb{C}^{n-1} : \forall_{s \geq k} : f_s(z') = 0\}$, $k \in \mathbb{Z}_+$; notice that each A_k is closed. Since f is separately polynomial, we get $\bigcup_{k=0}^{\infty} A_k = \mathbb{C}^{n-1}$. Consequently, Baire's theorem implies that $\operatorname{int} A_d \neq \varnothing$ for a d, which, by the identity principle for holomorphic functions, gives $f_s \equiv 0$, $s \geq d$. Thus $f(z', z_n) = \sum_{s=0}^{d} f_s(z') z_n^s$, $(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C}$. Observe (Exercise) that the functions $f_1, \dots, f_d$ must be separately polynomial in $\mathbb{C}^{n-1}$. Thus, by the inductive assumption, they are also polynomials. □

1.1.4 Counterexamples

Proposition 1.1.10 (a) is not true without the assumption that $f \in \mathcal{O}(D \times B)$ for some open set $\varnothing \neq B \subset G$ (even if f satisfies some additional regularity conditions). This was already observed by Hartogs [Har 1906]. Other counterexamples were constructed for example by Leja ([Lej 1950]) and Fuka ([Fuk 1983]).

Following T. T. Tuichiev [Tui 1985] (see also [Sad-Tui 2009]), let us consider the following more general problem. We are given a domain $D \subset \mathbb{C}^p$ and a function $f : D \times \mathbb{D}(R) \to \mathbb{C}$ $(0 < R \leq +\infty)$ such that $f(z, \cdot) \in \mathcal{O}(\mathbb{D}(R))$ for every $z \in D$. Write

$$f(z,w) = \sum_{k=0}^{\infty} f_k(z)w^k, \quad (z,w) \in D \times \mathbb{D}(R),$$

and *assume* that $f_k \in \mathcal{O}(D)$, $k \in \mathbb{N}$. Notice that this condition is automatically satisfied if $f \in \mathcal{O}(D \times \mathbb{D}(\delta))$ for certain $0 < \delta < R$ (cf. the proofs of Lemma 1.1.6).

Using Lemma 1.1.6 one can easily prove that $\mathcal{S}_{\mathcal{O}}(f) = S \times \mathbb{D}(R)$ (Exercise). In the sequel we will be able to show that S is nowhere dense (Theorem 4.1.1).

Remark 1.1.13. Observe that not every relatively closed nowhere dense set $S \subset D$ is such that $S \times \mathbb{D}(R) = \mathcal{S}_{\mathcal{O}}(f)$ for a certain f.

For example, it is excluded that there exists a domain $G \subset\subset D$ such that $S \cap G \neq \varnothing$ and $S \cap \partial G = \varnothing$.

In fact, since f is holomorphic in a neighborhood of $\partial G \times \mathbb{D}(R)$, we may assume that $|f| \leq C$ on $\partial G \times \mathbb{D}(r)$ for an $0 < r < R$. Consequently, by the Cauchy inequalities, $|f_k| \leq C/r^k$ on ∂G, $k \in \mathbb{N}$. Thus, by the maximum principle, $|f_k| \leq C/r^k$ in G, $k \in \mathbb{N}$. In particular, the series $\sum_{k=1}^{\infty} f_k(z)w^k$ converges locally uniformly in $G \times \mathbb{D}(r)$. Thus $f \in \mathcal{O}(G \times \mathbb{D}(r))$; a contradiction.

In particular, if $p = 1$, then $S \cap \mathbb{D}(a, \rho_0)$ is not polar for every disc $\mathbb{D}(a, \rho_0) \subset D$ such that $S \cap \mathbb{D}(a, \rho_0) \neq \varnothing$.

In fact, suppose that $S \cap \mathbb{D}(a, \rho_0)$ is polar. Then there exists a $\rho \in (0, \rho_0)$ such that $\partial\mathbb{D}(a, \rho) \cap S = \varnothing$ (cf. Proposition 2.3.21). Thus we may take $G := \mathbb{D}(a, \rho)$.

We are interested in those cases where $\mathcal{S}_{\mathcal{O}}(f) \neq \varnothing$. Following ideas from [Tui 1985], we get the following general result.

Example 1.1.14. Let $D := \mathbb{C}$, $R := +\infty$, and let $S \subset \mathbb{C}$ be a closed nowhere dense set such that $U := \mathbb{C} \setminus S$ is connected and simply connected. Then there exists a sequence of polynomials $(P_k)_{k=1}^{\infty} \subset \mathcal{P}(\mathbb{C})$ such that:

(a) the series $f(z,w) := \sum_{k=1}^{\infty} P_k(z)w^k$ pointwise convergent on $\mathbb{C} \times \mathbb{C}$, and locally uniformly convergent in $U \times \mathbb{C}$ (so $f \in \mathcal{O}(U \times \mathbb{C})$),

(b) $\mathcal{S}_{\mathcal{O}}(f) = S \times \mathbb{C}$.

Indeed, using the Riemann mapping theorem, we have a biholomorphic mapping $\varphi : U \to \mathbb{D}$. Put $K_k := \varphi^{-1}(\overline{\mathbb{D}}(\frac{k}{k+1}))$, $k \in \mathbb{N}$. Then K_k is compact, $U \setminus K_k$ is connected, $K_k \subset \operatorname{int} K_{k+1}$, and $U = \bigcup_{k=1}^{\infty} K_k$. Let $r_k > 0$ be such that $K_k \subset \overline{\mathbb{D}}(r_k)$

and $r_k \nearrow +\infty$. Let $S_k := S \cap \overline{\mathbb{D}}(r_k)$. Observe that for each $k \in \mathbb{N}$ the set $\mathbb{C} \setminus (K_k \cup S_k)$ is connected. For, take two points $a, b \in \mathbb{C} \setminus (K_k \cup S_k)$ and try to find a connecting curve $\gamma\colon [0,1] \to \mathbb{C} \setminus (K_k \cup S_k)$. If $a, b \in U$, then we may find such a curve because the set $U \setminus K_k$ is connected. Thus, suppose that e.g. $a \in S$. Then $|a| > r_k$. Since S is nowhere dense, there exists an $a' \notin S$ such that $[a, a'] \cap (K_k \cup \overline{\mathbb{D}}(r_k)) = \varnothing$ (in particular, $[a, a'] \subset U \setminus (K_k \cup S_k)$). If also $b \in S$, then we may construct in a similar way a point $b' \in U \setminus K_k$. If $b \in U$, then we take $b' := b$. Now it remains to connect a' with b' in $U \setminus K_k$.

Since S is nowhere dense, there exists a sequence $(a_k)_{k=1}^{\infty} \subset U$ such that $a_k \in U \setminus K_k$, $k \in \mathbb{N}$, and each point of S is an accumulation point of this sequence (Exercise).

Making use of Runge's approximation theorem, we find a sequence of polynomials $(P_k)_{k=1}^{\infty} \subset \mathcal{P}(\mathbb{C})$ such that for each $k \in \mathbb{N}$ we have $|P_k| \le 1/k!$ on $K_k \cup S_k$ and $|P_k(a_k)| \ge k! + \sum_{s=1}^{k-1} |P_s(a_k)|$.

Then (a) is obviously satisfied. To get (b), we only need to show that f is unbounded near every point $(z_0, 1) \in S_{k_0} \times \mathbb{C}_*$. Indeed, for $k > k_0$ we have

$$|f(a_k, 1)| \ge |P_k(a_k)| - \sum_{s=1}^{k-1} |P_s(a_k)| - \sum_{s=k+1}^{\infty} |P_s(a_k)| \ge k! - \sum_{s=k+1}^{\infty} \frac{1}{s!} \ge k! - e.$$

Note that T.T. Tuichiev in [Tui 1985] (see also [Sad-Tui 2009]) stated that an analogous example is possible for closed nowhere dense sets $S \subset \mathbb{C}$ such that each point from $\mathbb{C} \setminus S$ may be "connected to ∞ with a curve". As we have seen in Remark 1.1.13, under such weakened assumptions, the above example is in general impossible.

1.2 Separately continuous–holomorphic functions

Starting from Hartogs' theorem one may discuss the following more general problem:

Let $D \subset \mathbb{C}^p$, $G \subset \mathbb{C}^q$ be domains. We say that a function $f\colon D \times G \to \mathbb{C}$ belongs to $\mathcal{O}\mathcal{C}(D \times G)$ if:

- $f(z, \cdot) \in \mathcal{C}(G)$, $z \in D$,
- $f(\cdot, w) \in \mathcal{O}(D)$, $w \in G$.

The question now is under which conditions an $f \in \mathcal{O}\mathcal{C}(D \times G)$ belongs to $\mathcal{C}(D \times G)$.

Example 1.2.1. Let $D = G = \mathbb{D}$. By Runge's theorem we find a sequence of polynomials $(P_k)_{k \in \mathbb{N}}$ such that

- $P_k(z) \underset{k\to\infty}{\longrightarrow} 0$, $z \in \mathbb{D}$,
- for any $x \in \mathbb{D} \cap \mathbb{R}$, any $r \in [0, 1)$ the sequence $(P_k|_{\mathbb{D}(x,r)})_{k \in \mathbb{N}}$ is not uniformly convergent.

Put

$$f(z,w) := \begin{cases} P_k(z) & \text{if } w = 1/k,\ k \geq 2, \\ 0 & \text{if } w = 0, \\ tP_k(z) + (1-t)P_{k+1}(z) & \text{if } |w| = r(t,k),\ 0 < t < 1,\ k \in \mathbb{N}, \end{cases}$$

where $r(t,k) = t/k + (1-t)/(k+1)$. Then $f \in \mathcal{OC}(\mathbb{D} \times \mathbb{D})$ but $f \notin \mathcal{C}(\mathbb{D} \times \mathbb{D})$ (EXERCISE).

A characterization of those functions in $\mathcal{OC}(\mathbb{D} \times \mathbb{D})$ which are holomorphic is contained in the following result which is based on an observation by Rothstein in [Rot 1950].

Proposition 1.2.2 ([Wil 1969]). *Let $f \in \mathcal{OC}(\mathbb{D} \times \mathbb{D})$. Then $f \in \mathcal{C}(\mathbb{D}^2)$ if and only if $-R_f \in \mathcal{SH}(\mathbb{D})$ = the set of all functions subharmonic on $\mathbb{D}$, where*

$$R_f(z) := \inf\{|w| : w \in \overline{\mathbb{D}},\ (w \in \mathbb{D} \implies f \text{ is not continuous at } (z,w))\}, \quad z \in \mathbb{D}.$$

Proof. Assume that there is a point $z_0 \in \mathbb{D}$ with $R_f(z_0) = k < 1$ and fix $k' \in (k, 1)$.

Let $\mathbb{D}(z,r) \subset\subset \mathbb{D}$. Then the family of functions $(f(\cdot, w))_{|w|<k'}$ is pointwise bounded on $\overline{\mathbb{D}}(z,r)$. Therefore, there exists a disc $\mathbb{D}(z', r') \subset \mathbb{D}(z,r)$ and a positive constant c such that $|f| \leq c$ on $\mathbb{D}(z', r') \times \mathbb{D}(k')$. Hence, by Vitali's theorem, f is continuous there (EXERCISE).

Set

$$D := \operatorname{int}\{z \in \mathbb{D} : f \text{ is continuous at all points of } \{z\} \times \mathbb{D}(k')\}.$$

Then, D is an open dense subset of $\mathbb{D}$ and $R_f \geq k'$ on D. Denote by D_j, $j \in J \subset \mathbb{N}$, the different connected components of D. Take a disc $K_0 = \mathbb{D}(z_0, r_0) \subset\subset \mathbb{D}$. Put $M := \partial D \cap K_0$. Then $M \neq \varnothing$. Applying again the Baire argument, now for $\overline{M}$, leads to a disc $K_1 = \mathbb{D}(z_1, r_1) \subset K_0$ with $z_1 \in \overline{M}$ such that $|f| \leq d$ on $(\overline{M} \cap K_1) \times \mathbb{D}(k')$. Obviously there is a point $z^* \in K_1$ such that $R_f(z^*) < k'$.

Now fix an arbitrary point $a \in K_1$ with $R_f(a) < k'$. Take a sequence of points $(a_n)_n \subset K_1$ with $a_n \to a$ and choose connected components $D_{j_n} \ni a_n$. We need the following lemma which is due to Rothstein (see [Rot 1950]).

Lemma* 1.2.3. *Let $h\colon \mathbb{D} \to (0, +\infty)$ a superharmonic function (i.e. $-u \in \mathcal{SH}(\mathbb{D})$). Let $(z_j) \subset \mathbb{D}(s)$ with $z_j \to z_0 \in \mathbb{D}(s)$, $0 < s < 1$. Moreover, let $(w_j)_j \subset \mathbb{D}(s)$ with $|w_j - z_0| \geq d > 0$. Assume that there are Jordan arcs $J_j \subset \mathbb{D}(s)$ connecting z_j and w_j such that $u \geq \alpha$ on all J_j's. Then $u(z_0) \geq \alpha$.*

Applying this lemma it follows that $\operatorname{diam} D_{j_n} \to 0$ if $n \to +\infty$. Therefore, for large n, we have $\partial D_{j_n} \subset K_1$. By the maximum principle we conclude that $|f| \leq d$ on $D_{j_n} \times \mathbb{D}(k')$; in particular $|f(a_n, w)| \leq d$, $|w| < k'$. Therefore, $|f(a, \cdot)| \leq d$ on $\mathbb{D}(k')$. Since a was arbitrarily chosen, we conclude that $|f| \leq d$ on K_1. By Vitali's theorem, $R_f \geq k'$ on K_1, but $R_f(z^*) < k'$; a contradiction.

The converse implication is obvious. □

1.3 Separately pluriharmonic functions I

Definition 1.3.1. Let Ω be an open subset of $\mathbb{C}^n$. A function $u \in \mathcal{C}^2(\Omega, \mathbb{R})$ is *pluriharmonic* on Ω ($u \in \mathcal{PH}(\Omega)$) if

$$\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(z) = 0, \quad z \in \Omega, \ j, k = 1, \dots, n. \tag{†}$$

Remark 1.3.2. (a) If $n = 1$, then $\mathcal{PH}(\Omega) = \mathcal{H}(\Omega) =$ the space of all functions harmonic on Ω.

(b) Equations (†) in real variables mean

$$\frac{\partial^2 u}{\partial x_j \partial y_k}(z) = \frac{\partial^2 u}{\partial x_k \partial y_j}(z), \quad \frac{\partial^2 u}{\partial x_j \partial x_k}(z) + \frac{\partial^2 u}{\partial y_j \partial y_k}(z) = 0,$$
$$z \in \Omega, \ j, k = 1, \dots, n,$$

which in particular shows that each pluriharmonic function is harmonic and, consequently, of class $\mathcal{C}^\infty$.

(c) If $f = u + iv \in \mathcal{O}(\Omega)$, then $u \in \mathcal{PH}(\Omega)$.

Proposition 1.3.3. *If $D \subset \mathbb{C}^n$ is a starlike domain with respect to a point $a \in D$, then for any $u \in \mathcal{PH}(D)$ there exists an $f \in \mathcal{O}(D)$ such that $u = \operatorname{Re} f$.*

Consequently

- *any pluriharmonic function is locally the real part of a holomorphic function,*
- *if $F \in \mathcal{O}(\Omega', \Omega)$, where $\Omega' \subset \mathbb{C}^m$ is open, then $u \circ F \in \mathcal{PH}(\Omega')$ for any $u \in \mathcal{PH}(\Omega)$.*

A proof may be found in [Jar-Pfl 2008], Proposition 1.14.28.

In the case of pluriharmonic functions Proposition 1.1.10 gives the following result.

Proposition 1.3.4 (Cf. [Sad 2005]). *Let $u : \mathbb{P}_p(r) \times \mathbb{P}_q(r) \to \mathbb{R}$ be such that*

- *$u(a, \cdot) \in \mathcal{PH}(\mathbb{P}_q(r))$ for every $a \in \mathbb{P}_p(r)$,*
- *$u \in \mathcal{PH}(\mathbb{P}_p(r) \times \mathbb{P}_q(\delta))$ for some $0 < \delta < r$.*

Then $u \in \mathcal{PH}(\mathbb{P}_p(r) \times \mathbb{P}_q(r))$.

The result will be generalized in Proposition 9.3.1.

Proof. Using Proposition 1.3.3 we get a function $f : \mathbb{P}_p(r) \times \mathbb{P}_q(r) \to \mathbb{C}$ such that

- $f(a, \cdot) \in \mathcal{O}(\mathbb{P}_q(r))$ for every $a \in \mathbb{P}_p(r)$,
- $f \in \mathcal{O}(\mathbb{P}_p(r) \times \mathbb{P}_q(\delta))$,
- $u = \operatorname{Re} f$.

Using Proposition 1.1.10 we conclude that $f \in \mathcal{O}(\mathbb{P}_p(r) \times \mathbb{P}_q(r))$, which immediately implies that $u \in \mathcal{PH}(\mathbb{P}_p(r) \times \mathbb{P}_q(r))$. □

As a direct consequence we get (Exercise) the following result (cf. Proposition 1.1.10).

Proposition 1.3.5. (a) *Let $D \subset \mathbb{C}^p$, $G \subset \mathbb{C}^q$ be domains and let $\varnothing \neq B \subset G$ be open. Let $u\colon D \times G \to \mathbb{R}$ be such that*

- $u(a, \cdot) \in \mathcal{PH}(G)$, $a \in D$,
- $u \in \mathcal{PH}(D \times B)$.
- *Then* $u \in \mathcal{PH}(D \times G)$.

(b) *Let $D \subset \mathbb{C}^p$, $\Omega \subset D \times \mathbb{C}^q$ be domains such that for every $a \in D$ the fiber $\Omega_{(a,\cdot)}$ is connected. Let $U \subset \Omega$ be an open set such that for every $a \in D$ the fiber $U_{(a,\cdot)}$ is non-empty. Let $u\colon \Omega \to \mathbb{R}$ be such that:*

- $u(a, \cdot) \in \mathcal{PH}(\Omega_{(a,\cdot)})$, $a \in D$,
- $u \in \mathcal{PH}(U)$.

Then $u \in \mathcal{PH}(\Omega)$.

The reader is asked to formulate and solve an analogue of Exercise 1.1.11 for separately pluriharmonic functions.

1.4 Hukuhara and Shimoda theorems

Theorem 1.1.7 and Proposition 1.1.10 suggest the following problem, called the *Hukuhara problem*.

(S-$\mathcal{O}_{\mathrm{H}}$) Given two domains $D \subset \mathbb{C}^p$, $G \subset \mathbb{C}^q$, a non-empty test set $B \subset G$, and a function $f: D \times G \to \mathbb{C}$ that is *separately holomorphic* in the sense that

- $f(a, \cdot) \in \mathcal{O}(G)$ for every $a \in D$,
- $f(\cdot, b) \in \mathcal{O}(D)$ for every $b \in B$,

we ask whether $f \in \mathcal{O}(D \times G)$.

In the above situation we write $f \in \mathcal{O}_s(\boldsymbol{X})$ with $\boldsymbol{X} := (D\times G)\cup(D\times B)$. Notice that from the set-theoretic point of view the set $\boldsymbol{X}$ is nothing other than the Cartesian product $D\times G$ which is, of course, independent of B. Writing $\boldsymbol{X} = (D\times G)\cup(D\times B)$ we point out the role played by the set B.

Remark 1.4.1. (a) Theorem 1.1.7 and Proposition 1.1.10 (a) guarantee that the answer is positive (i.e. $\mathcal{O}_s(\boldsymbol{X}) = \mathcal{O}(D \times G)$) whenever B is open.

(b) Observe that the answer must be negative if B is too “thin”. For example, if $B := g^{-1}(0)$, where $g \in \mathcal{O}(G)$, $g \not\equiv 0$, then, for an *arbitrary* function $\varphi\colon D \to \mathbb{C}$, the function $f(z, w) := \varphi(z)g(w)$, $(z, w) \in D \times G$, belongs to $\mathcal{O}_s(\boldsymbol{X})$ (and, of course, may be not holomorphic on $D \times G$).

The discussion of (S-$\mathcal{O}_{\mathrm{H}}$) started with the paper by M. Hukuhara [Huk 1942].

Theorem 1.4.2 (Hukuhara). *If $p = q = 1$ and B has an accumulation point in G, then every locally bounded function $f \in \mathcal{O}_s(X)$ is holomorphic on $D \times G$.*

Below (Theorem 1.4.4) we present a more general result (cf. [Ter 1972]) whose proof uses the same ideas as the original proof by Hukuhara.

Definition 1.4.3. We say that a set $B \subset \mathbb{C}^q$ is *analytically thin at a point* $b_0 \in \overline{B}$ if there exist a connected open neighborhood U of b_0 and a function $\varphi \in \mathcal{O}(U)$, $\varphi \not\equiv 0$, such that $B \cap U \subset \varphi^{-1}(0)$.

Observe that (Exercise):

- if $q = 1$ and a set $B \subset \mathbb{C}$ has an accumulation point $b_0 \in \mathbb{C}$, then B is not analytically thin at b_0,
- if $B \subset \mathbb{C}^q$ is open and non-empty, then B is not analytically thin at each point $b_0 \in \overline{B}$,
- for every $b_0 \in \mathbb{C}^q$ there exists a sequence $(b_s)_{s=1}^{\infty}$ converging to b_0 such that the set $B := \{b_1, b_2, \dots\}$ is not analytically thin at b_0.

Theorem 1.4.4. *For arbitrary p and q, if B is not analytically thin at a point $b_0 \in G \cap \overline{B}$, then every locally bounded function $f \in \mathcal{O}_s(X)$ is holomorphic on $D \times G$.*

The following notion will be very useful in the sequel.

Definition 1.4.5. Let Ω be a topological space (e.g. an open set in $\mathbb{C}^n$). We say that a sequence $(\Omega_k)_{k=1}^{\infty}$ of open subsets of Ω is an *exhaustion sequence for* Ω if $\Omega_k \subset\subset \Omega_{k+1} \subset\subset \Omega$, $k \in \mathbb{N}$, and $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$. In the case where Ω is connected we additionally require that each Ω_k is connected.

Proof of Theorem 1.4.4. Let $(D_k)_{k=1}^{\infty}$ and $(G_k)_{k=1}^{\infty}$ be exhaustion sequences for D and G, respectively, with $b_0 \in G_1$. It suffices to prove that f is holomorphic on each $D_k \times G_k$. Thus, we may additionally assume that f is bounded.

Observe that f must be continuous. Indeed (cf. Remark 1.1.4), let

$$D \times G \ni (z_k, w_k) \to (z_0, w_0) \in D \times G$$

and $f(z_k, w_k) \to \alpha \in \mathbb{C}$. By a Montel argument we may assume that $f(z_k, \cdot) \to g$ locally uniformly in G with $g \in \mathcal{O}(G)$. In particular, $f(z_k, w_k) \to g(w_0) = \alpha$. Recall that if $b \in B$, then $f(\cdot, b) \in \mathcal{O}(D)$. Hence, $f(z_k, b) \to f(z_0, b) = g(b)$, $b \in B$. Since B is not analytically thin, we conclude that $f(z_0, \cdot) \equiv g$. Thus $\alpha = g(w_0) = f(z_0, w_0)$.

Fix an arbitrary polydisc $P = \mathbb{P}(a, r) \subset\subset D$ and define

$$\tilde{f}(z, w) := \frac{1}{(2\pi i)^p} \int_{\partial_0 P} \frac{f(\zeta, w)}{z - \zeta} d\zeta, \quad (z, w) \in P \times G.$$

Then $\tilde{f} \in \mathcal{O}_s(P \times G) \cap \mathcal{C}(P \times G)$ and so $\tilde{f} \in \mathcal{O}(P \times G)$. Moreover, by the Cauchy integral formula, $\tilde{f}(z, b) = f(z, b)$ for $(z, b) \in P \times B$. Since B is not analytically thin at b_0, we conclude that $\tilde{f} = f$ in $P \times G$, which finishes the proof. $\square$

It took another 25 years before I. Shimoda came back to the Hukuhara problem. He proved in [Shim 1957] an analogous result to the one of Osgood (Theorem 1.1.3 (b)).

Theorem 1.4.6 (Shimoda). *If $p = q = 1$ and B has an accumulation point in G, then for every function $f \in \mathcal{O}_s(X)$ the set $\mathcal{S}_{\mathcal{O}}(f)$ is nowhere dense.*

Below we present a more general result (cf. [Ter 1972]) whose proof follows the same ideas as the original proof by Shimoda.

Theorem 1.4.7. *For arbitrary p and q, if B is not analytically thin at a point $b_0 \in G \cap \overline{B}$, then for every function $f \in \mathcal{O}_s(X)$ the set $\Omega_0 := (D \times G) \setminus \mathcal{S}_{\mathcal{O}}(f)$ is dense in $D \times G$. Moreover, $\Omega_0 = U_0 \times G$, where U_0 is an open dense subset of D.*

Proof. First observe that if $\mathbb{P}_p(a, r) \times \mathbb{P}_q(b, r) \subset \Omega_0$ and $\mathbb{P}_p(a, r) \times \mathbb{P}_q(b, R) \subset D \times G$, then Proposition 1.1.10 (a) implies that $\mathbb{P}_p(a, r) \times \mathbb{P}_q(b, R) \subset \Omega_0$. Consequently, Ω_0 must be of the form $\Omega_0 = U_0 \times G$.

Take an arbitrary polydisc $P = \mathbb{P}(a, r) \subset D$ and a point $b \in G$. Let $G_0 \subset\subset G$ be a subdomain of G such that $b, b_0 \in G_0$. Define

$$A_k := \{z \in P : \forall_{w \in G_0} : |f(z, w)| \le k\}, \quad k \in \mathbb{N}.$$

Then $P = \bigcup_{k=1}^{\infty} A_k$. Moreover, each A_k is closed in P (Exercise – cf. the proof of Theorem 1.4.4). Now a Baire argument implies that there exists a k_0 such that $U := \operatorname{int} A_{k_0} \neq \varnothing$. Consequently, by Theorem 1.4.4, $f \in \mathcal{O}(U \times G_0)$. □

Chapter 2
Prerequisites

Summary. For the reader's convenience we collect various auxiliary results. Most of them may be found (with proofs) in [Jar-Pfl 2000]. The reader familiar with basic facts from several complex variables may continue reading with Chapter 3 and come back to Chapter 2 in the case of need.

2.1 Extension of holomorphic functions

Riemann domains appear in a very natural way while discussing problems related to holomorphic continuation. Recall that there exists an example of a bounded domain $D \subset \mathbb{C}^2$ such that every function $f \in \mathcal{O}(D)$ extends beyond D, but there is no domain $\widehat{D} \subset \mathbb{C}^2$ such that $D \subset \widehat{D}$ and each function $f \in \mathcal{O}(D)$ extends holomorphically to $\widehat{D}$ (cf. [Sha 1976]).

2.1.1 Riemann regions

See [Jar-Pfl 2000], § 1.1.

Definition 2.1.1. A pair (X, p) is called a *Riemann region over* $\mathbb{C}^n$ (in brief $(X, p) \in \mathfrak{R}(\mathbb{C}^n)$) if

- X is a topological Hausdorff space,
- $p\colon X \to \mathbb{C}^n$ is locally homeomorphic, i.e. each point $a \in X$ has an open neighborhood U such that $p(U)$ is open in $\mathbb{C}^n$ and $p|_U : U \to p(U)$ is homeomorphic.

For $z \in p(X)$ the set $p^{-1}(z)$ is called the *stalk* over z. A subset $A \subset X$ is said to be *univalent* if $p|_A\colon A \to p(A)$ is homeomorphic.

If X is connected, then we say that (X, p) is a *Riemann domain over* $\mathbb{C}^n$ ($(X, p) \in \mathfrak{R}_c(\mathbb{C}^n)$).

If X is σ-compact, i.e. $X = \bigcup_{\nu=1}^{\infty} K_\nu$, where each K_ν is compact, then we say that (X, p) is *countable at infinity* ($(X, p) \in \mathfrak{R}_\infty(\mathbb{C}^n)$).

We say that a Riemann region $(X, p) \in \mathfrak{R}(\mathbb{C}^n)$ is *relatively compact* ($(X, p) \in \mathfrak{R}_b(\mathbb{C}^n)$) if there exists $(X', p') \in \mathfrak{R}(\mathbb{C}^n)$ such that X is a relatively compact open set in X' and $p = p'|_X$.

Remark 2.1.2. (a) If $\Omega \subset \mathbb{C}^n$ is an open set, then $(\Omega, \mathrm{id}) \in \mathfrak{R}_\infty(\mathbb{C}^n)$. This is the standard identification of open sets in $\mathbb{C}^n$ with Riemann regions.

(b) If $(X, p) \in \mathfrak{R}(\mathbb{C}^n)$, then p is an open mapping. In particular, the set $p(X)$ is open in $\mathbb{C}^n$. For any $a \in p(X)$ the stalk $p^{-1}(a)$ is a discrete subset of X.

(c) If $(X, p) \in \mathfrak{R}(\mathbb{C}^n)$, then the family $(U, p|_U)_U$, where U runs over all univalent open subsets of X, introduces on X an atlas of an n-dimensional complex manifold.

(d) If $(X, p) \in \mathfrak{R}(\mathbb{C}^n)$, $(Y, q) \in \mathfrak{R}(\mathbb{C}^m)$, then $(X \times Y, p \times q) \in \mathfrak{R}(\mathbb{C}^{n+m})$, where $(p \times q)(x, y) := (p(x), q(y))$.

(e) Let $(X, p) \in \mathfrak{R}_c(\mathbb{C}^n)$ and let Y be an open univalent subset such that $p(Y) = p(X)$. Then $Y = X$.

(f) $\mathfrak{R}_c(\mathbb{C}^n) \subset \mathfrak{R}_\infty(\mathbb{C}^n)$. Consequently, a Riemann region is countable at infinity iff it has an at most countable number of connected components.

Definition 2.1.3. Let $(X, p) \in \mathfrak{R}(\mathbb{C}^n)$. For $a \in X$ and $0 < r \le +\infty$, we introduce on X the notion of a *polydisc centered at a of radius r* as an open univalent neighborhood $\widehat{\mathbb{P}}(a, r) = \widehat{\mathbb{P}}_X(a, r)$ of a such that $p(\widehat{\mathbb{P}}_X(a, r)) = \mathbb{P}_n(p(a), r)$. Notice that $\widehat{\mathbb{P}}_X(a, r)$ exists for small $r > 0$. We define

- the *distance to the boundary* $d_X : X \to (0, +\infty]$,

$$d_X(a) := \sup\{r \in (0, +\infty] : \widehat{\mathbb{P}}_X(a, r) \text{ exists}\}, \quad a \in X;$$

- the *maximal polydisc* centered at a point $a \in X$: $\widehat{\mathbb{P}}_X(a) = \widehat{\mathbb{P}}_X(a, d_X(a))$;
- $p_a := p|_{\widehat{\mathbb{P}}_X(a)}$;
- $d_X(A) := \inf\{d_X(a) : a \in A\}$, $A \subset X$;
- $A^{(r)} := \overline{\bigcup_{x \in A} \widehat{\mathbb{P}}_X(x, r)}$, $0 < r < d_X(A)$;
- $X_\infty := \{a \in X : d_X(a) = +\infty\}$.

Remark 2.1.4. (a) The set X_∞ is the union of all connected components $Y \subset X$ such that $p|_Y : Y \to \mathbb{C}^n$ is homeomorphic (cf. Remark 2.1.2 (e). Moreover,

$$|d_X(x) - d_X(a)| \le \|p(x) - p(a)\|_\infty, \quad a \in X \setminus X_\infty,\ x \in \widehat{\mathbb{P}}_X(a).$$

In particular, the function d_X is continuous.

(b) If $K \subset X$ is compact, then the set $K^{(r)}$ is compact for any $0 < r < d_X(K)$.

(c) If K is compact and univalent, then $K^{(r)}$ is univalent for small $r > 0$.

Definition 2.1.5. For $z, \xi \in \mathbb{C}^n$ and $0 < r \le +\infty$ let $\Delta_\xi(z, r) := z + \mathbb{D}(r)\xi$. For a point $a \in X$, $0 < r \le +\infty$, and $\xi \in \mathbb{C}^n$, we introduce on X the notion of a *disc in direction ξ centered at a of radius r* as a univalent set $\widehat{\Delta}_\xi(a, r)$ containing a such that $p(\widehat{\Delta}_\xi(a, r)) = \Delta_\xi(p(a), r)$. We define the *distance to the boundary in direction ξ*:

$$\delta_{X,\xi} : X \to (0, +\infty], \quad \delta_{X,\xi}(a) := \sup\{r > 0 : \widehat{\Delta}_\xi(a, r) \text{ exists}\}, \quad a \in X.$$

Observe that $\widehat{\Delta}_\xi(a, r)$ exists for small $r > 0$. Note that $\widehat{\Delta}_0(a, r) = \{a\}$ for every $r > 0$.

Remark 2.1.6. (a) The function

$$X \times \mathbb{C}^n \ni (x, \xi) \mapsto \delta_{X,\xi}(x) \in (0, +\infty]$$

is lower semicontinuous.

(b) The polydisc $\widehat{\mathbb{P}}_X(a, r)$ exists iff the disc $\widehat{\Delta}_\xi(a, r)$ exists for any ξ with $\|\xi\|_\infty = 1$. Moreover,

$$\widehat{\mathbb{P}}_X(a, r) = \bigcup_{\xi \in \mathbb{C}^n,\ \|\xi\|_\infty = 1} \widehat{\Delta}_\xi(a, r), \quad d_X = \inf\{\delta_{X,\xi} : \xi \in \mathbb{C}^n,\ \|\xi\|_\infty = 1\}.$$

Definition 2.1.7. For $f : X \to \mathbb{C}$ and $a \in X$, we define the *formal derivatives of f at a*

$$\frac{\partial f}{\partial z_j}(a) := \frac{\partial (f \circ p_a^{-1})}{\partial z_j}(p(a)), \quad \frac{\partial f}{\partial \bar{z}_j}(a) := \frac{\partial (f \circ p_a^{-1})}{\partial \bar{z}_j}(p(a)), \quad j = 1, \dots, n,$$

provided that the right-hand sides exist, where $\frac{\partial}{\partial z_j}$ and $\frac{\partial}{\partial \bar{z}_j}$ on the right-hand side are taken in the classical sense. If f is of class $\mathcal{C}^k$ in an open neighborhood of a and $\alpha, \beta \in \mathbb{Z}_+^n$ are such that $|\alpha| + |\beta| \le k$, then we may define the derivatives

$$D^{\alpha,\beta} f(a) := \left(\left(\frac{\partial}{\partial z_1} \right)^{\alpha_1} \circ \dots \circ \left(\frac{\partial}{\partial z_n} \right)^{\alpha_n} \circ \left(\frac{\partial}{\partial \bar{z}_1} \right)^{\beta_1} \circ \dots \circ \left(\frac{\partial}{\partial \bar{z}_n} \right)^{\beta_n} f \right)(a),$$
$$D^\alpha f(a) := D^{\alpha,0} f(a).$$

2.1.2 Holomorphic functions on Riemann regions

See [Jar-Pfl 2000], § 1.1.

Definition 2.1.8. Let $(X, p) \in \mathfrak{R}(\mathbb{C}^n)$. A function $f : X \to \mathbb{C}$ is said to be *holomorphic* ($f \in \mathcal{O}(X)$) if for each open univalent subset $U \subset X$ the function $f \circ (p|_U)^{-1}$ is holomorphic (in the standard sense) on the open set $p(U) \subset \mathbb{C}^n$.

For $k \in \mathbb{R}_+$ put

$$\mathcal{O}^{(k)}(X) := \{f \in \mathcal{O}(X) : \|f\|_{\mathcal{O}^{(k)}(X)} := \sup_{x \in X} |f(x)| \delta_X^k(x) < +\infty\},$$

where $\delta_X := \min\{d_X, (1 + \|p\|^2)^{-1/2}\}$. Functions from $\mathcal{O}^{(k)}(X)$ are called *tempered of order $\le k$* (see [Jar-Pfl 2000], Example 1.2.5).

If $(Y, q) \in \mathfrak{R}(\mathbb{C}^m)$, then a continuous mapping $F : X \to Y$ is said to be *holomorphic* ($F \in \mathcal{O}(X, Y)$) if $q \circ F \in \mathcal{O}(X, \mathbb{C}^m)$.

For $f \in \mathcal{O}(X)$ and $a \in X$, we define the *Taylor series* of f at a:

$$T_a f(z) := \sum_{\alpha \in \mathbb{Z}_+^n} \frac{D^\alpha f(a)}{\alpha!} (z - p(a))^\alpha = T_{p(a)}(f \circ p_a^{-1})(z), \quad z \in \mathbb{C}^n,$$

and its *radius of convergence* $T_a f$,

$$d(T_a f) := \sup\{r > 0 : T_a f(z) \text{ is convergent for } z \in \mathbb{P}(p(a), r)\}.$$

Notice that $d(T_a f) \geq d_X(a)$ and $f(x) = T_a f(p(x))$ for $x \in \widehat{\mathbb{P}}_X(a)$. Moreover,

$$\frac{1}{d(T_a f)} = \limsup_{k \to +\infty} \Big(\max_{\substack{\alpha \in \mathbb{Z}_+^n \\ |\alpha| = k}} \frac{1}{\alpha!} |D^\alpha f(a)| \Big)^{1/k}.$$

Proposition 2.1.9 (Identity principle). *Let* $(X, p) \in \mathfrak{R}_c(\mathbb{C}^n)$, $(Y, q) \in \mathfrak{R}(\mathbb{C}^m)$, F, $G \in \mathcal{O}(X, Y)$, *and assume that* $\operatorname{int}\{x \in X : F(x) = G(x)\} \neq \varnothing$. *Then* $F \equiv G$ *on* X.

2.1.3 Lebesgue measure on Riemann regions

Let $(X, p) \in \mathfrak{R}_\infty(\mathbb{C}^n)$. A set $A \subset X$ is called (*Lebesgue*) *measurable* if for any open univalent set $U \subset X$ the set $p(A \cap U)$ is Lebesgue measurable in $\mathbb{C}^n$ (in the classical sense). Then

- any Borel subset of X is measurable,
- a set $A \subset X$ is measurable iff any point $a \in X$ has an open univalent neighborhood U such that $p(A \cap U)$ is Lebesgue measurable in the classical sense.

Since X is countable at infinity we may write $X = \bigcup_{j=1}^\infty U_j$, where each U_j is open and univalent. Put $B_1 := U_1$, $B_j := U_j \setminus (U_1 \cup \cdots \cup U_{j-1})$, $j \in \mathbb{N}_2$. For any measurable set $A \subset X$ put

$$\mathcal{L}^X(A) := \sum_{j=1}^\infty \mathcal{L}^{2n}(p(A \cap B_j)),$$

where $\mathcal{L}^{2n}$ denotes the standard Lebesgue measure in $\mathbb{C}^n$. One can prove that $\mathcal{L}^X$ is a regular measure which is independent of the choice of the sequence $(U_j)_{j=1}^\infty$. It is called the *Lebesgue measure on* X. If $f : A \to [0, +\infty]$ is a measurable function, then

$$\int_A f d\mathcal{L}^X = \sum_{j=1}^\infty \int_{p(A \cap B_j)} f \circ (p|_{U_j})^{-1} d\mathcal{L}^{2n}.$$

If $U \subset X$ is univalent, then

$$\int_{U \cap A} f d\mathcal{L}^X = \int_{p(U \cap A)} f \circ (p|_U)^{-1} d\mathcal{L}^{2n}.$$

For $1 \le p < +\infty$, let $L^p(X)$ be the space of all measurable functions $f : X \to \mathbb{C}$ with $\int_X |f|^p d\mathcal{L}^X < +\infty$. Then $L^p(X)$ endowed with the norm

$$f \mapsto \|f\|_{L^p} := \left(\int_X |f|^p d\mathcal{L}^X \right)^{1/p}$$

is a Banach space. Moreover, $L^2(X)$ endowed with that scalar product

$$(f, g) \mapsto \int_X f \bar{g} d\mathcal{L}^X$$

is a Hilbert space. Put $L^p_h(X) := L^p(X) \cap \mathcal{O}(X)$. Then, $L^p_h(X)$ is closed in $L^p(X)$.

2.1.4 Sheaf of I-germs of holomorphic functions

See [Jar-Pfl 2000], Example 1.6.6.

Let I be an arbitrary non-empty set of indices. For $a \in \mathbb{C}^n$ define

$$\widetilde{\mathcal{O}}^I_a := \{(U, \boldsymbol{f}) : U \text{ is an open neighborhood of } a,\ \boldsymbol{f} = (f_i)_{i \in I} \subset \mathcal{O}(U)\}.$$

For $(U, \boldsymbol{f}), (V, \boldsymbol{g}) \in \widetilde{\mathcal{O}}^I_a$ we define an equivalence relation

$$(U, \boldsymbol{f}) \overset{a}{\simeq} (V, \boldsymbol{g}) :\iff \exists_{W - \text{neighborhood of } a} : W \subset U \cap V,\ f_i|_W = g_i|_W,\ i \in I.$$

Put

$$\mathcal{O}^I_a := \widetilde{\mathcal{O}}^I_a / \overset{a}{\simeq}.$$

The class $\widehat{\boldsymbol{f}}_a := [(U, \boldsymbol{f})]_{\overset{a}{\simeq}}$ is called the I-*germ* of $\boldsymbol{f}$ at a. Notice that the *value of* $\widehat{\boldsymbol{f}}_a$ *at* a understood as

$$\widehat{\boldsymbol{f}}_a(a) := (f_i(a))_{i \in I}$$

is well defined.

Let $\mathcal{R}^I_a$ be the ring of all families $(S_i)_{i \in I}$ of power series centered at a that are convergent in a common (independent of $i \in I$) neighborhood of a, which may depend on the family $(S_i)_{i \in I}$ (i.e. $\inf\{d(S_i) : i \in I\} > 0$). Then the mapping

$$\mathcal{O}^I_a \ni \widehat{\boldsymbol{f}}_a \to (T_a f_i)_{i \in I} \in \mathcal{R}^I_a \tag{2.1.1}$$

is an isomorphism. This gives an equivalent description of $\mathcal{O}^I_a$, which also introduces on $\mathcal{O}^I_a$ a structure of a commutative ring with the unit element – the *ring of* I-*germs of holomorphic functions at* a. Put

$$\mathcal{O}^I := \bigvee_{a \in \mathbb{C}^n} \mathcal{O}^I_a = \text{the disjoint union of the family } (\mathcal{O}^I_a)_{a \in \mathbb{C}^n}$$

and let $\pi^I : \mathcal{O}^I \to \mathbb{C}^n$ be given by the formula $\pi^I(\widehat{\boldsymbol{f}}_a) := a$.

For $\widehat{f}_a = [(U, f)]_{\underset{\sim}{a}}$ put

$$\mathbb{V}(\widehat{f}_a, U) := \{[(U, f)]_{\underset{\sim}{b}} : b \in U\}.$$

One may easily check that the system $\{\mathbb{V}(\widehat{f}_a, U) : \widehat{f}_a \in \mathcal{O}^I,\ (U, f) \in \widehat{f}_a\}$ is a neighborhood basis of a Hausdorff topology on $\mathcal{O}^I$ such that

$$\pi^I|_{\mathbb{V}(\widehat{f}_a, U)} \colon \mathbb{V}(\widehat{f}_a, U) \to U$$

is homeomorphic. Thus $(\mathcal{O}^I, \pi^I) \in \mathfrak{R}(\mathbb{C}^n)$. It is called the *sheaf of I-germs of holomorphic functions in $\mathbb{C}^n$*. One can easily prove that

$$d_{\mathcal{O}^I}(\widehat{f}_a) = \inf\{d(T_a f_i) : i \in I\}.$$

For $i_0 \in I$ define $\mathbb{F}_{i_0} \colon \mathcal{O}^I \to \mathbb{C}$, $\mathbb{F}_{i_0}(\widehat{f}_a) := f_{i_0}(a)$. Then

$$\mathbb{F}_{i_0} \circ (\pi^I|_{\mathbb{V}(\widehat{f}_a, U)})^{-1} = f_{i_0} \text{ on } U.$$

This shows that $\mathbb{F}_{i_0} \in \mathcal{O}(\mathcal{O}^I)$.

2.1.5 Holomorphic extension of Riemann regions

See [Jar-Pfl 2000], § 1.4.

Definition 2.1.10. Let $(X, p), (Y, q) \in \mathfrak{R}(\mathbb{C}^n)$. A continuous mapping $\varphi \colon X \to Y$ is said to be a *morphism* if $q \circ \varphi = p$.

If $\varphi \colon (X, p) \to (Y, q)$ is a morphism such that φ is bijective and $\varphi^{-1} \colon Y \to X$ is also a morphism, then we say that φ is an *isomorphism*.

Observe that if $\Omega_1, \Omega_2 \subset \mathbb{C}^n$ are open and $\varphi \colon (\Omega_1, \mathrm{id}) \to (\Omega_2, \mathrm{id})$ is a morphism, then $\Omega_1 \subset \Omega_2$ and φ is the inclusion operator.

Remark 2.1.11. Let $\varphi \colon (X, p) \to (Y, q)$ be a morphism.

(a) If $\psi \colon (X, p) \to (Y, q)$ is a morphism with $\varphi(a) = \psi(a)$ for some $a \in X$, then $\varphi = \psi$ on the connected component of X that contains a.

(b) φ is locally biholomorphic. In particular, φ is an open mapping.

(c) φ is an isomorphism iff φ is bijective.

(d) If $A \subset X$ is univalent, then $\varphi(A)$ is univalent. In particular:

- $\varphi(\widehat{\mathbb{P}}_X(a, r)) = \widehat{\mathbb{P}}_Y(\varphi(a), r)$, $a \in X$, $0 < r \le d_X(a)$,
- $d_Y \circ \varphi \ge d_X$,
- if φ is an isomorphism, then $d_Y \circ \varphi = d_X$.

(e) If every connected component of Y intersects $\varphi(X)$ and $d_Y \circ \varphi = d_X$, then $\varphi(X) = Y$.

(f) The mapping

$$\varphi^*: \mathcal{O}(Y) \to \mathcal{O}(X), \quad \varphi^*(g) := g \circ \varphi,$$

is injective iff every connected component of Y intersects $\varphi(X)$.

(g) $T_{\varphi(a)}g = T_a(g \circ \varphi)$, $g \in \mathcal{O}(Y)$, $a \in X$. In particular, $d(T_a f) \geq d_Y(\varphi(a))$ for any $a \in X$ and $f \in \varphi^*(\mathcal{O}(Y))$.

Definition 2.1.12. Let $(X, p) \in \mathfrak{R}(\mathbb{C}^n)$ and let $\varnothing \neq \mathcal{F} \subset \mathcal{O}(X)$. We say that a morphism $\varphi: (X, p) \to (Y, q)$ is an $\mathcal{F}$*-extension* if φ^* is injective and $\mathcal{F} \subset \varphi^*(\mathcal{O}(Y))$, i.e. for each $f \in \mathcal{F}$ there exists exactly one $g =: f^\varphi \in \mathcal{O}(Y)$ such that $g \circ \varphi = f$. Put

$$\mathcal{F}^\varphi := \{f^\varphi : f \in \mathcal{F}\} = \{g \in \mathcal{O}(Y) : g \circ \varphi \in \mathcal{F}\}.$$

Notice that if X is connected, then Y must be connected.

If $\mathcal{F} = \mathcal{O}(X)$, then we say that $\varphi: (X, p) \to (Y, q)$ is a *holomorphic extension*.

Remark 2.1.13. Let $(X, p) \in \mathfrak{R}(\mathbb{C}^n)$ and let $\varnothing \neq \mathcal{F} \subset \mathcal{O}(X)$. We define a morphism (cf. § 2.1.4 with $I := \mathcal{F}$):

$$\varphi = \varphi_{\mathcal{F}}: (X, p) \to (\mathcal{O}^{(\mathcal{F})}, \pi^{(\mathcal{F})}),$$
$$\varphi(x) := [(\mathbb{P}_n(p(x), d_X(x)), (f \circ p_x^{-1})_{f \in \mathcal{F}})]_{\underset{\simeq}{p(x)}}, \quad x \in X.$$

After the identification (2.1.1), the mapping φ may be written as

$$\varphi(x) := (T_x f)_{f \in \mathcal{F}}, \quad x \in X.$$

Then φ is a morphism and $\mathbb{F}_f \circ \varphi = f$ for any $f \in \mathcal{F}$. Consequently, if $\widehat{X}$ denotes the union of those connected components of $\mathcal{O}^{(\mathcal{F})}$ that intersect $\varphi(X)$ and $\hat{p} := \pi^{(\mathcal{F})}|_{\widehat{X}}$, then

$$\varphi: (X, p) \to (\widehat{X}, \hat{p})$$

is an $\mathcal{F}$-extension.

The following lemma will be frequently used in the sequel.

Lemma 2.1.14. *Let G be a Riemann domain, let $D \subset G$ be a subdomain, and let $A_0 \subset A \subset G$ be such that $A_0 \subset D$. Assume that A_0 is not analytically thin at a point $b_0 \in D \cap \overline{A}_0$ (cf. Definition 1.4.3). For a family $\mathcal{F}$ of functions $f: A \to \mathbb{C}$ consider the following conditions:*

(i) $\forall_{f \in \mathcal{F}} \exists_{\hat{f} \in \mathcal{O}(D)} : \hat{f} = f$ *on* A_0 *($\hat{f}$ is uniquely determined),*

(ii) $\forall_{f \in \mathcal{F}} \forall_{a \in \mathbb{C} \setminus f(A)} : \frac{1}{f-a} \in \mathcal{F}$,

(ii′) $\forall_{f \in \mathcal{F}} \forall_{a \in \mathbb{C}: |a| > \|f\|_A} : \frac{1}{f-a} \in \mathcal{F}$.

(a) *If* (i) *and* (ii) *are satisfied, then* $\hat{f}(D) \subset f(A)$ *for every* $f \in \mathcal{F}$ *(in particular,* $\|\hat{f}\|_D \le \|f\|_A$ *for every* $f \in \mathcal{F}$*).*

(b) *If* (i) *and* (ii′) *are satisfied, then* $\|\hat{f}\|_D \le \|f\|_A$ *for every* $f \in \mathcal{F}$*.*

In particular, if $\varphi\colon (X,p) \to (Y,q)$ is a holomorphic extension, then $f(X) = f^{\varphi}(Y)$ for every $f \in \mathcal{O}(X)$.

Observe that in the case (b) it may happen that $\hat{f}(D) \not\subset f(A)$ (Exercise).

Proof. (a) Suppose that there exists an $x_0 \in D$ such that $\hat{f}(x_0) \notin f(A)$. Then $g := \frac{1}{f-\hat{f}(x_0)} \in \mathcal{F}$. We have $\widehat{g}\cdot(\hat{f} - \hat{f}(x_0)) = 1$ on A_0. Thus $\widehat{g}\cdot(\hat{f} - \hat{f}(x_0)) \equiv 1$ on D; a contradiction.

(b) Suppose that there exists an $x_0 \in D$ such that $|\hat{f}(x_0)| > \|f\|_A$. Then $g := \frac{1}{f-\hat{f}(x_0)} \in \mathcal{F}$ and we continue as above. □

Proposition 2.1.15. *Let* $\varphi\colon (X,p) \to (Y,q)$ *be a holomorphic extension. Then the extension operator*

$$\mathcal{O}(X) \ni f \mapsto f^{\varphi} \in \mathcal{O}(Y)$$

is continuous in the topologies of locally uniform convergence. Moreover, for every compact $L \subset\subset Y$ *there exists a compact* $K \subset\subset X$ *such that*

$$\|f^{\varphi}\|_L \le \|f\|_K, \quad f \in \mathcal{O}(X).$$

2.1.6 Regions of existence

See [Jar-Pfl 2000], § 1.7.

Definition 2.1.16. Let $(X,p) \in \mathfrak{R}(\mathbb{C}^n)$ and let $\varnothing \ne \mathcal{F} \subset \mathcal{O}(X)$. We say that (X,p) is an *$\mathcal{F}$-region of existence* if

$$d_X(a) = \inf\{d(T_a f) : f \in \mathcal{F}\}, \quad a \in X;$$

equivalently, for any $r > d_X(a)$ there exists an $f \in \mathcal{F}$ such that $d(T_a f) < r$.

If $\mathcal{F} = \{f\}$, then we say that (X,p) is a *region of existence of* f.

If $\mathcal{F} = \mathcal{O}(X)$, then we say that (X,p) is a *region of existence*.

If X is connected, then we say that (X,p) is an *$\mathcal{F}$-domain of existence*, *domain of existence of* f, and *domain of existence*, respectively.

Remark 2.1.17. (a) (X,p) is an $\mathcal{F}$-region of existence if and only if for any $\mathcal{F}$-extension $\varphi\colon (X,p) \to (Y,q)$ we have $d_Y \circ \varphi \equiv d_X$ (i.e. φ is surjective – cf. Remark 2.1.11 (e)).

(b) If $(X,p) = (\Omega, \mathrm{id})$, where Ω is an open set in $\mathbb{C}^n$, then (Ω,id) is an $\mathcal{F}$-region of existence iff there are no domains $\Omega_0, \widetilde{\Omega} \subset \mathbb{C}^n$ with $\varnothing \ne \Omega_0 \subset \Omega \cap \widetilde{\Omega}$, $\widetilde{\Omega} \not\subset \Omega$, such that for each $f \in \mathcal{F}$ there exists an $\tilde{f} \in \mathcal{O}(\widetilde{\Omega})$ with $\tilde{f} = f$ on Ω_0.

(c) (X,p) is an $\mathcal{F}$-region of existence iff there exists a dense subset $A \subset X$ such that $d_X(a) = \inf\{d(T_a f) : f \in \mathcal{F}\}, a \in A$.

2.1.7 Maximal holomorphic extensions

See [Jar-Pfl 2000], § 1.8.

Definition 2.1.18. An $\mathcal{F}$-extension $\varphi\colon (X,p) \to (\hat{X},\hat{p})$ is called *maximal* if for any $\mathcal{F}$-extension $\psi\colon (X,p) \to (Y,q)$ there exists a morphism $\sigma\colon (Y,q) \to (\hat{X},\hat{p})$ such that $\sigma \circ \psi = \varphi$. The maximal $\mathcal{F}$-extension is uniquely determined up to an isomorphism. In the above situation we say that $\varphi\colon (X,p) \to (\hat{X},\hat{p})$ is the *$\mathcal{F}$-envelope of holomorphy of* (X,p). If $\mathcal{F} = \mathcal{O}(X)$, then we simply say that $\varphi\colon (X,p) \to (\hat{X},\hat{p})$ is the *envelope of holomorphy of* (X,p).

We say that (X,p) is an *$\mathcal{F}$-region of holomorphy* if for every $\mathcal{F}$-extension

$$\varphi\colon (X,p) \to (Y,q)$$

the mapping φ is an isomorphism.

If (X,p) is an $\mathcal{O}(X)$-region of holomorphy, then we say that (X,p) is a *region of holomorphy*. If X is connected, then we say that (X,p) is an *$\mathcal{F}$-domain of holomorphy* and *domain of holomorphy*, respectively.

Remark 2.1.19. If $\varphi\colon (X,p) \to (\hat{X},\hat{p})$ is the maximal $\mathcal{F}$-extension, then $(\hat{X},\hat{p})$ is an $\mathcal{F}^{\varphi}$-region of holomorphy.

Theorem 2.1.20 (Thullen theorem). *Let (X,p) be a Riemann region over $\mathbb{C}^n$ and let $\varnothing \neq \mathcal{F} \subset \mathcal{O}(X)$. Then (X,p) has an $\mathcal{F}$-envelope of holomorphy.*

Proof. It suffices to prove that the $\mathcal{F}$-extension

$$\varphi_{\mathcal{F}}\colon (X,p) \to (\hat{X},\hat{p}),$$

constructed in Remark 2.1.13, is maximal. Let $\psi\colon (X,p) \to (Y,q)$ be another $\mathcal{F}$-extension. By the same method as in Remark 2.1.13 we construct a morphism

$$\varphi_{\mathcal{F}^{\psi}}\colon (Y,q) \to (\mathcal{O}^{(\mathcal{F}^{\psi})}, \pi^{(\mathcal{F}^{\psi})}).$$

Observe that $(\mathcal{O}^{(\mathcal{F}^{\psi})}, \pi^{(\mathcal{F}^{\psi})}) \simeq (\mathcal{O}^{(\mathcal{F})}, \pi^{(\mathcal{F})})$. Moreover, $\varphi_{\mathcal{F}^{\psi}} \circ \psi = \varphi_{\mathcal{F}}$. Consequently, $\varphi_{\mathcal{F}^{\psi}}(Y) \subset \hat{X}$ (up to an isomorphism). □

Definition 2.1.21. We say that $\mathcal{F}$ *separates points* in X if for any $x_1, x_2 \in X$ with $x_1 \neq x_2$ there exists $f \in \mathcal{F}$ such that $f(x_1) \neq f(x_2)$.

We say that $\mathcal{F}$ *weakly separates points* in X if for any $x_1, x_2 \in X$ with $x_1 \neq x_2$ and $p(x_1) = p(x_2)$ there exist $f \in \mathcal{F}$ and $\alpha \in \mathbb{Z}_+^n$ such that $D^\alpha f(x_1) \neq D^\alpha f(x_2)$, i.e. there exists an $f \in \mathcal{F}$ such that $T_{x_1} f \neq T_{x_2} f$.

We say that $\mathcal{F}$ is *d-stable* if: $f \in \mathcal{F} \implies D^\alpha f \in \mathcal{F}, \alpha \in \mathbb{Z}_+^n$.

Observe that if $\mathcal{F}$ is d-stable and $p \in \mathcal{F}^n$, then $\mathcal{F}$ separates points in X iff $\mathcal{F}$ weakly separates points in X.

If (X,p) is univalent, then every family $\mathcal{F}$ weakly separates points in X.

Remark 2.1.22. The morphism $\varphi_{\mathcal{F}}$ is injective iff $\mathcal{F}$ weakly separates points in X. Recall that $d_{\widehat{X}}(\varphi_{\mathcal{F}}(x)) = \inf\{d(T_x f) : f \in \mathcal{F}\}$, $x \in X$.

Proposition 2.1.23. *Let $\varphi\colon (X,p) \to (Y,q)$ be an $\mathcal{F}$-extension such that (Y,q) is an $\mathcal{F}^{\varphi}$-domain of holomorphy. Then the extension is maximal.*

Remark 2.1.24. Let (X,p) be an $\mathcal{F}$-region of holomorphy and let $U \subset X$ be a univalent domain for which there exists a domain $V \supset p(U)$ such that for every $f \in \mathcal{F}$ there exists a function $F_f \in \mathcal{O}(V)$ with $F_f = f \circ (p|_U)^{-1}$ on $p(U)$. Then there exists a univalent domain $W \supset U$ with $p(W) = V$.

Indeed, by Thullen's theorem (Theorem 2.1.20) we may assume that (X,p) coincides with $(\widehat{X},\hat{p})$ constructed in Remark 2.1.13.

Proposition 2.1.25. *Let $(X,p) \in \mathfrak{R}(\mathbb{C}^n)$, $\varnothing \neq \mathcal{F} \subset \mathcal{O}(X)$. Then the following conditions are equivalent:*

(i) *(X,p) is an $\mathcal{F}$-region of holomorphy;*

(ii) *$\mathcal{F}$ weakly separates points in X and (X,p) is an $\mathcal{F}$-region of existence;*

(iii) *there exists a dense subset $A \subset X$ with $A = p^{-1}(p(A))$ such that*

- *for any $x', x'' \in A$ with $x' \neq x''$ and $p(x') = p(x'')$ there exists an $f \in \mathcal{F}$ such that $T_{x'} f \neq T_{x''} f$,*
- *$d_X(x) = \inf\{d(T_x f) : f \in \mathcal{F}\}$, $x \in A$.*

Proposition 2.1.26. *If $(X,p) \in \mathfrak{R}_\infty(\mathbb{C}^n)$ is an $\mathcal{F}$-region of holomorphy, then there exists a finite or countable subfamily $\mathcal{F}_0 \subset \mathcal{F}$ such that (X,p) is an $\mathcal{F}_0$-region of holomorphy.*

Proof. We may assume that X is connected. The case where $(X,p) \simeq (\mathbb{C}^n, \mathrm{id})$ is trivial. Thus assume that $d_X(x) < +\infty$, $x \in X$. Let $A \subset X$ be a countable dense subset such that $A = p^{-1}(p(A))$. By Proposition 2.1.25, for any $x \in A$ and $r > d_X(x)$ there exists an $f_{x,r} \in \mathcal{F}$ such that $d(T_x f_{x,r}) < r$, and for $x', x'' \in A$, with $x' \neq x''$ and $p(x') = p(x'')$, there exists an $f_{x',x''} \in \mathcal{F}$ such that $T_{x'} f_{x',x''} \neq T_{x''} f_{x',x''}$. Now we may take

$$\begin{aligned}\mathcal{F}_0 := \{f_{x,r} : x \in A,\ \mathbb{Q} \ni r > d_X(x)\}\\ \cup \{f_{x',x''} : x', x'' \in A,\ x' \neq x'',\ p(x') = p(x'')\}.\end{aligned}$$ □

Proposition 2.1.27. *Let $(X,p) \in \mathfrak{R}_\infty(\mathbb{C}^n)$. Then the following conditions are equivalent:*

(i) *(X,p) is a region of holomorphy;*

(ii) *$\mathfrak{N}(\mathcal{O}(X)) := \{f \in \mathcal{O}(X) : (X,p)$ is an $\{f\}$-domain of existence$\} \neq \varnothing$;*

(iii) *$\mathfrak{N}(\mathcal{O}(X))$ is of the second Baire category in $\mathcal{O}(X)$.*

Remark 2.1.28. The above result remains true if we substitute $\mathcal{O}(X)$ by a *natural Fréchet space* $\mathcal{F}$, i.e. a vector space $\mathcal{F} \subset \mathcal{O}(X)$ endowed with a structure of a Fréchet space such that if $f_k \to f$ in $\mathcal{F}$, then $f_k \to f$ locally uniformly in X. For example: $\mathcal{F} = L_h^\infty(X) :=$ the space of all bounded holomorphic functions on X with the topology of uniform convergence.

? Recall (cf. [Jar-Pfl 2008], Remark 1.10.6(b)) that it is an open problem whether there exist Fréchet spaces $\mathcal{F} \subset \mathcal{O}(X)$ that are not natural ?

2.2 Holomorphic convexity

See [Jar-Pfl 2000], § 1.10.

Definition 2.2.1. Let $(X, p) \in \mathfrak{R}(\mathbb{C}^n)$. For a compact set $K \subset\subset X$ put

$$\widehat{K}^{\mathcal{O}(X)} := \{x \in X : \forall_{f \in \mathcal{O}(X)} : |f(x)| \le \|f\|_K\}.$$

We say that K is *holomorphically convex* if $K = \widehat{K}^{\mathcal{O}(X)}$. We say that (X, p) is *holomorphically convex* if $\widehat{K}^{\mathcal{O}(X)}$ is compact for every compact $K \subset\subset X$.

Proposition 2.2.2. *Let $(X, p) \in \mathfrak{R}_\infty(\mathbb{C}^n)$. Then X is holomorphically convex iff there exists a sequence $(K_j)_{j=1}^\infty$ of holomorphically convex compact sets such that $K_j \subset \operatorname{int} K_{j+1}$, $j \in \mathbb{N}$, and $X = \bigcup_{j=1}^\infty K_j$.*

Theorem 2.2.3 (Cartan–Thullen theorem). *Let $(X, p) \in \mathfrak{R}_\infty(\mathbb{C}^n)$. Then the following conditions are equivalent:*

(i) *(X, p) is a region of holomorphy;*

(ii) *$\mathcal{O}(X)$ separates points in X and $d_X(\widehat{K}^{\mathcal{O}(X)}) = d_X(K)$ for every compact $K \subset\subset X$;*

(iii) *$\mathcal{O}(X)$ separates points in X and $d_X(\widehat{K}^{\mathcal{O}(X)}) > 0$ for every compact $K \subset\subset X$;*

(iv) *$\mathcal{O}(X)$ separates points in X and for any set $A \subset X$ with $d_X(A) = 0$ there exists an $f \in \mathcal{O}(X)$ such that $\sup_A |f| = +\infty$;*

(v) *$\mathcal{O}(X)$ separates points in X and X is holomorphically convex;*

(vi) *$\mathcal{O}(X)$ separates points in X and for any infinite set $A \subset X$ with no limit points in X there exists an $f \in \mathcal{F}$ such that $\sup_A |f| = +\infty$.*

Notice that in fact, if X is holomorphically convex, then $\mathcal{O}(X)$ separates points in X – cf. Theorem 2.5.9.

Definition 2.2.4. Any $(X, p) \in \mathfrak{R}_\infty(\mathbb{C}^n)$ satisfying (vi) is called a *Riemann–Stein region over $\mathbb{C}^n$*.

2.3 Plurisubharmonic functions

See [Jar-Pfl 2000], § 2.1.

Let $(X, p) \in \mathfrak{R}_\infty(\mathbb{C}^n)$ (notice that in fact the majority of results remains true for arbitrary $(X, p) \in \mathfrak{R}(\mathbb{C}^n)$).

Definition 2.3.1. For $u\colon X \to \mathbb{R}_{-\infty} := [-\infty, +\infty)$, $a \in X$, and $\xi \in \mathbb{C}^n$, we put

$$\lambda \overset{u_{a,\xi}}{\mapsto} (u \circ p_a^{-1})(p(a) + \lambda\xi).$$

A function $u\colon X \to \mathbb{R}_{-\infty}$ is called *plurisubharmonic* (*psh*) in X ($u \in \mathcal{PSH}(X)$) if

- u is upper semicontinuous on X,
- for every $a \in X$ and $\xi \in \mathbb{C}^n$ the function $u_{a,\xi}$ is subharmonic in a neighborhood of zero (as a function of one complex variable).

Notice that the above definition has a local character. Consequently, whenever we are interested in local properties of psh functions, we may assume that $(X, p) = (D, \mathrm{id})$, where D is a domain in $\mathbb{C}^n$.

We say that a function $u\colon X \to \mathbb{R}_+$ is *logarithmically plurisubharmonic* (*log-psh*) if $\log u \in \mathcal{PSH}(X)$.

For $I \subset \mathbb{R}_{-\infty}$ we put $\mathcal{PSH}(X, I) := \{u \in \mathcal{PSH}(X) : u(X) \subset I\}$.

Remark 2.3.2. (a) For an upper semicontinuous function $u\colon X \to \mathbb{R}_{-\infty}$ the following conditions are equivalent:

(i) $u \in \mathcal{PSH}(X)$;

(ii) $\forall_{a \in X}\ \forall_{\xi \in \mathbb{C}^n : \|\xi\|_\infty = 1}\ \exists_{0 < R \le d_X(a)}$:

$$u(a) = u_{a,\xi}(0) \le \frac{1}{2\pi} \int_0^{2\pi} u_{a,\xi}(re^{i\theta})d\theta, \quad 0 < r < R;$$

(iii) $\forall_{a \in X}\ \forall_{\xi \in \mathbb{C}^n : \|\xi\|_\infty = 1}\ \exists_{0 < R \le d_X(a)}$:

$$u(a) \le \frac{1}{\pi r^2} \int_{\mathbb{D}(r)} u_{a,\xi}(\zeta)d\mathcal{L}^2(\zeta), \quad 0 < r < R;$$

(iv) $\forall_{a \in X}\ \forall_{\xi \in \mathbb{C}^n : \|\xi\|_\infty = 1}\ \exists_{0 < R \le d_X(a)}\ \forall_{0<r<R}\ \forall_{f \in \mathcal{P}(\mathbb{C})}$: if $u_{a,\xi} \le \operatorname{Re} f$ on $\partial\mathbb{D}(r)$, then $u(a) \le \operatorname{Re} f(0)$ (where $\mathcal{P}(\mathbb{C})$ stands for the space of all complex polynomials of one complex variable);

(v) $\forall_{a \in X}\ \forall_{\xi \in \mathbb{C}^n : \|\xi\|_\infty = 1}\ \exists_{0 < R \le d_X(a)}\ \forall_{0<r<R}\ \forall_{h \in \mathcal{H}(\mathbb{D}(r)) \cap \mathcal{C}(\bar{\mathbb{D}}(r))}$: if $u_{a,\xi} \le h$ on $\partial\mathbb{D}(r)$, then $u(a) \le h(0)$;

(vi) for any $a \in X$ and $\xi \in \mathbb{C}^n$ the function

$$\mathbb{D}(\delta_{X,\xi}(a)) \ni \lambda \mapsto (u \circ (p|_{\hat{\Delta}(a,\xi)})^{-1})(p(a) + \lambda\xi)$$

is subharmonic;

(vii) $u \circ (p|_U)^{-1} \in \mathcal{PSH}(p(U))$ for any univalent open set $U \subset X$.

(b) $\mathcal{PSH}(X) + \mathcal{PSH}(X) = \mathcal{PSH}(X)$, $\quad \mathbb{R}_{>0} \cdot \mathcal{PSH}(X) = \mathcal{PSH}(X)$.

(c) $|f|$ is log-psh on X for any $f \in \mathcal{O}(X)$.

(d) If $(u_\nu)_{\nu=1}^\infty \subset \mathcal{PSH}(X)$ and $u_\nu \searrow u$ pointwise on X, then $u \in \mathcal{PSH}(X)$. In particular, if $(u_\nu)_{\nu=1}^\infty \subset \mathcal{PSH}(X, [-\infty, 0])$, then $\sum_{\nu=1}^\infty u_\nu \in \mathcal{PSH}(X)$.

(e) If $(u_\nu)_{\nu=1}^\infty \subset \mathcal{PSH}(X)$ and $u_\nu \to u$ locally uniformly in X, then $u \in \mathcal{PSH}(X)$.

(f) If $u_1, \dots, u_N \in \mathcal{PSH}(X)$, then $\max\{u_1, \dots, u_N\} \in \mathcal{PSH}(X)$ (cf. Proposition 2.3.11).

(g) (Liouville type theorem) If $u \in \mathcal{PSH}(\mathbb{C}^n)$ and $\sup_{\mathbb{C}^n} u < +\infty$, then $u \equiv \text{const}$.

(h) Let $I \subset \mathbb{R}$ be an open interval and let $\varphi\colon I \to \mathbb{R}$ be convex and increasing. Then $\varphi \circ u \in \mathcal{PSH}(X)$ for every $u \in \mathcal{PSH}(X, I)$. Consequently:

If $u \in \mathcal{PSH}(X)$, then $e^u \in \mathcal{PSH}(X)$ (in particular, any log-psh function is psh).

If $u \in \mathcal{PSH}(X, \mathbb{R}_+)$, then $u^p \in \mathcal{PSH}(X)$ for every $p \geq 1$.

(i) If u_1, u_2 are log-psh, then $u_1 + u_2$ is log-psh.

(j) (Maximum principle) If X is connected, $u \in \mathcal{PSH}(X)$, and $u \leq u(a)$ for some $a \in X$, then $u \equiv u(a)$. Consequently, if $Y \subset\subset X$ is a domain, $u \in \mathcal{PSH}(Y)$, and $u \not\equiv \text{const}$, then

$$u(x) < \sup\{\limsup_{Y \ni y \to \zeta} u(y) : \zeta \in \partial Y\}, \quad x \in Y.$$

Let $\Omega \subset \mathbb{C}^n$ be open and let $u \in \mathcal{C}^2(\Omega, \mathbb{R})$. We define the *Levi form of u at a*:

$$\mathcal{L}u(a;\xi) := \sum_{j,k=1} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(a)\xi_j \bar{\xi}_k, \quad a \in \Omega,\ \xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n.$$

Observe that

$$\mathcal{L}u(a;\xi) = \frac{\partial^2 u_{a,\xi}}{\partial\lambda\partial\bar{\lambda}}(0).$$

If $u \in \mathcal{C}^2(X, \mathbb{R})$, then we put

$$\mathcal{L}u(a;\xi) := \mathcal{L}(u \circ p_a^{-1})(p(a);\xi), \quad a \in X,\ \xi \in \mathbb{C}^n.$$

Consequently, we have the following

Proposition 2.3.3. *Let $u \in \mathcal{C}^2(X, \mathbb{R})$. Then*

$$u \in \mathcal{PSH}(X) \iff \forall_{a \in X,\ \xi \in \mathbb{C}^n} : \mathcal{L}u(a;\xi) \geq 0.$$

Remark 2.3.4. Let $(Y, q) \in \mathfrak{R}(\mathbb{C}^m)$, $F \in \mathcal{O}(Y, X)$, $u \in \mathcal{C}^2(X, \mathbb{R})$. Then

$$\mathcal{L}(u \circ F)(b;\eta) = \mathcal{L}u(F(b);(p \circ F)'(b)(\eta)), \quad b \in Y,\ \eta \in \mathbb{C}^m.$$

Consequently, if $u \in \mathcal{PSH}(X) \cap \mathcal{C}^2(X, \mathbb{R})$, then $u \circ F \in \mathcal{PSH}(Y)$ – cf. Proposition 2.3.16.

Definition 2.3.5. We say that a function $u \in \mathcal{C}^2(X, \mathbb{R})$ is *strictly plurisubharmonic* if

$$\forall_{a \in X,\, \xi \in (\mathbb{C}^n)_*} :\ \mathcal{L}u(a;\xi) > 0.$$

Proposition 2.3.6. *Let $Y \subset X$ be open, $v \in \mathcal{PSH}(Y)$, $u \in \mathcal{PSH}(X)$. Assume that*

$$\limsup_{Y \ni y \to \zeta} v(y) \le u(\zeta), \quad \zeta \in \partial Y.$$

Put

$$\tilde{u}(x) := \begin{cases} \max\{v(x), u(x)\}, & x \in Y, \\ u(x), & x \in X \setminus Y. \end{cases}$$

Then $\tilde{u} \in \mathcal{PSH}(X)$.

To simplify notation we will use the following abbreviations:

$$e^z := (e^{z_1}, \dots, e^{z_n}), \quad z \cdot w := (z_1 w_1, \dots, z_n w_n),$$
$$z = (z_1, \dots, z_n),\ w = (w_1, \dots, w_n) \in \mathbb{C}^n.$$

Let $a = (a_1, \dots, a_n) \in \mathbb{C}^n$, $\boldsymbol{r} = (r_1, \dots, r_n) \in \mathbb{R}^n_{>0}$. If

$$\partial_0 \mathbb{P}(a, \boldsymbol{r}) \xrightarrow{u} \mathbb{R}_{-\infty}$$

is bounded from above and measurable, i.e. the function

$$[0, 2\pi)^n \ni \theta \mapsto u(a + \boldsymbol{r} \cdot e^{i\theta})$$

is Lebesgue measurable, then we define

$$\boldsymbol{P}(u; a, \boldsymbol{r}; z) := \frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} \Big(\prod_{j=1}^n \frac{r_j^2 - |z_j - a_j|^2}{|r_j e^{i\theta_j} - (z_j - a_j)|^2} \Big) u(a + \boldsymbol{r} \cdot e^{i\theta}) d\mathcal{L}^n(\theta),$$
$$z = (z_1, \dots, z_n) \in \mathbb{P}(a, \boldsymbol{r}),$$

$$\boldsymbol{J}(u; a, \boldsymbol{r}) := \boldsymbol{P}(u; a; \boldsymbol{r}; a) = \frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} u(a + \boldsymbol{r} \cdot e^{i\theta}) d\mathcal{L}^n(\theta).$$

If $u \colon \mathbb{P}(a, \boldsymbol{r}) \to \mathbb{R}_{-\infty}$ is bounded from above and measurable, then we define

$$\boldsymbol{A}(u; a, \boldsymbol{r}) := \frac{1}{(\pi r_1^2) \dots (\pi r_n^2)} \int_{\mathbb{P}(a,\boldsymbol{r})} u\, d\mathcal{L}^{2n} = \frac{1}{\mathcal{L}^{2n}(\mathbb{P}(a, \boldsymbol{r}))} \int_{\mathbb{P}(a,\boldsymbol{r})} u\, d\mathcal{L}^{2n}.$$

Proposition 2.3.7. *Let $\Omega \subset \mathbb{C}^n$ be open, $u \in \mathcal{PSH}(\Omega)$, $a \in \Omega$. Then*

$$\boldsymbol{J}(u; a, \boldsymbol{r}) \searrow u(a), \quad \boldsymbol{A}(u; a, \boldsymbol{r}) \searrow u(a) \ \text{ when } \boldsymbol{r} \searrow 0.$$

Proposition 2.3.8. *Let $u_1, u_2 \in \mathcal{PSH}(X)$. If $u_1 \le u_2$ almost everywhere in X, then $u_1 \le u_2$ everywhere.*

Proposition 2.3.9. *Let $\Omega \subset \mathbb{C}^n$ be open, $u \in \mathcal{PSH}(\Omega)$, $\mathbb{P}(a, \boldsymbol{r}) \subset\subset \Omega$, $\boldsymbol{r} \in \mathbb{R}^n_{>0}$. Then*

$$\begin{aligned} u(z) &\le \boldsymbol{P}(u; a, \boldsymbol{r}; z), \quad z \in \mathbb{P}(a, \boldsymbol{r}),\\ u(a) &\le \boldsymbol{J}(u; a, \boldsymbol{r}),\\ u(a) &\le \boldsymbol{A}(u; a, \boldsymbol{r}). \end{aligned}$$

Proposition 2.3.10. *If X is connected, $u \in \mathcal{PSH}(X)$, and $u \not\equiv -\infty$, then u is locally integrable; in particular, the set $u^{-1}(-\infty)$ is of zero measure.*

Proposition 2.3.11. *If a family $(u_i)_{i\in I} \subset \mathcal{PSH}(X)$ is locally bounded from above, then the function $u := (\sup_{i\in I} u_i)^*$ is psh in X.*

Here v^* denotes the *upper regularization* of v, $v^*(x) := \limsup_{y\to x} v(y)$, $x \in X$.

Proposition 2.3.12. *If a sequence $(u_\nu)_{\nu=1}^\infty \subset \mathcal{PSH}(X)$ is locally bounded from above, then the function $u := (\limsup_{\nu\to\infty} u_\nu)^*$ is psh on X.*

Proposition 2.3.13 (Hartogs' lemma for plurisubharmonic functions). *Assume that a sequence $(u_k)_{k=1}^\infty \subset \mathcal{PSH}(X)$ is locally bounded from above and, for some $m \in \mathbb{R}$, we have $\limsup_{k\to+\infty} u_k \le m$. Then for every compact subset $K \subset X$ and for every $\varepsilon > 0$, there exists a k_0 such that*

$$\max_K u_k \le m + \varepsilon, \quad k \ge k_0.$$

Definition 2.3.14 (Regularization). Let

$$\Phi(z_1, \dots, z_n) := \Psi(z_1) \cdot \dots \cdot \Psi(z_n), \quad z = (z_1, \dots, z_n) \in \mathbb{C}^n,$$

where $\Psi \in \mathcal{C}_0^\infty(\mathbb{C}, \mathbb{R}_+)$ is such that:

$$\operatorname{supp} \Psi = \overline{\mathbb{D}}, \quad \Psi(z) = \Psi(|z|),\ z \in \mathbb{C}, \quad \int \Psi d\mathcal{L}^2 = 1.$$

Put

$$\Phi_\varepsilon(z) := \frac{1}{\varepsilon^{2n}} \Phi(\frac{z}{\varepsilon}),\ z \in \mathbb{C}^n, \quad X_\varepsilon := \{x \in X : d_X(x) > \varepsilon\}, \quad \varepsilon > 0.$$

For every function $u \in L^1(X, \mathrm{loc})$, define

$$\begin{aligned} u_\varepsilon(x) &:= \int_{\widehat{\mathbb{P}}_X(x)} u(y)\Phi_\varepsilon(p(x) - p(y)) d\mathcal{L}^X(y)\\ &= \int_{\mathbb{D}^n} (u \circ p_x^{-1})(p(x) + \varepsilon w)\Phi(w) d\mathcal{L}^{2n}(w), \quad x \in X_\varepsilon. \end{aligned}$$

The function u_ε is called the *ε-regularization* of u.

Proposition 2.3.15. *If $u \in \mathcal{PSH}(X)$, $u \not\equiv -\infty$, then $u_\varepsilon \in \mathcal{PSH}(X_\varepsilon) \cap \mathcal{C}^\infty(X_\varepsilon)$ and $u_\varepsilon \searrow u$ pointwise in X when $\varepsilon \searrow 0$.*

Proposition 2.3.16. *Let $(Y, q) \in \mathfrak{R}(\mathbb{C}^m)$, $F \in \mathcal{O}(Y, X)$. Then $u \circ F \in \mathcal{PSH}(Y)$ for any $u \in \mathcal{PSH}(X)$.*

Corollary 2.3.17. *Let $u : X \to \mathbb{R}_{-\infty}$ be upper semicontinuous. Then u is psh on X iff for any analytic disc $\varphi : \mathbb{D} \to X$ the function $u \circ \varphi$ is subharmonic in $\mathbb{D}$.*

Proposition 2.3.18. (a) *Let $Q : \mathbb{C}^n \to \mathbb{R}_+$ be an arbitrary complex seminorm. Then* $\log Q \in \mathcal{PSH}(\mathbb{C}^n)$.

(b) *Let $h : \mathbb{C}^n \to \mathbb{R}_+$ be an upper semicontinuous function with*

$$h(\lambda z) = |\lambda| h(z), \quad \lambda \in \mathbb{C},\ z \in \mathbb{C}^n.$$

Then $h \in \mathcal{PSH}(\mathbb{C}^n)$ iff $\log h \in \mathcal{PSH}(\mathbb{C}^n)$.

Definition 2.3.19. A set $M \subset X$ is called (*locally*) *pluripolar* ($M \in \mathcal{PLP}$) if any point $a \in M$ has a connected neighborhood U_a and a function $v_a \in \mathcal{PSH}(U_a)$ with $v_a \not\equiv -\infty$, $M \cap U_a \subset v_a^{-1}(-\infty)$.

If $n = 1$, then pluripolar sets are called *polar*. For $A \subset X$ put

$$\mathcal{PLP}(A) := \{P \in \mathcal{PLP}(X) : P \subset A\}.$$

By Proposition 2.3.10, if M is pluripolar, then $\mathcal{L}^X(M) = 0$.

Definition 2.3.20. A set $M \subset X$ is *thin in X* if for any $a \in X$ there exist a connected neighborhood $U \subset X$ of a and a holomorphic function $\varphi \in \mathcal{O}(U)$, $\varphi \not\equiv 0$, such that $P \cap U \subset \varphi^{-1}(0)$, i.e. M is analytically thin at each point $a \in \bar{M}$ (cf. Definition 1.4.3).

Note that every thin set is pluripolar.

Proposition 2.3.21 ([Arm-Gar 2001], Theorem 7.3.9 (cf. Remark 3.6.2 (e))). *If $M \subset \mathbb{D}$ is polar, then there exists an $r \in (0, 1)$ such that $M \cap \partial\mathbb{D}(r) = \varnothing$.*

Proposition 2.3.22. (a) *Let $(u_i)_{i \in I} \subset \mathcal{PSH}(X)$ be locally bounded from above. Put $u := \sup_{i \in I} u_i$. Then the set $\{x \in X : u(x) < u^*(x)\}$ is of zero measure.*

(b) *Let $(u_\nu)_{\nu \in \mathbb{N}} \subset \mathcal{PSH}(X)$ be a sequence locally bounded from above. Put $u := \limsup_{\nu \to +\infty} u_\nu$. Then the set $\{x \in X : u(x) < u^*(x)\}$ is of zero measure.*

Notice that in fact the set $\{x \in X : u(x) < u^*(x)\}$ is pluripolar – cf. Theorem 2.3.33.

Theorem 2.3.23 (Josefson's theorem; cf. [Jos 1978]). *If $M \subset \mathbb{C}^n$ is pluripolar, then there exists a $v \in \mathcal{PSH}(\mathbb{C}^n)$, $v \not\equiv -\infty$, such that $M \subset v^{-1}(-\infty)$.*

Proposition 2.3.24. *Let $M_j \subset \mathbb{C}^n$ be pluripolar, $j \in \mathbb{N}$. Then $M := \bigcup_{j=1}^\infty M_j$ is pluripolar.*

Theorem 2.3.25. *Let $M \subset X$ be pluripolar. Then there exists a $v \in \mathcal{PSH}(X)$, $v \not\equiv -\infty$, such that $M \subset v^{-1}(-\infty)$.*

Proof. We may assume that X is connected. Let $X = \bigcup_{k=1}^{\infty} U_k$ be an open covering by univalent sets (cf. Remark 2.1.2 (f). Then each set $A_k := p(M \cap U_k)$ is pluripolar, and consequently, by Proposition 2.3.24, the set $A := \bigcup_{k=1}^{\infty} A_k$ is pluripolar. Hence, by the Josefson theorem (Theorem 2.3.23), there exists a $u \in \mathcal{PSH}(\mathbb{C}^n)$, $u \not\equiv -\infty$, such that $u = -\infty$ on A. By Proposition 2.3.10, $u|_{p(X)} \not\equiv -\infty$. Now we only need to put $v := u \circ p$. Then $v \in \mathcal{PSH}(X)$ (Proposition 2.3.16), $v \not\equiv -\infty$, and $v = -\infty$ on M. □

Exercise 2.3.26. Let X be a countable at infinity complex manifold on which global holomorphic functions give local coordinates. Using the method of the above proof, show that every locally pluripolar set $M \subset X$ is globally pluripolar.

Remark 2.3.27. Theorems 2.3.23 and 2.3.25 may be strengthened by requiring that the plurisubharmonic function v has restricted growth – cf. [El 1980], [Sic 1983].

Proposition 2.3.28. *Let $M_j \subset X$ be pluripolar, $j \in \mathbb{N}$. Then $M := \bigcup_{j=1}^{\infty} M_j$ is pluripolar.*

Proposition 2.3.29 (Removable singularities of psh functions). *Let M be a closed pluripolar subset of X.*

(a) *Let $u \in \mathcal{PSH}(X \setminus M)$ be locally bounded from above in X, i.e. every point $a \in X$ has a neighborhood V_a such that u is bounded from above in $V_a \setminus M$. Define*

$$\tilde{u}(z) := \limsup_{X \setminus M \ni w \to z} u(w), \quad z \in X$$

(*notice that $\tilde{u}$ is well defined*). *Then $\tilde{u} \in \mathcal{PSH}(X)$.*

(b) *For every function $u \in \mathcal{PSH}(X)$ we have*

$$u(z) = \limsup_{X \setminus M \ni w \to z} u(w), \quad z \in X.$$

(c) *The set $X \setminus M$ is connected.*

Corollary 2.3.30. *Let M be a closed pluripolar subset of X. Let $f \in \mathcal{O}(X \setminus M)$ be locally bounded in X. Then f extends holomorphically to X.*

Proposition 2.3.31. *Let $(Y, q) \in \mathfrak{R}_\infty(\mathbb{C}^m)$.*

(a) *If $A \subset X \times Y$ is pluripolar, then*

$$P := \{z \in X : A_{(z,\cdot)} \notin \mathcal{PLP}(Y)\} \in \mathcal{PLP}(X),$$

where

$$A_{(z,\cdot)} := \{w \in Y : (z, w) \in A\}.$$

(b) *If* $A \subset X \times Y$ *is thin, then*

$$P := \{z \in X : A_{(z,\cdot)} \text{ is not thin in } Y\} \in \mathcal{PLP}(X).$$

(c) *Let* $Q \subset X \times Y$ *be such that* $Q_{(a,\cdot)} \in \mathcal{PLP}(Y)$, $a \in X$. *Let* $C \subset X \times Y$ *be such that*

$$\{z \in X : C_{(z,\cdot)} \notin \mathcal{PLP}(Y)\} \notin \mathcal{PLP}(X)$$

(*e.g.* $C = C' \times C'' \subset X \times Y$, *where* $C' \notin \mathcal{PLP}(X)$, $C'' \notin \mathcal{PLP}(Y)$). *Then* $C \setminus Q \notin \mathcal{PLP}(X \times Y)$.

Proof. We may assume that $X = D$ and $Y = G$ are domains in $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively.

(a) Let $v \in \mathcal{PSH}(D \times G)$, $v \not\equiv -\infty$, be such that $A \subset v^{-1}(-\infty)$ (Theorem 2.3.23). Fix a compact $K \subset\subset G$ with $\operatorname{int} K \neq \varnothing$. Define

$$u(z) := \sup\{v(z, w) : w \in K\}, \quad z \in D.$$

Then $u \in \mathcal{PSH}(D)$ and $u \not\equiv -\infty$. If $z \in P$, then $A_{(z,\cdot)} \notin \mathcal{PLP}$. Hence, $v(z, \cdot) \equiv -\infty$, and consequently, $u(z) = -\infty$. Thus $A \subset u^{-1}(-\infty)$.

(b) Using the definition of a thin set, we get

$$A \subset \bigcup_{k=1}^{\infty} \{(z, w) \in U_k \times V_k : \varphi_k(z, w) = 0\},$$

where $U_k \times V_k \subset D \times G$ is connected, $\varphi_k \in \mathcal{O}(U_k \times V_k)$, and $\varphi_k \not\equiv 0$. Observe that for any k the set

$$P_k := \{z \in U_k : \varphi_k(z, \cdot) \equiv 0\} = \bigcap_{w \in V_k} \{z \in U_k : \varphi_k(z, w) = 0\}$$

is analytic in U_k. Hence the set $P_0 := \bigcup_{k=1}^{\infty} P_k$ is pluripolar. If $a \notin P_0$, then

$$A_{(a,\cdot)} \subset \bigcup_{k \in \mathbb{N}:\, a \in U_k} \{w \in V_k : \varphi_k(a, w) = 0\},$$

and consequently, the set $A_{(a,\cdot)}$ is thin.

(c) Suppose that $C \setminus Q$ is pluripolar. Then, by (a), there exists a pluripolar set $P \subset D$ such that the fiber $(C \setminus Q)_{(a,\cdot)}$ is pluripolar, $a \in D \setminus P$. Consequently, the fiber $C_{(a,\cdot)}$ is pluripolar, $a \in D \setminus P$; a contradiction. □

Exercise 2.3.32. The set P in Proposition 2.3.31 (b) need not be thin. Complete details of the following example. Let $X = Y := \mathbb{D}$,

$$A := (\{0\} \times \{|w| = 1/4\}) \cup \bigcup_{k \in \mathbb{N}_2} \{1/k\} \times \{|w| = 1 - 1/k\}.$$

Then $P = \{0\} \cup \{1/k : k \in \mathbb{N},\ k \geq 2\}$.

Theorem 2.3.33 (Bedford–Taylor theorem; cf. [Kli 1991], Theorem 4.7.6). (a) *Assume that a family* $(u_i)_{i\in I} \subset \mathcal{PSH}(X)$ *is locally bounded from above. Put* $u := \sup_{i\in I} u_i$. *Then the set* $\{x \in X : u(x) < u^*(x)\}$ *is pluripolar.*

(b) *Assume that a sequence* $(u_\nu)_{\nu=1}^{\infty} \subset \mathcal{PSH}(X)$ *is locally bounded from above. Put* $u := \limsup_{\nu\to+\infty} u_\nu$. *Then the set* $\{x \in X : u(x) < u^*(x)\}$ *is pluripolar.*

2.4 Singular sets

See [Jar-Pfl 2000], § 3.4.

Let $(X, p) \in \mathfrak{R}(\mathbb{C}^n)$, let M be a closed subset of X satisfying the condition

for any domain $D \subset X$ the set $D \setminus M$ is connected and dense in D,

and let $\varnothing \neq \mathcal{F} \subset \mathcal{O}(X \setminus M)$.

Notice that

- int $M = \varnothing$;
- every pluripolar set (cf. Definition 2.3.19) satisfies (2.4);
- consequently, every thin set (cf. Definition 2.3.20) satisfies (2.4);
- in particular, every analytic set of dimension $\leq n-1$ satisfies (2.4).

Definition 2.4.1. We say that a point $a \in M$ is *non-singular with respect to* $\mathcal{F}$ ($a \in M_{ns,\mathcal{F}}$) if there exists an open neighborhood U of a such that for each $f \in \mathcal{F}$ there exists a function $\tilde{f} \in \mathcal{O}(U)$ with $\tilde{f} = f$ on $U \setminus M$.

If $a \in M_{s,\mathcal{F}} := M \setminus M_{ns,\mathcal{F}}$, then we say that a is *singular with respect to* $\mathcal{F}$. If $M_{ns,\mathcal{F}} = \varnothing$, i.e. $M_{s,\mathcal{F}} = M$, then we say that M *is singular with respect to* $\mathcal{F}$. If $\mathcal{F} = \mathcal{O}(X \setminus M)$, then we simply say that M *is singular* and we skip the index $\mathcal{F}$.

Remark 2.4.2. Notice the difference between the notion of "the singular analytic subset M of X" and "the singular points $\operatorname{Sing}(M)$ of an analytic subset M of X". Recall that if M is an analytic subset of X, then a point $a \in M$ is called *regular* ($a \in \operatorname{Reg}(M)$) if there exists an open neighborhood U of a such that $M \cap U$ is a complex manifold. If $a \in \operatorname{Sing}(M) := M \setminus \operatorname{Reg}(M)$, then we say that a is *singular* – cf. [Chi 1989], § 2.3.

Remark 2.4.3. (a) The set $M_{s,\mathcal{F}}$ is closed in M and satisfies (2.4).

(b) Each function $f \in \mathcal{F}$ has a holomorphic extension $\tilde{f} \in \mathcal{O}(X \setminus M_{s,\mathcal{F}})$.

(c) $M_{s,\mathcal{F}} = (M_{s,\mathcal{F}})_{s,\tilde{\mathcal{F}}}$, where $\tilde{\mathcal{F}} := \{\tilde{f} : f \in \mathcal{F}\}$, i.e. $M_{s,\mathcal{F}}$ is singular with respect to $\tilde{\mathcal{F}}$.

(d) $M_{s,\mathcal{F}} \cap U = (M \cap U)_{s,\mathcal{F}|_{U\setminus M}}$ for every open set $U \subset X$.

(e) If M is an analytic subset of X, then $\{a \in M : \dim_a M \leq n-2\} \subset M_{ns}$ (cf. [Chi 1989], Appendix I). In other words, if $M \neq \varnothing$ is singular, then M is of pure codimension 1.

Proposition 2.4.4. *Let $M \subset X$ be an analytic subset of pure dimension $(n-1)$, and let $M = \bigcup_{i \in I} M_i$ be the decomposition of M into irreducible components (cf. [Chi 1989], Section 5.4). Then*

$$M_{s,\mathcal{F}} = \bigcup_{i: M_i \subset M_{s,\mathcal{F}}} M_i = \bigcup_{i: M_i \cap \operatorname{Reg}(M) \cap M_{s,\mathcal{F}} \neq \varnothing} M_i.$$

In particular:

- *the set $M_{s,\mathcal{F}}$ is also analytic,*
- *if all functions from $\mathcal{F}$ extend holomorphically through one point $a_0 \in M_{i_0}$, then they extend through every point from M_{i_0}.*

Proposition 2.4.5. *Let X and M be as in Proposition 2.4.4. If, in addition, X is a domain of holomorphy and $J \subset I$, then $X \setminus \bigcup_{i \in J} M_i$ is a domain of holomorphy.*

Proposition 2.4.6. *If M is a closed thin set (cf. Definition 2.3.20), then $M_{s,\mathcal{F}}$ is analytic.*

Proof. Take an $a \in M$ and a connected neighborhood $U \subset X$ of a such that $M \cap U \subset \varphi^{-1}(0) =: S$ for a $\varphi \in \mathcal{O}(U)$, $\varphi \not\equiv 0$. By Remark 2.4.3 (e) and Proposition 2.4.4, the set $S_{s,\mathcal{F}|_{U \setminus S}}$ is analytic in U. By Remark 2.4.3 (d), $M_{s,\mathcal{F}} \cap U = (M \cap U)_{s,\mathcal{F}|_{U \setminus M}}$. It remains to observe that $(M \cap U)_{s,\mathcal{F}|_{U \setminus M}} = S_{s,\mathcal{F}|_{U \setminus S}}$. □

2.5 Pseudoconvexity

See [Jar-Pfl 2000], § 2.2.

Definition 2.5.1. Let $\mathcal{S} \subset \mathcal{PSH}(X)$. For a compact set $K \subset X$ we put

$$\widetilde{K}^{\mathcal{S}} := \{x \in X : \forall_{u \in \mathcal{S}} : u(x) \le \sup_K u\}.$$

We say that a Riemann region $(X, p) \in \mathfrak{R}_\infty(\mathbb{C}^n)$ is *pseudoconvex* if for any compact set $K \subset X$ the set $\widetilde{K}^{\mathcal{PSH}(X)}$ is relatively compact.

We say that (X, p) is *hyperconvex* if there exists a function $u \in \mathcal{PSH}(X, \mathbb{R}_-)$ such that

$$\{x \in X : u(x) < t\} \subset\subset X, \quad t < 0.$$

Remark 2.5.2. (a) $\widetilde{K}^{\mathcal{PSH}(X)} \subset \widehat{K}^{\mathcal{O}(X)}$. Consequently, if $(X, p) \in \mathfrak{R}_\infty(\mathbb{C}^n)$ is holomorphically convex, then (X, p) is pseudoconvex.

(b) If (X, p) is hyperconvex, then (X, p) is pseudoconvex.

2.5.1 Smooth regions

Let $(X, p) \in \mathfrak{R}_\infty(\mathbb{C}^n)$, let $\Omega \subset\subset X$ be open, and let $k \in \mathbb{N} \cup \{\infty\} \cup \{\omega\}$.

Definition 2.5.3. We say that $\partial\Omega$ is *smooth of class* $\mathcal{C}^k$ (or $\mathcal{C}^k$*-smooth*) *at a point* $a \in \partial\Omega$ if there exist an open neighborhood U of a and a function $u \in \mathcal{C}^k(U, \mathbb{R})$ such that

$$\Omega \cap U = \{x \in U : u(x) < 0\}, \quad U \setminus \bar{\Omega} = \{x \in U : u(x) > 0\},$$
$$\operatorname{grad} u(x) \neq 0, \quad x \in U \cap \partial\Omega,$$

where

$$\operatorname{grad} u(x) := \left(\frac{\partial u}{\partial \bar{z}_1}(x), \dots, \frac{\partial u}{\partial \bar{z}_n}(x)\right).$$

The function u is called a *local defining function for* Ω *at* a. We say that Ω is $\mathcal{C}^k$*-smooth* if $\partial\Omega$ is $\mathcal{C}^k$-smooth at any point $a \in \partial\Omega$. Put

$$T_x^{\mathbb{C}}(\partial\Omega) := \left\{\xi \in \mathbb{C}^n : \textstyle\sum_{j=1}^n \frac{\partial u}{\partial z_j}(x)\xi_j = 0\right\}, \quad x \in U \cap \partial\Omega.$$

The space $T_x^{\mathbb{C}}(\partial\Omega)$ is called the *complex tangent space to* $\partial\Omega$ *at* x. The definition of $T_x^{\mathbb{C}}(\partial\Omega)$ is independent of u. If $n = 1$, then $T_x^{\mathbb{C}}(\partial\Omega) = \{0\}$.

We say that $\partial\Omega$ is *strongly pseudoconvex at a point* $a \in \partial\Omega$ if there exist an open neighborhood U of a and a local defining function $u \in \mathcal{C}^2(U, \mathbb{R})$ such that

$$\mathcal{L}u(x; \xi) > 0, \quad x \in U \cap \partial\Omega, \ \xi \in T_x^{\mathbb{C}}(\partial\Omega) \setminus \{0\}.$$

The definition is independent of u. We say that Ω is *strongly pseudoconvex* if $\partial\Omega$ is strongly pseudoconvex at any point $a \in \partial\Omega$. If $n = 1$, then any $\mathcal{C}^2$-smooth open set $\Omega \subset\subset X$ is strongly pseudoconvex.

2.5.2 Pseudoconvexity in terms of the boundary distance

Theorem 2.5.4. *Let* $(X, p) \in \mathfrak{R}_\infty(\mathbb{C}^n)$. *Then the following conditions are equivalent:*

(i) *for any compact* $K \subset X$ *the set* $\widetilde{K}^{\mathcal{PSH}(X) \cap \mathcal{C}^\infty(X)}$ *is compact;*

(ii) (X, p) *is pseudoconvex;*

(iii) *for any* $\xi \in \mathbb{C}^n$ *the function* $-\log \delta_{X,\xi}$ *is psh on* X*;*

(iv) $-\log d_X \in \mathcal{PSH}(X)$*;*

(v) *there exists an* ***exhaustion function*** $u \in \mathcal{PSH}(X) \cap \mathcal{C}(X)$, *i.e. for any* $t \in \mathbb{R}$ *the set* $\{x \in X : u(x) < t\}$ *is relatively compact;*

(vi) *there exists an exhaustion function* $u \in \mathcal{PSH}(X)$*;*

(vii) *there exists a strictly psh exhaustion function* $u \in \mathcal{C}^\omega(X)$ (*cf. Definition* 2.3.5).

2.5.3 Basic properties of pseudoconvex domains

Theorem 2.5.5. *Let $(X, p) \in \mathfrak{R}_\infty(\mathbb{C}^n)$, $(Y, q) \in \mathfrak{R}_\infty(\mathbb{C}^m)$.*

(a) *If $X_0 = \bigcup_{\nu\in\mathbb{N}} X_\nu$, where X_ν is a pseudoconvex open subset of X with $X_\nu \subset X_{\nu+1}$, $\nu \in \mathbb{N}$, then X_0 is pseudoconvex.*

(b) *If $X_0 = \operatorname{int} \bigcap_{\nu\in\mathbb{N}} X_\nu$, where X_ν is a pseudoconvex open subset of X, $\nu \in \mathbb{N}$, then X_0 is pseudoconvex.*

(c) *If $(X_j, p_j) \in \mathfrak{R}_\infty(\mathbb{C}^{n_j})$ is pseudoconvex, $j = 1, \dots, N$, then $X_1 \times \dots \times X_N$ is pseudoconvex.*

(d) *Any Riemann region over $\mathbb{C}$ is pseudoconvex.*

(e) *If X is pseudoconvex and $u \in \mathcal{PSH}(X)$, then $\{x \in X : u(x) < 0\}$ is pseudoconvex.*

(f) *If X is pseudoconvex and $X_0 \subset X$ is an open set such that for any point $a \in \partial X_0$ there exists an open neighborhood U_a such that $X_0 \cap U_a$ is pseudoconvex, then X_0 is pseudoconvex.*

(g) *If X is pseudoconvex and M is an analytic subset of X of pure codimension* 1*, then $X \setminus M$ is pseudoconvex.*

(h) *If X is pseudoconvex and $H \subset \mathbb{C}^n$ is a complex ℓ-dimensional affine subspace, then $(p^{-1}(H), T \circ p)$ is a pseudoconvex Riemann region over $\mathbb{C}^\ell$, where $T\colon H \to \mathbb{C}^\ell$ is an arbitrary affine isomorphism identifying H with $\mathbb{C}^\ell$.*

(i) *If $Z \subset X \times Y$ is pseudoconvex, then for any $y_0 \in Y$ the fiber $Z_{y_0} := \{x \in X : (x, y_0) \in Z\}$ is a pseudoconvex open subset of X.*

(j) *If X is pseudoconvex, $f\colon X \to Y$ be holomorphic, and $Z \subset Y$ is open pseudoconvex, then $f^{-1}(Z)$ is pseudoconvex.*

2.5.4 Smooth pseudoconvex domains

So far, pseudoconvex domains were characterized by the plurisubharmonicity of the function $-\log d_X$. In the case of smooth open subsets $\Omega \subset\subset X$ we can say more, namely:

Theorem 2.5.6. *Let $(X, p) \in \mathfrak{R}_\infty(\mathbb{C}^n)$ and let $\Omega \subset\subset X$ be a $\mathcal{C}^2$-smooth open set. Then $(\Omega, p|_\Omega)$ is pseudoconvex iff any local defining function $u \in \mathcal{C}^2(U, \mathbb{R})$ satisfies the* ***Levi condition***

$$\mathcal{L}u(x;\xi) \geq 0, \quad x \in U \cap \partial\Omega,\ \xi \in T_x^{\mathbb{C}}(\partial\Omega).$$

Theorem 2.5.7. *Let $\Omega \subset\subset X$ be strongly pseudoconvex.*

(a) *If Ω is $\mathcal{C}^k$-smooth ($k \geq 2$), then there exist an open neighborhood U of $\overline{\Omega}$ and a strictly psh defining function $u \in \mathcal{C}^k(U, \mathbb{R})$.*

In particular, any strongly pseudoconvex open set is hyperconvex.

(b) *For any open neighborhood U of $\overline{\Omega}$ there exists a strongly pseudoconvex $\mathcal{C}^\infty$-smooth open set Ω' such that $\overline{\Omega} \subset \Omega' \subset U$. Consequently, every function $f \in$*

$\mathcal{O}(\Omega)$ may be approximated locally uniformly in Ω by functions holomorphic in a neighborhood of $\overline{\Omega}$ (cf. [Jar-Pfl 2000], *Proposition* 2.7.7).

Theorem 2.5.8. *Let (X, p) be pseudoconvex and let $u\colon X \to \mathbb{R}$ be a real analytic exhaustion function (cf. Theorem* 2.5.4 (vii)). *Then there exists a sequence $t_k \nearrow +\infty$ such that $t_k \notin u(\{x \in X : \operatorname{grad} u(x) = 0\})$, $k \in \mathbb{N}$. In particular, the sets $\Omega_k := \{x \in X : u(x) < t_k\}$, $k \in \mathbb{N}$, give an exhaustion of X by strongly pseudoconvex open sets with real analytic boundaries.*

2.5.5 Levi problem

See [Jar-Pfl 2000], §§ 2.5, 2.7.

In view of Remark 2.5.2 (a) it is natural to ask whether any pseudoconvex Riemann region is a region of holomorphy. This is the famous *Levi Problem*. The problem, formulated by E. E. Levi in 1910, was solved by Oka only in 1942 for $n = 2$ and in 1954 by Oka, Norguet, and Bremermann for $n > 2$.

Theorem 2.5.9 (Solution of the Levi problem). *Let $(X, p) \in \mathfrak{R}_\infty(\mathbb{C}^n)$. Then the following conditions are equivalent:*

(i) *(X, p) is a region of holomorphy;*

(ii) *$\mathcal{O}(X)$ separates points in X and (X, p) is holomorphically convex;*

(iii) *(X, p) is holomorphically convex;*

(iv) *(X, p) is pseudoconvex.*

Moreover, if (X, p) is pseudoconvex, then for every compact set $K \subset\subset X$ we have

$$\widehat{K}^{\mathcal{O}(X)} = \widetilde{K}^{\mathcal{PSH}(X)} = \widetilde{K}^{\mathcal{PSH}(X) \cap \mathcal{C}^\infty(X)}.$$

Theorem 2.5.10. *Let $(X, p) \in \mathfrak{R}_c(\mathbb{C}^n)$ be a pseudoconvex Riemann domain. Then X is a $\mathcal{O}^{(k)}(X)$-domain of holomorphy for any $k > 6n$.*

This means that already the tempered holomorphic functions determine the envelope of holomorphy.

Proposition 2.5.11. *If $(X, p) \in \mathfrak{R}_c(\mathbb{C}^n)$ is pseudoconvex, then every $u \in \mathcal{PSH}(X)$ is a* ***Hartogs plurisubharmonic function****, i.e. there exists a sequence $(f_k)_{k=1}^\infty \subset \mathcal{O}(X)$ such that*

- *the sequence $(|f_k|^{1/k})_{k=1}^\infty$ is locally bounded in X,*
- *$u = v^*$, where $v := \limsup_{k \to +\infty} (1/k) \log |f_k|$.*

Proof. The Hartogs domain

$$Y := \{(z, w) \in X \times \mathbb{C} : |w| < e^{-u(z)}\}$$

is pseudoconvex (cf. Theorem 2.5.5 (e)). Let $f \in \mathcal{O}(Y)$ be non-continuable beyond Y. Write f in form of the Hartogs series

$$f(z,w) = \sum_{k=0}^{\infty} f_k(z)w^k, \quad (z,w) \in Y,$$

where $f_k \in \mathcal{O}(X)$, $k \in \mathbb{N}$. Obviously $v := \limsup_{k\to+\infty}(1/k)\log|f_k| \le u$. In particular, $v^* \le u$ and, by the Hartogs lemma (Proposition 2.3.13), the sequence $(|f_k|^{1/k})_{k=1}^{\infty}$ is locally bounded in X. Suppose that $v^*(a) < u(a)$. Then $v(z) \le v^*(z) < -\log R < u(a)$, $z \in \widehat{\mathbb{P}}_X(a,r) \subset\subset X$. Thus $f(z,\cdot)$ extends holomorphically to $\mathbb{D}(R)$ for every $z \in \widehat{\mathbb{P}}_X(a,r)$. Consequently, by Lemma 1.1.6, the function f extends holomorphically to $\widehat{\mathbb{P}}_X(a,r) \times \mathbb{D}(R)$. Since f is non-continuable, we conclude that $R \le e^{-u(a)}$; a contradiction. □

2.6 The Grauert boundary of a Riemann domain

See [Jar-Pfl 2000], § 1.5.

Let $(X,p), (Y,q) \in \mathfrak{R}_c(\mathbb{C}^n)$ and let $\varphi\colon X \to Y$ be a morphism. Our aim is to define an abstract boundary $\overset{=\varphi}{\partial} X$ of X with respect to the morphism φ. The idea of such an abstract boundary is due to H. Grauert (cf. [Gra 1956], [Gra-Rem 1956], [Gra-Rem 1957], [Doc-Gra 1960]).

In the case where $(X,p) = (G,\mathrm{id})$ (G is a domain in $\mathbb{C}^n$), $(Y,q) = (\mathbb{C}^n,\mathrm{id})$, $\varphi = \mathrm{id}$, the abstract boundary $\overset{=\mathrm{id}}{\partial} G$ coincides with the set of, so-called, prime ends of G.

For $a \in X$ let $\mathfrak{B}_c(a)$ denote the family of all open connected neighborhoods U of a.

Recall that a non-empty family $\mathfrak{F}$ of subsets of a topological space X is a *filter* if

- $A \in \mathfrak{F},\ A \subset B \implies B \in \mathfrak{F}$,
- $A_1, A_2 \in \mathfrak{F} \implies A_1 \cap A_2 \in \mathfrak{F}$,
- $\varnothing \notin \mathfrak{F}$.

A non-empty family $\mathfrak{P}$ of non-empty subsets of X is said to be a *filter basis* if

- $\forall_{A_1,A_2\in\mathfrak{P}}\ \exists_{A\in\mathfrak{P}} : A \subset A_1 \cap A_2$.

It is clear that for each filter basis $\mathfrak{P}$ the family $\mathfrak{F}_{\mathfrak{P}} := \{A \subset X : \exists_{B\in\mathfrak{P}} : B \subset A\}$ is a filter.

We say that a filter $\mathfrak{F}$ is *convergent* to a point $a \in X$ if each neighborhood of a belongs to $\mathfrak{F}$. We write in brief $a \in \lim \mathfrak{F}$.

We say that a filter basis $\mathfrak{P}$ is convergent to a if $a \in \lim \mathfrak{F}_{\mathfrak{P}}$ (equivalently, each neighborhood of a contains an element of $\mathfrak{P}$); we put $\lim \mathfrak{P} := \lim \mathfrak{F}_{\mathfrak{P}}$.

We say that a is an *accumulation point* of a filter $\mathfrak{F}$ (resp. filter basis $\mathfrak{P}$) if $a \in \overline{A}$ for any $A \in \mathfrak{F}$ (resp. $A \in \mathfrak{P}$).

Let us recall a few elementary properties of filters:

- If $\mathfrak{F} \subset \mathfrak{F}'$ are filters and if a is an accumulation point of $\mathfrak{F}'$, then a is an accumulation point of $\mathfrak{F}$.
- If $a \in \lim \mathfrak{F}$, then $a \in \lim \mathfrak{F}'$ for any filter $\mathfrak{F}' \supset \mathfrak{F}$.
- If a is an accumulation point of $\mathfrak{F}$, then there exists a filter $\mathfrak{F}' \supset \mathfrak{F}$ such that $a \in \lim \mathfrak{F}'$.
- $a \in \overline{A}$ iff there exists a filter basis $\mathfrak{P}$ consisting of subsets of A such that $a \in \lim \mathfrak{P}$.
- Let Y be another topological space and let $\varphi\colon X \to Y$. Then φ is continuous iff for any filter basis $\mathfrak{P}$ in X the filter basis $\varphi(\mathfrak{P}) := \{\varphi(A) : A \in \mathfrak{P}\}$ satisfies the relation: $\varphi(\lim \mathfrak{P}) \subset \lim \varphi(\mathfrak{P})$.
- X is Hausdorff iff any filter in X converges to at most one point. If X is a Hausdorff space and $\lim \mathfrak{F} = \{a\}$, then we write $\lim \mathfrak{F} = a$.

Definition 2.6.1. We say that a filter basis $\mathfrak{a}$ of subdomains of X is a *φ-boundary point of X* if

- $\mathfrak{a}$ has no accumulation points in X,
- there exists a point $y_0 \in Y$ such that $\lim \varphi(\mathfrak{a}) = y_0$,
- for any $V \in \mathfrak{B}_c(y_0)$ there exists exactly one connected component $U =: \mathfrak{C}(\mathfrak{a}, V)$ of $\varphi^{-1}(V)$ such that $U \in \mathfrak{a}$,
- for any $U \in \mathfrak{a}$ there exists a $V \in \mathfrak{B}_c(y_0)$ such that $U = \mathfrak{C}(\mathfrak{a}, V)$.

Let $\overset{=\varphi}{\partial} X$ denote the set of all φ-boundary points of X. We put

$$\overset{=\varphi}{X} := X \cup \overset{=\varphi}{\partial} X$$

and we extend φ to $\overline{\overline{\varphi}}\colon \overset{=\varphi}{X} \to Y$ by putting $\overline{\overline{\varphi}}(\mathfrak{a}) := y_0$ if $\mathfrak{a}$ and y_0 are as above. Moreover, we put $\overset{=\varphi}{p} := q \circ \overline{\overline{\varphi}}$. We endow $\overset{=\varphi}{X}$ with a Hausdorff topology which coincides with the initial topology on X and is such that the mapping $\overline{\overline{\varphi}}$ is continuous: by an open neighborhood of a point $\mathfrak{a} \in \overset{=\varphi}{\partial} X$ we mean any set of the form

$$\widehat{U}_{\mathfrak{a}} := U \cup \{\mathfrak{b} \in \overset{=\varphi}{\partial} X : U \text{ belongs to the filter generated by } \mathfrak{b}\},$$

where $U \in \mathfrak{a}$.

Proposition 2.6.2. *For any $\mathfrak{a} \in \overset{=\varphi}{\partial} X$ and for any neighborhood $\widehat{U}_{\mathfrak{a}} \subset \overset{=\varphi}{X}$ there exists a neighborhood $\widehat{W}_{\mathfrak{a}} \subset \widehat{U}_{\mathfrak{a}}$ such that $d_X = d_U$ on W. In particular,*

$$\lim_{X \ni y \to \mathfrak{a}} d_X(y) = 0.$$

Let $\mathfrak{P}(A)$ denote the family of all relatively closed pluripolar subsets of A.

Proposition 2.6.3. (a) *Assume that* $\mathfrak{a} \in \overset{=\varphi}{\partial X}$ *is such that there exists a neighborhood* $U \subset \overset{=\varphi}{X}$ *of* $\mathfrak{a}$ *with the properties*

- $V := \overline{\overline{\varphi}}(U)$ *is open in* Y,
- $P := \overline{\overline{\varphi}}(U \cap \overset{=\varphi}{\partial X}) \in \mathfrak{P}(V)$,
- $\varphi\colon U \setminus \overset{=\varphi}{\partial X} \to V \setminus P$ *is biholomorphic.*

Then the mapping $\overline{\overline{\varphi}}|_U : U \to V$ *is homeomorphic.*

(b) *Let* Σ *denote the set of all points* $\mathfrak{a} \in \overset{=\varphi}{\partial X}$ *which satisfies the above conditions. Put*

$$\overset{*\varphi}{X} := X \cup \Sigma.$$

Then

- $(\overset{*\varphi}{X}, \overset{=\varphi}{p}|_{\overset{*\varphi}{X}})$ *is a Riemann domain over* $\mathbb{C}^n$,
- $\overline{\overline{\varphi}}|_{\overset{*\varphi}{X}} : (\overset{*\varphi}{X}, \overset{=\varphi}{p}|_{\overset{*\varphi}{X}}) \to (Y, q)$ *is a morphism,*
- $\Sigma \in \mathfrak{P}(\overset{*\varphi}{X})$.

The following proposition shows that $\overset{*\varphi}{X}$ is in some sense maximal.

Proposition 2.6.4. *Suppose that* $W \subset X$ *is an open subset such that*

- $\varphi(W) = V \setminus P$, *where* V *is an open subset of* Y *and* $P \in \mathfrak{P}(V)$,
- $\varphi\colon W \to V \setminus P$ *is biholomorphic.*

Then there exists an open set $U \subset \overset{*\varphi}{X}$ *such that* $W \subset U$ *and* $\overline{\overline{\varphi}}\colon U \to V$ *is biholomorphic.*

2.7 The Docquier–Grauert criteria

See [Jar-Pfl 2000], § 2.9.

The aim of this section is to localize the description of pseudoconvexity. The main local criteria for pseudoconvexity are contained in the following theorem.

Theorem 2.7.1. *Let* $(X, p) \in \mathfrak{R}_c(\mathbb{C}^n)$. *Then the following conditions are equivalent:*

(i) (X, p) *is a Riemann–Stein domain;*

(ii) *for any continuous mappings* $f\colon [0,1]\times\overline{\mathbb{D}} \to \overline{\overline{X}} := \overline{\overline{X}}^p$ (*cf.* § 2.6) *with* $(\overline{\overline{p}}\circ f)(t,\cdot)\in\mathcal{O}(\mathbb{D})$ *for any* $t\in[0,1)$, *the following implication is true:* *if* $f(([0,1)\times\overline{\mathbb{D}})\cup(\{1\}\times\partial\mathbb{D}))\subset X$, *then* $f([0,1]\times\mathbb{D})\subset X$;

(iii) *for any continuous mapping* $f\colon[0,1]\times\mathbb{D}\to\overline{\overline{X}}$ *for which the mapping* $\overline{\overline{p}}\circ f$ *extends to a holomorphic mapping* $g\colon D\times\mathbb{D}\to\mathbb{C}^n$, *where* $D\subset\mathbb{C}$ *is a neighborhood of* $[0,1]$, *the following implication is true:*

if $f([0,1)\times\mathbb{D})\subset X$, *then* $f(\{1\}\times\mathbb{D})\subset\overline{\overline{\partial}}X$ *or* $f(\{1\}\times\mathbb{D})\subset X$;

(iv) *for any domain* $T=T_{r,\rho}=(\mathbb{P}_{n-1}(r)\times\mathbb{D})\cup(\mathbb{D}^{n-1}\times\mathbb{A}(\rho,1))$, $0<r,\rho<1$, *any biholomorphic mapping* $f\colon T\to f(T)\subset X$ *extends to a holomorphic mapping* $\hat{f}\colon\mathbb{D}^n\to X$;

(v) *there exist a Riemann–Stein domain* (Y,q) *over* $\mathbb{C}^n$ *and a morphism* $\varphi\colon (X,p)\to(Y,q)$ *such that for any domain* $T=T_{r,\rho}$ *and a biholomorphic mapping* $f\colon T\to f(T)\subset X$ *such that* $\varphi\circ f$ *extends to a biholomorphic mapping* $\widehat{g}\colon\mathbb{D}^n\to\widehat{g}(\mathbb{D}^n)\subset Y$, *there exists a holomorphic mapping* $\hat{f}\colon\mathbb{D}^n\to X$ *such that* $\hat{f}=f$ *on* T;

(vi) *there exist a Riemann–Stein domain* (Y,q) *over* $\mathbb{C}^n$ *and a morphism*

$$\varphi\colon(X,p)\to(Y,q)$$

such that there exists a neighborhood U *of* $\overline{\overline{\partial}}^{\varphi}X$ *with* $-\log d_X\in\mathcal{PSH}(U\cap X)$.

2.8 Meromorphic functions

See [Jar-Pfl 2000], § 3.6.

Let $(X,p)\in\mathfrak{R}_c(\mathbb{C}^n)$.

Definition 2.8.1. A function $f\colon X\setminus S\to\mathbb{C}$, where $S=\mathcal{S}(f)$ is a closed subset of X with (2.4), is said to be *meromorphic on* X ($f\in\mathcal{M}(X)$) if

(a) $f\in\mathcal{O}(X\setminus S)$ and S is singular for f in the sense of § 2.4,

(b) for any point $a\in S$ there exist an open connected neighborhood U of a and functions $\varphi,\psi\in\mathcal{O}(U)$, $\psi\not\equiv 0$, such that $\psi f=\varphi$ on $U\setminus S$.

We say that (φ,ψ) is a *local representation* of f at a. Note that in view of (a) we must have $\psi(a)=0$. Consequently, either $S=\varnothing$ or S is an $(n-1)$-dimensional analytic set of pure codimension 1.

The set $\mathcal{R}(f):=X\setminus\mathcal{S}(f)$ is called the set of *regular* points of f.

We say that a point $a\in S$ is a *pole* of f ($a\in\mathcal{P}(f)$) if there exists a local representation (φ,ψ) of f at a such that $\varphi(a)\neq 0$.

We say that a point $a\in S$ is a *point of indeterminacy* of f ($a\in\mathcal{I}(f)$) if for every local representation (φ,ψ) of f at a we have $\varphi(a)=0$.

Obviously, $\mathcal{S}(f) = \mathcal{P}(f) \cup \mathcal{I}(f)$ and $\mathcal{P}(f) \cap \mathcal{I}(f) = \varnothing$. Moreover, $\mathcal{I}(f)$ is an analytic set of dimension $\le n-2$. In particular, if $n = 1$, then $\mathcal{I}(f) = \varnothing$.

Theorem 2.8.2. *Let X be as above and let $M \subset X$ be an analytic subset of codimension ≥ 2. Then any $f \in \mathcal{M}(X \setminus M)$ extends to an $\tilde{f} \in \mathcal{M}(X)$.*

Based on the former result we obtain an interpretation of tempered functions outside of an analytic subset.

Proposition 2.8.3. *Let $M \subset X$ be an analytic subset of pure codimension 1 and let $f \in \mathcal{O}^{(k)}(X \setminus M)$ be a tempered function (see Definition 2.1.8). Then there is an $\tilde{f} \in \mathcal{M}(X)$ such that $\mathcal{S}(\tilde{f}) \subset M$ and $f = \tilde{f}$ on $X \setminus M$.*

That means that every tempered function on $X \setminus M$ may be thought as a meromorphic function on the whole of X.

Proof. Fix a regular point a of M. Then there exist an open neighborhood $U = U(a)$ and a $g \in \mathcal{O}(U)$, $g'(x) \ne 0$ for all $x \in U$, such that $U \cap M = \{x \in U : g(x) = 0\}$. We may assume that $U = \mathbb{P}_X(a, \varepsilon)$. Let $s := (p|_U)^{-1}$, $V := s(\mathbb{P}(p(a), \varepsilon/4))$. Then, for each $x \in V \setminus M$, there is a $b \in p(U \cap M)$ such that

$$\operatorname{dist}(p(x), p(M \cap U)) = \|p(x) - b\|_\infty = d_{X \setminus M}(x),$$
$$\|g(x) - g \circ s(b)\|_\infty \le C\|p(x) - b\|_\infty$$

(the constant C may be chosen to be independent on x). Hence we have

$$|(fg^k)(x)| \le \frac{\|f\|_{\mathcal{O}^{(k)}(X \setminus M)}}{d^k_{X \setminus M}(x)} |g(x)|^k \le C\|f\|_{\mathcal{O}^{(k)}(X \setminus M)}.$$

Consequently, fg^k extends to a holomorphic function h on V. Thus f may be thought of as a meromorphic function $\hat{f}$ on $X \setminus \operatorname{Sing}(M)$. What remains is to recall that $\operatorname{Sing}(M)$ is of codimension ≥ 2 and therefore, due to Theorem 2.8.2, $\hat{f}$ extends to a meromorphic function $\tilde{f}$ on X. □

The theory of extension of holomorphic mappings developed in § 2.1 may be repeated word for word for meromorphic functions and leads to the following Thullen theorem (cf. Theorem 2.1.20).

Theorem 2.8.4 (Thullen theorem). *Let $\varnothing \ne \mathcal{F} \subset \mathcal{M}(X)$. Then (X, p) has an $\mathcal{F}$-envelope of meromorphy $\alpha\colon (X, p) \to (\widetilde{X}, \tilde{p})$ such that $(\widetilde{X}, \tilde{p})$ is a Riemann–Stein domain. In particular, the envelope of meromorphy of (X, p) coincides with its envelope of holomorphy.*

Theorem 2.8.5. *Let $f \in \mathcal{M}(X)$. Then there exist $\varphi, \psi \in \mathcal{O}(X)$, $\psi \not\equiv 0$, such that $f = \varphi/\psi$.*

2.9 Reinhardt domains

See [Jar-Pfl 2008], §§ 1.11, 1.12, 1.14.

Definition 2.9.1. Let $A \subset \mathbb{C}^n$. We say that A is a *Reinhardt* (*n-circled*) *set* if for every $(a_1, \dots, a_n) \in A$ we have

$$\{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_j| = |a_j|,\ j = 1, \dots, n\} \subset A.$$

Put

$$\begin{aligned}
V_j &:= \{(z_1, \dots, z_n) \in \mathbb{C}^n : z_j = 0\},\\
V_0 &:= V_1 \cup \dots \cup V_n = \{(z_1, \dots, z_n) \in \mathbb{C}^n : z_1 \cdots z_n = 0\},\\
\log A &:= \{(\log|z_1|, \dots, \log|z_n|) : (z_1, \dots, z_n) \in A \setminus V_0\}, \quad A \subset \mathbb{C}^n,\\
\exp S &:= \{(z_1, \dots, z_n) \in \mathbb{C}_*^n : (\log|z_1|, \dots, \log|z_n|) \in S\}, \quad S \subset \mathbb{R}^n,\\
A^* &:= \operatorname{int}(\overline{\exp(\log A)}), \quad A \subset \mathbb{C}^n.
\end{aligned}$$

We say that a set $A \subset \mathbb{C}^n$ is *logarithmically convex* (*log-convex*) if $\log A$ is convex.

Theorem 2.9.2. *Let $D \subset \mathbb{C}^n$ be a Reinhardt domain. Then the following conditions are equivalent:*

(i) *D is a domain of holomorphy;*

(ii) *D is log-convex and $D = D^* \setminus \bigcup_{\substack{j \in \{1,\dots,n\}\\ D \cap V_j = \varnothing}} V_j$.*

Theorem 2.9.3. *For every Reinhardt domain $D \subset \mathbb{C}^n$ its envelope of holomorphy $\widehat{D}$ is a Reinhardt domain.*

Corollary 2.9.4. *Let $D \subset \mathbb{C}^n$ be a Reinhardt domain and let $\widehat{D}$ be its envelope of holomorphy. Then:*

(a) *$V_j \cap \widehat{D} = \varnothing$ iff $V_j \cap D = \varnothing$,*

(b) *$\log \widehat{D} = \operatorname{conv}(\log D)$.*

Consequently, by Theorem 2.9.3,

$$\widehat{D} = \operatorname{int}(\overline{\exp(\operatorname{conv}(\log D))}) \setminus \bigcup_{\substack{j \in \{1,\dots,n\}\\ D \cap V_j = \varnothing}} V_j =: \widetilde{D}.$$

Proof. (a) If $V_j \cap D = \varnothing$, then the function $D \ni z_j \mapsto 1/z_j$ is holomorphic on D. Thus, it must be holomorphically continuable to $\widehat{D}$, which means that $V_j \cap \widehat{D} = \varnothing$.

(b) First observe that $\log \widetilde{D} = \operatorname{conv}(\log D)$. Consequently, $\widetilde{D}$ is a domain of holomorphy with $D \subset \widetilde{D}$. Hence, $\widehat{D} \subset \widetilde{D}$. Finally, $\log D \subset \log \widehat{D} \subset \log \widetilde{D} = \operatorname{conv}(\log D)$. □

Proposition 2.9.5. *Let $D \subset (\mathbb{C}_*)^n$ be a Reinhardt domain and let $u\colon D \to \mathbb{R}_{-\infty}$ be such that*

$$u(z_1, \dots, z_n) = u(|z_1|, \dots, |z_n|), \quad (z_1, \dots, z_n) \in D.$$

Put $\tilde{u}(x) := u(e^{x_1}, \dots, e^{x_n})$, $x = (x_1, \dots, x_n) \in \log D$. Then $u \in \mathcal{PSH}(D)$ iff $\tilde{u}$ is a convex function on $\log D$.

Chapter 3
Relative extremal functions

Summary. The relative extremal function $\boldsymbol{h}^*_{A,X}$ will play the fundamental role in the sequel. First, to get an intuition, we present in § 3.1 the convex extremal function, which may be considered as a prototype of the relative extremal function. The main properties of the relative extremal function are presented in § 3.2. Special geometric situations of balanced and Reinhardt sets are discussed in §§ 3.3, 3.4, and 3.5. Section 3.6 contains a list of basic properties of so-called plurithin sets, which will be needed only in Chapter 11. Section 3.7 presents the relative boundary extremal function which will be used only in Chapter 8.

3.1 Convex extremal function

This section is based on [Jar-Pfl 2010a].

Recall that there is a strict connection between envelopes of holomorphy of Reinhardt domains and convex hulls of their logarithmic images. In this context, in Proposition 3.1.3 we will consider the following elementary geometric problem:

Given $\varnothing \neq S_j \subset U_j \subset \mathbb{R}^{n_j}$, where U_j is a convex domain and $\operatorname{int} S_j \neq \varnothing$, $j = 1, \dots, N$, $N \geq 2$, describe the convex envelope of the set

$$\bigcup_{j=1}^{N} S_1 \times \cdots \times S_{j-1} \times U_j \times S_{j+1} \times \cdots \times S_N.$$

It has appeared that the convex extremal functions defined below are very useful to solve this problem.

Definition 3.1.1. Let $\varnothing \neq S \subset U \subset \mathbb{R}^n$, where U is a convex domain. Define the *convex extremal function*

$$\Phi_{S,U} := \sup\{\varphi : \varphi \in \mathcal{CVX}(U),\ \varphi \leq 1,\ \varphi|_S \leq 0\},$$

where $\mathcal{CVX}(U)$ stands for the family of all convex functions $\varphi\colon U \to \mathbb{R}_{-\infty}$.

Observe that $\Phi_{S,U} \in \mathcal{CVX}(U)$, $0 \leq \Phi_{S,U} \leq 1$, and $\Phi_{S,U} = 0$ on S.

Remark 3.1.2. (a) $\Phi_{S,U} = 0$ on $\operatorname{conv}(S)$ and $\Phi_{S,U}(x) < 1$, $x \in U$.

(b) $\Phi_{\operatorname{conv}(S),U} \equiv \Phi_{S,U} \equiv \Phi_{\bar{S}\cap U,U}$.

(c) For $0 < \mu < 1$, let $U_\mu := \{x \in U : \Phi_{S,U}(x) < \mu\}$ (observe that U_μ is a convex domain with $S \subset U_\mu$). Then $\Phi_{S,U_\mu} = (1/\mu)\Phi_{S,U}$ on U_μ.

Indeed, the inequality "$\geq$" is obvious. To prove the opposite inequality, let

$$\varphi := \begin{cases} \max\{\Phi_{S,U},\ \mu\Phi_{S,U_\mu}\} & \text{on } U_\mu, \\ \Phi_{S,U} & \text{on } U \setminus U_\mu. \end{cases}$$

Then $\varphi \in \mathcal{CVX}(U)$ (Exercise), $\varphi < 1$, and $\varphi = 0$ on S. Thus $\varphi \leq \Phi_{S,U}$ and hence $\Phi_{S,U_\mu} \leq (1/\mu)\Phi_{S,U}$ in U_μ.

(d) Let $\varnothing \neq S_j \subset U_j \subset \mathbb{R}^{n_j}$, where U_j is a convex domain, $j = 1, \dots, N$, $N \geq 2$. Put

$$W := \left\{(x_1, \dots, x_N) \in U_1 \times \dots \times U_N : \textstyle\sum_{j=1}^N \Phi_{S_j,U_j}(x_j) < 1\right\}$$

(observe that W is a convex domain with $S_1 \times \dots \times S_N \subset W$). Then

$$\Phi_{S_1\times\dots\times S_N,W}(x) = \sum_{j=1}^N \Phi_{S_j,U_j}(x_j), \quad x = (x_1, \dots, x_N) \in W.$$

Indeed, the inequality "$\geq$" is obvious. To prove the opposite inequality we use induction on $N \geq 2$.

Let $N = 2$: To simplify notation write $A := S_1$, $U := U_1$, $B := S_2$, $V := U_2$. Observe that $T := (A \times V) \cup (U \times B) \subset W$ and directly from the definition we get

$$\Phi_{A\times B,W}(x, y) \leq \Phi_{A,U}(x) + \Phi_{B,V}(y), \quad (x, y) \in T.$$

Fix a point $(x_0, y_0) \in W \setminus T$. Let

$$\mu := 1 - \Phi_{A,U}(x_0) \in (0, 1], \quad V_\mu := \{y \in V : \Phi_{B,V}(y) < \mu\},$$
$$\varphi := \frac{1}{\mu}(\Phi_{A\times B,W}(x_0, \cdot) - \Phi_{A,U}(x_0)).$$

Then φ is a well-defined convex function on V_μ, $\varphi < 1$ on V_μ, and $\varphi \leq 0$ on B. Thus, by Remark 3.1.2 (c), $\varphi(y_0) \leq \Phi_{B,V_\mu}(y_0) = \frac{1}{\mu}\Phi_{B,V}(y_0)$, which finishes the proof.

Now assume that the formula is true for $N - 1 \geq 2$. Put $S' := S_1 \times \dots \times S_{N-1}$,

$$W' := \left\{(x_1, \dots, x_{N-1}) \in U_1 \times \dots \times U_{N-1} : \textstyle\sum_{j=1}^{N-1} \Phi_{S_j,U_j}(x_j) < 1\right\}.$$

Then, by the inductive hypothesis, we have

$$\Phi_{S',W'}(x') = \sum_{j=1}^{N-1} \Phi_{S_j,U_j}(x_j), \quad x' = (x_1, \dots, x_{N-1}) \in W'.$$

Consequently,

$$W = \{(x', x_N) \in W' \times U_N : \Phi_{S',W'}(x') + \Phi_{S_N,U_N}(x_N) < 1\}.$$

Hence, using the case $N = 2$ (for $S' \subset W'$ and $S_N \subset U_N$), we get

$$\Phi_{S_1\times\cdots\times S_N,W}(x) = \Phi_{S',W'}(x') + \Phi_{S_N,U_N}(x_N) = \sum_{j=1}^{N} \Phi_{S_j,U_j}(x_j),$$
$$x = (x', x_N) = (x_1, \dots, x_N) \in W.$$

(e) If $\varnothing \neq S_k \subset U_k \subset \mathbb{R}^n$, U_k a convex domain, $k \in \mathbb{N}$, $S_k \nearrow S$, and $U_k \nearrow U$, then $\Phi_{S_k,U_k} \searrow \Phi_{S,U}$.

Indeed, it is clear that $\Phi_{S_k,U_k} \geq \Phi_{S_{k+1},U_{k+1}} \geq \Phi_{S,U}$ on U_k. Define $\varphi := \lim_{k\to+\infty} \Phi_{S_k,U_k}$. Then $\varphi \in \mathcal{CVX}(U)$, $\varphi < 1$, and $\varphi = 0$ on S. Thus $\varphi \leq \Phi_{S,U}$.

Proposition 3.1.3. *Let $\varnothing \neq S_j \subset U_j \subset \mathbb{R}^{n_j}$, where U_j is a convex domain and* $\operatorname{int} S_j \neq \varnothing$, *$j = 1, \dots, N$, $N \geq 2$, and define the* ***cross***

$$T := \bigcup_{j=1}^{N} S_1 \times \cdots \times S_{j-1} \times U_j \times S_{j+1} \times \cdots \times S_N.$$

Then

$$\operatorname{conv}(T) = \Big\{(x_1, \dots, x_N) \in U_1 \times \cdots \times U_N : \textstyle\sum_{j=1}^{N} \Phi_{S_j,U_j}(x_j) < 1\Big\} =: W.$$

Proof. We may assume that S_j is convex, $j = 1, \dots, N$ (cf. Remark 3.1.2 (b)). The inclusion "$\subset$" is obvious. Let

$$T_j := S_1 \times \cdots \times S_{j-1} \times U_j \times S_{j+1} \times \cdots \times S_N, \quad j = 1, \dots, N,$$
$$T' := \bigcup_{j=1}^{N-1} S_1 \times \cdots \times S_{j-1} \times U_j \times S_{j+1} \times \cdots \times S_{N-1}, \quad S' := S_1 \times \cdots \times S_{N-1}.$$

Recall (cf. [Roc 1972], Theorem 3.3) that

$$\begin{aligned} \operatorname{conv}(T) &= \bigcup_{\substack{t_1,\dots,t_N \geq 0 \\ t_1+\cdots+t_N=1}} t_1T_1 + \cdots + t_NT_N \\ &= \operatorname{conv}((\operatorname{conv}(T') \times S_N) \cup (S' \times U_N)). \end{aligned} \tag{$*$}$$

We use induction on N.

$N = 2$: To simplify notation write $A := S_1$, $U := U_1$, $p := n_1$, $B := S_2$, $V := U_2$, $q := n_2$. Using Remark 3.1.2 (e), we may assume that U, V are bounded.

Since $\operatorname{conv}(T)$ is open (Exercise) and $\operatorname{conv}(T) \subset W$, we only need to show that for every $(x_0, y_0) \in \partial(\operatorname{conv}(T)) \cap (U \times V)$ we have $\Phi_{A,U}(x_0) + \Phi_{B,V}(y_0) = 1$. Since U, V are bounded, we have $\overline{\operatorname{conv}(T)} = \operatorname{conv}(\overline{T})$ (cf. [Roc 1972], Theorem 17.2) and therefore, $(x_0, y_0) = t(x_1, y_1) + (1 - t)(x_2, y_2)$, where $t \in [0, 1]$, $(x_1, y_1) \in \overline{A} \times \overline{V}$,

$(x_2, y_2) \in \overline{U} \times \overline{B}$. Notice that in general $\overline{\mathrm{conv}(T)} \supsetneq \mathrm{conv}(\overline{T})$, e.g. $p = q = 1$, $A = B := (0,1)$, $U := (0,2)$, $V := \mathbb{R}$.

First observe that $t \in (0,1)$.

Indeed, suppose for instance that $(x_0, y_0) \in U \times (\overline{B} \cap V)$. Take an arbitrary $x_* \in \mathrm{int}\, A$ and let $r > 0$, $\varepsilon > 0$ be such that $\mathbb{B}^{\mathbb{R}}_{p+q}((x_*, y_0), r) \subset A \times V$ and $x_{**} := x_* + \varepsilon(x_0 - x_*) \in U$, where $\mathbb{B}^{\mathbb{R}}_k(a, r)$ stands for the Euclidean ball in $\mathbb{R}^k$. Then

$$(x_0, y_0) \in \mathrm{int}(\mathrm{conv}(\mathbb{B}_{p+q}((x_*, y_0), r) \cup \{(x_{**}, y_0)\})) \subset \mathrm{int}(\mathrm{conv}(\overline{T})) = \mathrm{int}(\overline{\mathrm{conv}(T)}) = \mathrm{conv}(T);$$

a contradiction.

Let $L \colon \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}$ be a linear form such that $L(x_0, y_0) = 1$ and $L \le 1$ on T. Since $1 = L(x_0, y_0) = tL(x_1, y_1) + (1-t)L(x_2, y_2)$, we conclude that $L(x_1, y_1) = L(x_2, y_2) = 1$. Write $L(x, y) = P(x) + Q(y)$, where $P \colon \mathbb{R}^p \to \mathbb{R}$, $Q \colon \mathbb{R}^q \to \mathbb{R}$ are linear forms.

Put $P_C := \sup_C P$, $C \subset \mathbb{R}^p$, $Q_D := \sup_D Q$, $D \subset \mathbb{R}^q$. Since $L \le 1$ on T and $L(x_1, y_1) = L(x_2, y_2) = 1$, we conclude that

$$P_A + Q_V = 1,$$
$$P_U + Q_B = 1.$$

In particular, $P_A = P_U$ iff $Q_B = Q_V$. Consider the following two cases:

- $P_A < P_U$ and $Q_B < Q_V$: Then

$$\frac{P - P_A}{P_U - P_A} \le \Phi_{A,U}, \qquad \frac{Q - Q_B}{Q_V - Q_B} \le \Phi_{B,V}.$$

  Hence

$$\Phi_{A,U}(x_0) + \Phi_{B,V}(y_0) \ge \frac{P(x_0) - P_A}{1 - Q_B - P_A} + \frac{Q(y_0) - Q_B}{1 - P_A - Q_B} = 1.$$

- $P_A = P_U$ and $Q_B = Q_V$: Then $P_U + Q_V = 1$, which implies that $(x_0, y_0) \in U \times V \subset \{L < 1\}$; a contradiction.

Now assume that the result is true for $N - 1 \ge 2$. In particular,

$$\mathrm{conv}(T') = \Big\{(x_1, \dots, x_{N-1}) \in U_1 \times \dots \times U_{N-1} : \textstyle\sum_{j=1}^{N-1} \Phi_{S_j, U_j}(x_j) < 1\Big\} =: W'.$$

Using $(*)$, the case $N = 2$, and Remark 3.1.2 (d), we get

$$\mathrm{conv}(T) = \mathrm{conv}((W' \times S_N) \cup ((S' \times U_N)) = \{(x', x_N) \in W' \times U_N : \Phi_{S',W'}(x') + \Phi_{S_N,U_N}(x_N) < 1\} = W. \qquad \square$$

3.2 Relative extremal function

$\boxed{>}$ § 2.3.

Let $(X, p) \in \mathfrak{R}_\infty(\mathbb{C}^n)$, $A \subset X$.

Definition 3.2.1. The *relative extremal function of A with respect to X* is defined as the upper semicontinuous regularization $\boldsymbol{h}^*_{A,X}$ of the function

$$\boldsymbol{h}_{A,X} := \sup\{u : u \in \mathcal{PSH}(X),\ u \le 1,\ u|_A \le 0\}.$$

For an open set $Y \subset X$ we put $\boldsymbol{h}_{A,Y} := \boldsymbol{h}_{A\cap Y,Y}$, $\boldsymbol{h}^*_{A,Y} := \boldsymbol{h}^*_{A\cap Y,Y}$.

Obviously

- $\boldsymbol{h}_{A,X} = \sup\{u : u \in \mathcal{PSH}(X),\ u \le 1,\ \sup_A u < 0\}$,
- $0 \le \boldsymbol{h}_{A,X} \le \boldsymbol{h}^*_{A,X} \le 1$,
- $\boldsymbol{h}_{A,X} = 0$ on A,
- if A is open, then $\boldsymbol{h}_{A,X} = \boldsymbol{h}^*_{A,X} = 0$ on A,
- $\boldsymbol{h}_{\varnothing,X} \equiv 1$.

Proposition 3.2.2. (a) *If Y is a connected component of X, then $\boldsymbol{h}_{A,X} = \boldsymbol{h}_{A,Y}$ and $\boldsymbol{h}^*_{A,X} = \boldsymbol{h}^*_{A,Y}$ on Y.*

(b) $\boldsymbol{h}^*_{A,X} \in \mathcal{PSH}(X)$. *In particular, for every connected component Y of X, either $\boldsymbol{h}^*_{A,X} \equiv 1$ on Y or $\boldsymbol{h}^*_{A,X}(z) < 1$, $z \in Y$. Moreover, either $\boldsymbol{h}^*_{A,X} \equiv 0$ or $\sup_X \boldsymbol{h}^*_{A,X} = 1$.*

(c) *If $Y_1 \subset Y_2 \subset X$ are open, $A_1 \subset Y_1$, and $A_1 \subset A_2 \subset Y_2$, then $\boldsymbol{h}_{A_2,Y_2} \le \boldsymbol{h}_{A_1,Y_1}$ and $\boldsymbol{h}^*_{A_2,Y_2} \le \boldsymbol{h}^*_{A_1,Y_1}$ on Y_1.*

(d) *The set $P := \{z \in A : \boldsymbol{h}_{A,X}(z) < \boldsymbol{h}^*_{A,X}(z)\}$ is pluripolar.*

Proof. (a) and (c) follow directly from the definition.

(b) By Proposition 2.3.11, $\boldsymbol{h}^*_{A,X} \in \mathcal{PSH}(X)$. Let $C := \sup_X \boldsymbol{h}^*_{A,X}$. Suppose that $0 < C < 1$. Let $u \in \mathcal{PSH}(X)$, $u \le 1$, $u|_A \le 0$. Then $u \le \boldsymbol{h}^*_{A,X} \le C$. Hence $(1/C)u \le \boldsymbol{h}_{A,X}$. Thus $(1/C)\boldsymbol{h}^*_{A,X} \le \boldsymbol{h}^*_{A,X}$; a contradiction.

(d) follows from Theorem 2.3.33 (a). $\square$

Proposition 3.2.3. *If $A \subset \mathbb{C}^n$, $A \notin \mathcal{PLP}$, then $\boldsymbol{h}^*_{A,\mathbb{C}^n} \equiv 0$.*

Proof. Let $u := \boldsymbol{h}^*_{A,\mathbb{C}^n}$. Then $u \in \mathcal{PSH}(\mathbb{C}^n)$ and $u \le 1$. Thus $u \equiv \text{const}$ (cf. Remark 2.3.2 (g)). Since $A \notin \mathcal{PLP}$, Proposition 3.2.2 (d) implies that there exists an $a \in A$ such that $u(a) = 0$. $\square$

Proposition 3.2.4 (Two constants theorem). *Let $A \subset X$ be such that A is not analytically thin at a point $b_0 \in \overline{A}$. Then for every $f \in \mathcal{O}(X)$ we have*

$$|f(z)| \le \|f\|_A^{1-\boldsymbol{h}^*_{A,X}(z)} \|f\|_X^{\boldsymbol{h}^*_{A,X}(z)}, \quad z \in X.$$

Proof. Put $m := \|f\|_A$, $M := \|f\|_X$. We may assume that $0 < m < M < +\infty$. Define

$$u := \frac{\log \frac{|f|}{m}}{\log \frac{M}{m}}.$$

Then $u \in \mathcal{PSH}(X)$, $u \le 1$, and $u \le 0$ on A. Consequently, $u \le \boldsymbol{h}^*_{A,D}$, which is equivalent to the required inequality. □

Proposition 3.2.4 gives the following extension of Lemma 2.1.14.

Lemma 3.2.5. *Let* $(G, D, A_0, A, \mathcal{F})$ *be as in Lemma 2.1.14. Assume that the conditions* (i) *and* (ii′) *from Lemma 2.1.14 are satisfied. Then*

$$|\hat{f}(z)| \le \|f\|_{A_0}^{1-\boldsymbol{h}^*_{A_0,D}(z)} \|f\|_A^{\boldsymbol{h}^*_{A_0,D}(z)}, \quad z \in D,\ f \in \mathcal{F}.$$

Notice that the above inequality is in fact equivalent to the inequality $\|\hat{f}\|_D \le \|f\|_A$, $f \in \mathcal{F}$.

Proof. By Lemma 2.1.14 we have $\|\hat{f}\|_D \le \|f\|_A$. Now, by Proposition 3.2.4 with $(X, A) = (D, A_0)$, we get

$$\begin{aligned} |\hat{f}(z)| &\le \|\hat{f}\|_{A_0}^{1-\boldsymbol{h}^*_{A_0,D}(z)} \|\hat{f}\|_D^{\boldsymbol{h}^*_{A_0,D}(z)} \\ &\le \|f\|_{A_0}^{1-\boldsymbol{h}^*_{A_0,D}(z)} \|f\|_A^{\boldsymbol{h}^*_{A_0,D}(z)}, \quad z \in D,\ f \in \mathcal{F}. \end{aligned}$$ □

Example 3.2.6. Let $\varphi\colon (X, p) \to (\hat{X}, \hat{p})$ be the maximal holomorphic extension (cf. § 2.1.7) and let $u \in \mathcal{PSH}(\hat{X})$, $u \le C$ on $\varphi(X)$. Then $u \le C$ on $\hat{X}$. In particular, $\boldsymbol{h}^*_{\varphi(X),\hat{X}} \equiv 0$.

Indeed, for every compact $L \subset\subset \hat{X}$ there exists a compact $K \subset\subset X$ such that

$$L \subset \widehat{\varphi(K)}^{\mathcal{O}(\hat{X})} = \widetilde{\varphi(K)}^{\mathcal{PSH}(\hat{X})}$$

(cf. Proposition 2.1.15, Theorem 2.5.9). Thus, $\sup_L u \le \sup_{\varphi(K)} u \le C$.

The following lemma will be useful in Chapter 6.

Lemma 3.2.7. *Let* $T \subset \overline{\mathbb{D}}$ *be a relatively open subset. Then*

$$\boldsymbol{h}^*_{T\cap\mathbb{D},\mathbb{D}}(0) \le \frac{1}{2\pi}\int_0^{2\pi} \chi_{\mathbb{T}\setminus T}(e^{it})\,dt,$$

where χ_B *stands for the characteristic function of* B.

Proof. For $r \in (0,1)$, the mean value inequality for subharmonic functions implies

$$\boldsymbol{h}^*_{T\cap\mathbb{D},\mathbb{D}}(0) \le \frac{1}{2\pi}\int_0^{2\pi} \boldsymbol{h}^*_{T\cap\mathbb{D},\mathbb{D}}(re^{it})dt \le \frac{1}{2\pi}\Big(\int_{Q_1} dt + \int_{Q_2} \boldsymbol{h}^*_{T\cap\mathbb{D},\mathbb{D}}(re^{it})dt\Big),$$

where $Q_1 := \{t \in [0,2\pi] : e^{it} \notin T\}$ and $Q_2 := [0,2\pi] \setminus Q_1$. Observe now that $\boldsymbol{h}^*_{T\cap\mathbb{D},\mathbb{D}}(re^{it}) \underset{r\to 1}{\longrightarrow} 0$, whenever $t \in Q_2$. It remains to apply the Lebesgue Theorem. □

Definition 3.2.8. We say that a set $A \subset X$ is *pluriregular at a point* $a \in \overline{A}$ if $\boldsymbol{h}^*_{A,U}(a) = 0$ for any open neighborhood U of a. Observe that A is pluriregular at a iff there exists a basis $\mathcal{U}(a)$ of neighborhoods of a such that $\boldsymbol{h}^*_{A,U}(a) = 0$ for every $U \in \mathcal{U}(a)$. Define

$$A^* = A^{*,X} := \{a \in \overline{A} : A \text{ is pluriregular at } a\}.$$

We say that A is *locally pluriregular* if $A \neq \varnothing$ and A is pluriregular at every point $a \in A$, i.e. $\varnothing \neq A \subset A^*$.

If $n = 1$, then we say *locally regular* instead of locally pluriregular.

Remark 3.2.9. (a) If $\varnothing \neq Y \subset X$ is open, then Y is locally pluriregular.

(b) If $\varnothing \neq B \subset A \subset X$, $B \subset Y$, where Y is open, then $B^{*,Y} \subset A^{*,X} \cap Y = (A \cap Y)^{*,Y}$.

(c) $\boldsymbol{h}^*_{A,X} = 0$ on A^*.

(d) $\boldsymbol{h}^*_{A,X} \le \boldsymbol{h}_{A^*,X} \le \boldsymbol{h}^*_{A^*,X}$.

(e) If A is locally pluriregular, then $\boldsymbol{h}^*_{A,X} \le \boldsymbol{h}_{A^*,X} \le \boldsymbol{h}_{A,X} \le \boldsymbol{h}^*_{A,X}$, $\boldsymbol{h}^*_{A,X} \le \boldsymbol{h}_{A^*,X} \le \boldsymbol{h}^*_{A^*,X} \le \boldsymbol{h}^*_{A,X}$, and therefore, $\boldsymbol{h}^*_{A,X} \equiv \boldsymbol{h}_{A,X} \equiv \boldsymbol{h}^*_{A^*,X} \equiv \boldsymbol{h}_{A^*,X}$.

Proposition 3.2.10. $A \setminus A^* \in \mathcal{PLP}$.

Proof. Let $(U_k)_{k=1}^{\infty}$ be a basis of the topology of X. Put

$$P_k := \{z \in A \cap U_k : \boldsymbol{h}_{A,U_k}(z) < \boldsymbol{h}^*_{A,U_k}(z)\}, \quad P := \bigcup_{k=1}^{\infty} P_k.$$

Then $P \in \mathcal{PLP}$ (cf. Proposition 3.2.2 (d)) and $\boldsymbol{h}^*_{A,U_k}(a) = 0$, $a \in (A \setminus P) \cap U_k$, $k \in \mathbb{N}$. Thus $A \setminus A^* \subset P$. □

Proposition 3.2.11 ([Ale-Hec 2004]). *Assume that X is connected. The following conditions are equivalent:*

(i) *for any $A \subset X$ and $P \in \mathcal{PLP}(X)$ we have $\boldsymbol{h}^*_{A\cup P,X} \equiv \boldsymbol{h}^*_{A,X}$;*

(ii) *for every $P \in \mathcal{PLP}(X)$ we have $\boldsymbol{h}^*_{P,X} \equiv 1$;*

(iii) *for every $P \in \mathcal{PLP}(X)$ there exists a $v \in \mathcal{PSH}(X)$, $v \not\equiv -\infty$, $v \le 0$, such that $P \subset v^{-1}(-\infty)$;*

(iv) *for every $A \subset X$ we have $\boldsymbol{h}^*_{A,X} \equiv \boldsymbol{h}^*_{A^*,X}$ (cf. Remark 3.2.9).*

Moreover

- *by Theorem* 2.3.25*, condition* (iii) (*and, consequently, each other condition*) *is always satisfied if* $X \in \mathfrak{R}_b(\mathbb{C}^n)$ (*cf. Definition* 2.1.1);
- *conditions* (i), (ii), (iii) *are also equivalent for a fixed pluripolar set* $P \subset X$.

Proof. (i) $\Longrightarrow$ (ii): $\boldsymbol{h}^*_{P,X} \equiv \boldsymbol{h}^*_{\varnothing,X} \equiv 1$.

(ii) $\Longrightarrow$ (iii): By Proposition 2.3.22 (a) there exists an $a \in X$ such that $\boldsymbol{h}_{P,X}(a) = 1$. Take a sequence $(u_k)_{k=1}^{\infty} \subset \mathcal{PSH}(X)$ with $u_k \le 1$, $u_k \le 0$ on P, and $u_k(a) \ge 1 - 1/2^k$, $k \in \mathbb{N}$. Define $v := \sum_{k=1}^{\infty}(u_k - 1)$. Then $v \in \mathcal{PSH}(X)$, $v \le 0$, $P \subset v^{-1}(-\infty)$, and $v(a) \ge -1$.

(iii) $\Longrightarrow$ (i): Let $u \in \mathcal{PSH}(X)$, $u \le 1$ on X, $u \le 0$ on A. Then, for every $\varepsilon > 0$, we get $u + \varepsilon v \le 1$ on X and $u + \varepsilon v \le 0$ on $A \cup P$. Thus $u + \varepsilon v \le \boldsymbol{h}_{A \cup P,X}$ and hence $u + \varepsilon v \le \boldsymbol{h}^*_{A\cup P,X}$. Thus $u \le \boldsymbol{h}^*_{A\cup P,X}$ on $X \setminus v^{-1}(-\infty)$. Consequently, by Proposition 2.3.8, $u \le \boldsymbol{h}^*_{A\cup P,X}$ and, finally, $\boldsymbol{h}^*_{A,X} \le \boldsymbol{h}^*_{A\cup P,X}$.

(iv) $\Longrightarrow$ (ii): $\boldsymbol{h}^*_{P,X} = \boldsymbol{h}^*_{P^*,X} = \boldsymbol{h}^*_{\varnothing,X} \equiv 1$.

(i) $\Longrightarrow$ (iv): The inequality "$\le$" follows from Remark 3.2.9. By Proposition 3.2.10 and (i) we get $\boldsymbol{h}^*_{A,X} = \boldsymbol{h}^*_{A\cap A^*,X} \ge \boldsymbol{h}^*_{A^*,X}$. □

Corollary 3.2.12. *Let* $X \in \mathfrak{R}_b(\mathbb{C}^n)$. *Then*

- $\boldsymbol{h}^*_{A\cup P,X} \equiv \boldsymbol{h}^*_{A,X}$ *for any* $A \subset X$ *and* $P \in \mathcal{PLP}(X)$.
- *A set* $P \subset X$ *is pluripolar iff* $\boldsymbol{h}^*_{P,X} \equiv 1$.

Corollary 3.2.13. $(A \setminus P)^* = A^*$ *for arbitrary* $P \in \mathcal{PLP}$. *In particular*

- *if* $P \in \mathcal{PLP}$, *then* $P^* = \varnothing$;
- *if* A *is locally pluriregular, then* $A \setminus P$ *is locally pluriregular for arbitrary* $P \in \mathcal{PLP}$;
- $(A \cap A^*)^* = A^*$;
- $A \cap A^*$ *is locally pluriregular* (*cf. Proposition* 3.2.10).

Proof. We only need to show that $A^* \subset (A \setminus P)^*$. Fix a point $a \in A^*$. If U is an open relatively compact neighborhood of a, then we have $\boldsymbol{h}^*_{A\setminus P,U}(a) = \boldsymbol{h}^*_{A,U}(a) = 0$. It remains to observe that $a \in \overline{A \setminus P}$ – otherwise, a would have a relatively compact neighborhood U such that $U \cap (A \setminus P) = \varnothing$, which implies that $1 = \boldsymbol{h}_{A\setminus P,U}(a) = \boldsymbol{h}^*_{A,U}(a) = 0$; a contradiction. □

Proposition 3.2.14. *Let* $A \subset X$ *be locally pluriregular and let* $P', P'' \subset X$ *be pluripolar such that* $A \setminus P' \subset X \setminus P''$. *Then* $\boldsymbol{h}^*_{A\setminus P',X\setminus P''} = \boldsymbol{h}^*_{A,X}$ *on* $X \setminus P''$.

Proof. By Proposition 2.3.29 we get $\boldsymbol{h}^*_{A\setminus P',X\setminus P''} = \boldsymbol{h}^*_{A\setminus P',X}$ on $X \setminus P''$. Now, by Remark 3.2.9 (e) and Corollary 3.2.13, we have $\boldsymbol{h}^*_{A\setminus P',X} = \boldsymbol{h}^*_{(A\setminus P')^*,X} = \boldsymbol{h}^*_{A^*,X} = \boldsymbol{h}^*_{A,X}$. □

Proposition 3.2.15 (cf. [Bło 2000]). *Let* $X \in \mathfrak{R}_b(\mathbb{C}^n)$, $A \subset X$. *Put*

$$\Delta(\varepsilon) := \{z \in X : \boldsymbol{h}^*_{A,X}(z) < \varepsilon\}, \quad 0 < \varepsilon < 1.$$

Then

$$\frac{\boldsymbol{h}^*_{A,X} - \varepsilon}{1 - \varepsilon} \le \boldsymbol{h}^*_{\Delta(\varepsilon),X} \le \boldsymbol{h}^*_{A,X}$$

(*cf. Lemma* 7.2.9). *Consequently,* $\boldsymbol{h}^*_{\Delta(\varepsilon),X} \nearrow \boldsymbol{h}^*_{A,X}$ *as* $\varepsilon \searrow 0$.

? We do not know whether the result is true for arbitrary $X \in \mathfrak{R}_c(\mathbb{C}^n)$? Notice that $\boldsymbol{h}^*_{\Delta(\varepsilon),X} \equiv \boldsymbol{h}_{\Delta(\varepsilon),X}$ (cf. Remark 3.2.9).

Proof. It is clear that $\boldsymbol{h}_{\Delta(\varepsilon),X} \ge \frac{\boldsymbol{h}^*_{A,X} - \varepsilon}{1-\varepsilon}$. Put

$$P := \{z \in A : 0 = \boldsymbol{h}_{A,X}(z) < \boldsymbol{h}^*_{A,X}(z)\} \in \mathcal{PLP}$$

(cf. Proposition 3.2.2 (d)). Since $X \in \mathfrak{R}_b(\mathbb{C}^n)$, we get $\boldsymbol{h}^*_{A \setminus P,X} \equiv \boldsymbol{h}^*_{A,X}$ (cf. Proposition 3.2.11). Observe that $A \setminus P \subset \Delta(\varepsilon)$. Thus $\boldsymbol{h}_{\Delta(\varepsilon),X} \le \boldsymbol{h}^*_{A \setminus P,X} \equiv \boldsymbol{h}^*_{A,X}$ on $\Delta(\varepsilon)$. □

Theorem* 3.2.16 (Product property; [NTV-Sic 1991], [Edi-Pol 1997], [Edi 2002], Theorem 4.1). *Let* $X_j \in \mathfrak{R}_\infty(\mathbb{C}^{n_j})$, $A_j \subset X_j$, $j = 1, 2$. *Assume that* A_1, A_2 *are open or* A_1, A_2 *are compact. Then*

$$\boldsymbol{h}_{A_1 \times A_2, X_1 \times X_2}(z_1, z_2) = \max\{\boldsymbol{h}_{A_1,X_1}(z_1),\ \boldsymbol{h}_{A_2,X_2}(z_2)\}, \quad (z_1, z_2) \in X_1 \times X_2.$$

The proof of the above theorem is based on deep results from Poletsky's theory of holomorphic discs.

Observe that the inequality "$\ge$" is elementary and it holds for arbitrary sets A_1, A_2.

Theorem 3.2.17 (Product property; [NTV-Sic 1991], [Edi-Pol 1997], [Edi 2002], Theorem 4.1). *If* $X_j \in \mathfrak{R}_b(\mathbb{C}^{n_j})$, $j = 1, 2$, *then for arbitrary subsets* $A_1 \subset X_1$, $A_2 \subset X_2$, *we have*

$$\boldsymbol{h}^*_{A_1 \times A_2, X_1 \times X_2}(z_1, z_2) = \max\{\boldsymbol{h}^*_{A_1,X_1}(z_1),\ \boldsymbol{h}^*_{A_2,X_2}(z_2)\}, \quad (z_1, z_2) \in X_1 \times X_2.$$

We need the following lemma.

Lemma 3.2.18. *If* $X_j \in \mathfrak{R}_\infty(\mathbb{C}^{n_j})$, $j = 1, 2$, *then for arbitrary subsets* $A_1 \subset X_1$, $A_2 \subset X_2$, *we have*

$$\boldsymbol{h}_{A_1 \times A_2, X_1 \times X_2}(z_1, z_2) \le 1 - (1 - \boldsymbol{h}_{A_1,X_1}(z_1))(1 - \boldsymbol{h}_{A_2,X_2}(z_2)), \quad (z_1, z_2) \in X_1 \times X_2.$$

Proof. Fix $(z_1^0, z_2^0) \in X_1 \times X_2$. The inequality is trivial if $\boldsymbol{h}_{A_1,X_1}(z_1^0) = 1$ or $\boldsymbol{h}_{A_2,X_2}(z_2^0) = 1$. Assume that $h_{A_j,X_j}(z_j^0) < 1$, $j = 1, 2$. Fix a $u \in \mathcal{PSH}(X_1 \times X_2)$ with $u \le 1$, $u|_{A_1 \times A_2} \le 0$. For $(a_1, a_2) \in X_1 \times X_2$ with $\boldsymbol{h}_{A_j,X_j}(a_j) < 1$, $j = 1, 2$, define

$$v_{a_1} := \frac{u(a_1, \cdot) - \boldsymbol{h}_{A_1,X_1}(a_1)}{1 - \boldsymbol{h}_{A_1,X_1}(a_1)}, \quad v^{a_2} := \frac{u(\cdot, a_2) - \boldsymbol{h}_{A_2,X_2}(a_2)}{1 - \boldsymbol{h}_{A_2,X_2}(a_2)}.$$

Observe that

$$\begin{aligned}
\boldsymbol{h}_{A_1 \times A_2, X_1 \times X_2}(a_1, a_2) \le 1 - (1 - \boldsymbol{h}_{A_1,X_1}(a_1))(1 - \boldsymbol{h}_{A_2,X_2}(a_2)) \\
\iff v_{a_1}(a_2) \le \boldsymbol{h}_{A_2,X_2}(a_2) \\
\iff v^{a_2}(a_1) \le \boldsymbol{h}_{A_1,X_1}(a_1).
\end{aligned}$$

It is clear that $v_{a_1} \in \mathcal{PSH}(X_2)$, $v_{a_1} \le 1$, $v^{a_2} \in \mathcal{PSH}(X_1)$, $v^{a_2} \le 1$. If $a_2 \in A_2$, then $v^{a_2} \le 0$ on A_1. Thus, if $a_2 \in A_2$, then $v^{a_2} \le \boldsymbol{h}_{A_1,X_1}$. In particular, $v_{z_1^0}(a_2) \le \boldsymbol{h}_{A_2,X_2}(a_2)$, $a_2 \in A_2$. Hence $v_{z_1^0} \le 0$ on A_2, which gives $v_{z_1^0}(z_2^0) \le \boldsymbol{h}_{A_2,X_2}(z_2^0)$. □

Proof of Theorem 3.2.17. The inequality "≥" is elementary. Let

$$\begin{aligned}
\varDelta_j(\varepsilon) &:= \{z_j \in X_j : \boldsymbol{h}^*_{A_j,X_j}(z_j) < \varepsilon\}, \quad j = 1, 2, \\
\varDelta(\varepsilon) &:= \{(z_1, z_2) \in X_1 \times X_2 : \boldsymbol{h}^*_{A_1 \times A_2, X_1 \times X_2}(z_1, z_2) \le 1 - (1 - \varepsilon)^2\}, \quad 0 < \varepsilon < 1.
\end{aligned}$$

By Lemma 3.2.18 we get $\boldsymbol{h}_{\varDelta_1(\varepsilon) \times \varDelta_2(\varepsilon), X_1 \times X_2} < 1 - (1 - \varepsilon)^2$ on $\varDelta_1(\varepsilon) \times \varDelta_2(\varepsilon)$. Hence $\boldsymbol{h}^*_{\varDelta_1(\varepsilon) \times \varDelta_2(\varepsilon), X_1 \times X_2} \le 1 - (1 - \varepsilon)^2$ on $\varDelta_1(\varepsilon) \times \varDelta_2(\varepsilon)$, which implies that $\varDelta_1(\varepsilon) \times \varDelta_2(\varepsilon) \subset \varDelta(\varepsilon)$. Now, by Theorem 3.2.16, we have

$$\begin{aligned}
\max\{\boldsymbol{h}_{\varDelta_1(\varepsilon),X_1}(z_1), \ \boldsymbol{h}_{\varDelta_2(\varepsilon),X_2}(z_2)\} &= \boldsymbol{h}_{\varDelta_1(\varepsilon) \times \varDelta_2(\varepsilon), X_1 \times X_2}(z_1, z_2) \\
&= \boldsymbol{h}^*_{\varDelta_1(\varepsilon) \times \varDelta_2(\varepsilon), X_1 \times X_2}(z_1, z_2) \\
&\ge \boldsymbol{h}^*_{\varDelta(\varepsilon), X_1 \times X_2}(z_1, z_2).
\end{aligned}$$

To finish the proof, it remains to observe that by Proposition 3.2.15 we get $\boldsymbol{h}_{\varDelta_j(\varepsilon),X_j} \nearrow \boldsymbol{h}^*_{A_j,X_j}$, $j = 1, 2$, and $\boldsymbol{h}^*_{\varDelta(\varepsilon), X_1 \times X_2} \nearrow \boldsymbol{h}^*_{A_1 \times A_2, X_1 \times X_2}$ when $\varepsilon \searrow 0$. □

Exercise 3.2.19. Prove that if $X \in \mathfrak{R}_\infty(\mathbb{C}^n)$, $Y \in \mathfrak{R}_\infty(\mathbb{C}^m)$, $A \subset X$, $B \subset Y$, then $(A \times B)^* = A^* \times B^*$. In particular, if A and B are locally pluriregular, then so is $A \times B$.

Example 3.2.20. (a)

$$\boldsymbol{h}^*_{(-1,1),\mathbb{D}}(z) = \frac{2}{\pi} \left| \operatorname{Arg} \frac{1+z}{1-z} \right|, \quad z \in \mathbb{D}.$$

Indeed, we only need to observe that the function $\varphi_\pm(z) = \pm \frac{2}{\pi} \operatorname{Arg} \frac{1+z}{1-z}$ is the solution of the Dirichlet problem in $\mathbb{D}_\pm := \{z \in \mathbb{D} : \pm \operatorname{Im} z > 0\}$ with $\varphi_\pm = 0$ on $(-1, 1)$ and $\varphi_\pm = 1$ on $\{z \in \mathbb{T} : \pm \operatorname{Im} z > 0\}$.

(b) Let $S := \{z \in \mathbb{C} : |\operatorname{Im} z| < 1\}$. Then $\boldsymbol{h}^*_{\mathbb{R},S}(z) = |\operatorname{Im} z|$, $z \in S$.

(c) Every non-empty open set $U \subset \mathbb{R}^n \subset \mathbb{R}^n + i\mathbb{R}^n \simeq \mathbb{C}^n$ is locally pluriregular (as a subset of $\mathbb{C}^n$).

In fact, we only need to show that for all $a \in \mathbb{R}^n$ and $r > 0$ we have

$$\boldsymbol{h}^*_{\mathbb{P}_n(a,r)\cap\mathbb{R}^n,\mathbb{P}_n(a,r)}(a) = 0.$$

Using a suitable transformation of coordinates and the product property (Theorem 3.2.17) one can easily reduce the problem to the equality $\boldsymbol{h}^*_{(-1,1),\mathbb{D}}(0) = 0$, which follows directly from (a).

(d) If $L \subset \partial\mathbb{D}(a,r) \subset \mathbb{C}$ is a non-empty open arc, then L is locally pluriregular.

In fact, there exists a homography h such that $h(L)$ is an open subset of $\mathbb{R}$. Thus the result follows from (c).

Proposition 3.2.21. *Let $X \in \mathfrak{R}_\infty(\mathbb{C}^n)$, $Y \in \mathfrak{R}_\infty(\mathbb{C}^m)$, $P \subset A \times B \subset X \times Y$. Assume that B is locally pluriregular and for any $a \in A$ the fiber $P_{(a,\cdot)}$ is pluripolar (we do not assume that P is pluripolar). Then for any relatively compact open set $V \times W \subset\subset X \times Y$ we have $\boldsymbol{h}^*_{(A\times B)\setminus P,\, V\times W} = \boldsymbol{h}^*_{A\times B,\, V\times W}$. In particular, if A is also locally pluriregular, then $(A \times B) \setminus P$ is locally pluriregular.*

Proof. Take a $u \in \mathcal{PSH}(V \times W)$, $u \le 1$, with $u \le 0$ on $((A \times B) \cap (V \times W)) \setminus P$. Then for any $a \in A \cap V$ we have $u(a,\cdot) \le 0$ on $(B \cap W) \setminus P_{(a,\cdot)}$. Hence $u(a,b) \le \boldsymbol{h}^*_{B\setminus P_{(a,\cdot)},W}(b) = \boldsymbol{h}^*_{B,W}(b) = 0$, $b \in B \cap W$. Thus $u \le \boldsymbol{h}_{A\times B,V\times W}$. Finally, $\boldsymbol{h}^*_{(A\times B)\setminus P,V\times W} \le \boldsymbol{h}^*_{A\times B,V\times W}$. □

Proposition 3.2.22 ([Ale-Hec 2004]). (a) *Let $G \subset \mathbb{C}^{n-k}$ be an arbitrary domain and let $A \subset \mathbb{C}^k \times G =: D$. Then $\boldsymbol{h}_{A,D}(z,w) = \boldsymbol{h}_{\widetilde{A},G}(w)$, $(z,w) \in D$, where*

$$\widetilde{A} := \operatorname{pr}_G(A) = \{w \in G : \exists_{z\in\mathbb{C}^k} : (z,w) \in A\}.$$

(b) *Let $G \subset \mathbb{C}^{n-1}$ be an arbitrary bounded domain, let $C \subset \mathbb{C}$ be polar, and let $\widetilde{A} \subset G$, $\widetilde{A} \notin \mathcal{PLP}$. Put $D := \mathbb{C} \times G \subset \mathbb{C}^n$, $A := C \times \widetilde{A}$. Then $\boldsymbol{h}^*_{A,D}(z,w) < \boldsymbol{h}^*_{A^*,D}(z,w) = 1$, $(z,w) \in D$ (cf. Proposition 3.2.11* (iv)).

Proof. (a) It is clear that $\boldsymbol{h}_{\widetilde{A},G}(w) \le \boldsymbol{h}_{A,D}(z,w)$. Conversely, if $u \in \mathcal{PSH}(D)$, $u \le 1$ on D and $u \le 0$ on A, then $u(z,w) = v(w)$ with $v \in \mathcal{PSH}(G)$. Obviously, $v \le 1$ on G and $v \le 0$ on $\widetilde{A}$. Hence $v \le \boldsymbol{h}_{\widetilde{A},G}$.

(b) By (a) we have $\boldsymbol{h}^*_{A,D}(z,w) = \boldsymbol{h}_{\widetilde{A},G}(w) < 1$, $(z,w) \in D$ (cf. Corollary 3.2.12). On the other hand, $A^* = C^* \times (\widetilde{A})^* = \varnothing \times (\widetilde{A})^* = \varnothing$ (cf. Exercise 3.2.19), which gives $\boldsymbol{h}^*_{A^*,D} \equiv 1$. □

The following approximation property of relative extremal function will play the fundamental role in the sequel – see also Propositions 3.2.24 and 3.2.25.

Proposition 3.2.23. *Let $X_k \nearrow X \subset\subset Y$ and let $A_k \subset X_k$, $A_k \nearrow A$. Then $\boldsymbol{h}^*_{A_k,X_k} \searrow \boldsymbol{h}^*_{A,X}$.*

Proof. Let $u_k := \boldsymbol{h}_{A_k,X_k}$. Obviously, $u_{k+1} \le u_k$. Let $v := \lim_{k\to+\infty} u_k^*$. Then $v \in \mathcal{PSH}(X)$ and $\boldsymbol{h}^*_{A,X} \le v \le 1$. Put $P_k := \{z \in A_k : u_k(z) < u_k^*(z)\}$, $P := \bigcup_{k=1}^\infty P_k$. Then $P \in \mathcal{PLP}$. Observe that $v = \lim_{k\to+\infty} u_k \le 0$ on $A \setminus P$. Consequently, by Corollary 3.2.12, $v \le \boldsymbol{h}^*_{A\setminus P,X} = \boldsymbol{h}^*_{A,X}$. □

Proposition 3.2.24 ([Kli 1991], Proposition 4.5.10). *If $(K_\nu)_{\nu=1}^\infty$ is a sequence of compact subsets of X such that $K_\nu \searrow K$, then $\boldsymbol{h}_{K_\nu,X} \nearrow \boldsymbol{h}_{K,X}$. In particular, $\boldsymbol{h}_{K^{(\varepsilon)},X} \nearrow \boldsymbol{h}_{K,X}$ as $\varepsilon \searrow 0$, where $K^{(\varepsilon)}$ is as in Definition 2.1.3.*

Proof. Obviously, $\boldsymbol{h}_{K_\nu,X} \le \boldsymbol{h}_{K_{\nu+1},X} \le \boldsymbol{h}_{K,X}$. Let $u \in \mathcal{PSH}(X)$ be such that $u \le 1$ and $u < 0$ on K. Then $u < 0$ on K_ν for $\nu \gg 1$. Hence $u \le \boldsymbol{h}_{K_\nu,X}$ for $\nu \gg 1$ and, consequently, $\boldsymbol{h}_{K,X} \le \lim_{\nu\to+\infty} \boldsymbol{h}_{K_\nu,X}$. □

Proposition 3.2.25. *Let $A_k \subset X_k \subset X$, X_k open, $X_k \nearrow X$, $A_k \nearrow A$. Then $\boldsymbol{h}^*_{A_k^*,X_k} \searrow \boldsymbol{h}^*_{A^*,X}$. In particular, if $(X_k)_{k=1}^\infty$ is an exhaustion of X (Definition 1.4.5), then $\boldsymbol{h}^*_{A,X_k} \searrow \boldsymbol{h}^*_{A^*,X}$.*

Proof. Put $v_k := \boldsymbol{h}^*_{A_k^*,X_k}$, $k \in \mathbb{N}$. It is clear that $\boldsymbol{h}^*_{A^*,X} \le v_{k+1} \le v_k$. Let $v := \lim_{k\to+\infty} v_k$. Then $v \in \mathcal{PSH}(X)$ and $v \ge \boldsymbol{h}^*_{A^*,X}$. We know that $v_k = 0$ on $A_k \setminus P_k$, where $P_k \in \mathcal{PLP}$ (Propositions 3.2.2 (d) and 3.2.10). Put $P := \bigcup_{k=1}^\infty P_k$. Then $v = 0$ on $A \setminus P$. Consequently, $v \le \boldsymbol{h}^*_{A\setminus P,X} \le \boldsymbol{h}^*_{(A\setminus P)^*,X} = \boldsymbol{h}^*_{A^*,X}$. □

Remark 3.2.26. Recall that

(a) $\boldsymbol{h}_{A,X} \le \boldsymbol{h}^*_{A^*,X}$ (see Remark 3.2.9). In general $\boldsymbol{h}^*_{A,X} \not\equiv \boldsymbol{h}^*_{A^*,X}$ (cf. Proposition 3.2.22 (b)),

(b) if A is locally pluriregular, then $\boldsymbol{h}^*_{A^*,X} \equiv \boldsymbol{h}^*_{A,X}$ (Remark 3.2.9),

(c) if $X \in \mathfrak{R}_b(\mathbb{C}^n)$, then $\boldsymbol{h}^*_{A^*,X} \equiv \boldsymbol{h}^*_{A,X}$ (Proposition 3.2.11).

Proposition 3.2.27. *Let $A \subset X \in \mathfrak{R}_c(\mathbb{C}^n) \cap \mathfrak{R}_b(\mathbb{C}^n)$, $A \notin \mathcal{PLP}$, $0 < \mu < 1$, and*

$$\Delta(\mu) := \{z \in X : \boldsymbol{h}^*_{A,X}(z) < \mu\}.$$

Then $\boldsymbol{h}^*_{A,\Delta(\mu)} = (1/\mu)\boldsymbol{h}^*_{A,X}$ on $\Delta(\mu)$. In particular, $\boldsymbol{h}^*_{A,\Delta(\mu)}(z) < 1$, $z \in \Delta(\mu)$, which implies (cf. Corollary 3.2.12) that $A \cap S \notin \mathcal{PLP}$ for any connected component S of $\Delta(\mu)$.

In the case where A is locally pluriregular, the result remains true for arbitrary $X \in \mathfrak{R}_c(\mathbb{C}^n)$.

Proof. We know that $\boldsymbol{h}^*_{A,X} = 0$ on $A \setminus P$, where $P \in \mathcal{PLP}$. Consequently, $A \setminus P \subset \Delta(\mu)$ and hence $\boldsymbol{h}^*_{A\cap\Delta(\mu),X} = \boldsymbol{h}^*_{A,X}$ and $\boldsymbol{h}^*_{A,\Delta(\mu)} = \boldsymbol{h}^*_{A\setminus P,\Delta(\mu)} \ge (1/\mu)\boldsymbol{h}^*_{A,X}$ on $\Delta(\mu)$.

Let $u \in \mathcal{PSH}(\Delta(\mu))$, $u \le 1$, $u \le 0$ on $A \cap \Delta(\mu)$. Define

$$v := \begin{cases} \max\{\mu u, \boldsymbol{h}^*_{A,X}\} & \text{on } \Delta(\mu), \\ \boldsymbol{h}^*_{A,X} & \text{on } X \setminus \Delta(\mu), \end{cases}$$

and observe that for every $z_0 \in \partial(\Delta(\mu))$ we get $\limsup_{\Delta(\mu)\ni z\to z_0} \mu u(z) \le \mu \le \boldsymbol{h}^*_{A,X}(z_0)$. This implies that $v \in \mathcal{PSH}(X)$ (cf. Proposition 2.3.6) and hence $v \le \boldsymbol{h}^*_{A\cap\Delta(\mu),X} = \boldsymbol{h}^*_{A,X}$. In particular, $\boldsymbol{h}^*_{A,\Delta(\mu)} \le (1/\mu)\boldsymbol{h}^*_{A,X}$ on $\Delta(\mu)$.

If X is an arbitrary Riemann domain and A is locally pluriregular, then $\boldsymbol{h}^*_{A,X} = 0$ on A. Thus $A \subset \Delta(\mu)$ and we may repeat the above proof (with $P := \varnothing$). □

Proposition 3.2.28. *Let D_j be a Riemann domain over $\mathbb{C}^{n_j}$ and let $A_j \subset D_j$ be locally pluriregular, $j = 1, \dots, N$. Put*

$$\widehat{X} := \Big\{(z_1, \dots, z_N) \in D_1 \times \cdots \times D_N : \textstyle\sum_{j=1}^N \boldsymbol{h}^*_{A_j,D_j}(z_j) < 1\Big\}.$$

Then

$$\boldsymbol{h}^*_{A_1\times\cdots\times A_N,\widehat{X}}(z) = \sum_{j=1}^N \boldsymbol{h}^*_{A_j,D_j}(z_j), \quad z = (z_1, \dots, z_N) \in \widehat{X}.$$

Proof. The inequality "$\ge$" is obvious. To get the opposite inequality we proceed by induction on $N \ge 2$.

Let $N = 2$ (cf. [Sic 1981a]): Put $u := \boldsymbol{h}^*_{A_1\times A_2,\widehat{X}} \in \mathcal{PSH}(\widehat{X})$ and fix a point $(a_1, a_2) \in \widehat{X}$. If $a_1 \in A_1$, then $u(a_1, \cdot) \in \mathcal{PSH}(D_2)$, $u(a_1, \cdot) \le 1$, and $u(a_1, \cdot) \le 0$ on A_2. Therefore,

$$u(a_1, \cdot) \le \boldsymbol{h}^*_{A_2,D_2} = \boldsymbol{h}^*_{A_1,D_1}(a_1) + \boldsymbol{h}^*_{A_2,D_2} \ \text{ on } D_2.$$

In particular, $u(a_1, a_2) \le \boldsymbol{h}^*_{A_1,D_1}(a_1) + \boldsymbol{h}^*_{A_2,D_2}(a_2)$. The same argument works if $a_2 \in A_2$. If $a_1 \notin A_1$, then let $\mu := 1 - \boldsymbol{h}^*_{A_1,D_1}(a_1) \in (0, 1]$. Put

$$(D_2)_\mu := \{z_2 \in D_2 : \boldsymbol{h}^*_{A_2,D_2}(z_2) < \mu\}.$$

It is clear that $A_2 \subset (D_2)_\mu \ni a_2$. Put

$$v := \frac{1}{\mu}\big(u(a_1, \cdot) - \boldsymbol{h}^*_{A_1,D_1}(a_1)\big) \in \mathcal{PSH}((D_2)_\mu).$$

Then $v \le 1$ and $v \le 0$ on A_2. Therefore, by Proposition 3.2.27,

$$v \le \boldsymbol{h}^*_{A_2,(D_2)_\mu} = \frac{1}{\mu}\boldsymbol{h}^*_{A_2,D_2} \ \text{ on } (D_2)_\mu.$$

Consequently, $u(a_1, a_2) \le \boldsymbol{h}^*_{A_1,D_1}(a_1) + \boldsymbol{h}^*_{A_2,D_2}(a_2)$, which finishes the proof for $N = 2$.

Now assume that the formula is true for $N - 1 \ge 2$. Put

$$\widehat{Y} := \Big\{(z_1, \dots, z_{N-1}) \in D_1 \times \cdots \times D_{N-1} : \textstyle\sum_{j=1}^{N-1} \boldsymbol{h}^*_{A_j,D_j}(z_j) < 1\Big\}.$$

By the inductive hypothesis, we conclude that

$$\boldsymbol{h}^*_{A_1\times\cdots\times A_{N-1},\widehat{Y}}(z') = \sum_{j=1}^{N-1} \boldsymbol{h}^*_{A_j,D_j}(z_j), \quad z' = (z_1,\dots,z_{N-1}) \in \widehat{Y}.$$

Now we apply the case $N = 2$ to the following situation:

$$\widehat{Z} := \{(z', z_N) \in \widehat{Y} \times D_N : \boldsymbol{h}^*_{A_1\times\cdots\times A_{N-1},\widehat{Y}}(z') + \boldsymbol{h}^*_{A_N,D_N}(z_N) < 1\}.$$

So

$$\boldsymbol{h}^*_{A_1\times\cdots\times A_{N-1},\widehat{Y}}(z') + \boldsymbol{h}^*_{A_N,D_N}(z_N) = \boldsymbol{h}^*_{A_1\times\cdots\times A_N,\widehat{Z}}(z), \quad z = (z', z_N) \in \widehat{Z}.$$

It remains to observe that $\widehat{Z} = \widehat{X}$. □

Finally, we present (without proof) a few more developed results which will be important in the sequel.

Theorem* 3.2.29 ([Bed-Tay 1976], [Bed 1981], [Kli 1991]). *Let $X \in \mathfrak{R}_\infty(\mathbb{C}^n)$. There exists a **Monge–Ampère operator***

$$\mathcal{PSH}(X) \cap L^\infty(X, \mathrm{loc}) \ni u \mapsto (dd^c u)^n \in \mathfrak{M}(X),$$

where $\mathfrak{M}(X)$ denotes the space of all non-negative Borel measure on X such that

(a) *if $u \in \mathcal{PSH}(X) \cap \mathcal{C}^2(X, \mathbb{R})$, then $(dd^c u)^n = 4^n n! \det\big[\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\big]_{j,k=1,\dots,n} \mathcal{L}^X$;*

(b) *if $\mathcal{PSH}(X) \cap L^\infty(X, \mathrm{loc}) \ni u_\nu \searrow u \in \mathcal{PSH}(X) \cap L^\infty(X, \mathrm{loc})$, then we get $(dd^c u_\nu)^n \to (dd^c u)^n$ in the weak sense.*

Definition 3.2.30. The measure $\mu_{A,X} := (dd^c \boldsymbol{h}^*_{A,X})^n$ is called the *equilibrium measure for A*.

Theorem* 3.2.31 ([Bed 1981], [Zer 1986], [Kli 1991], [Ale-Zer 2001]). *Let $Y \in \mathfrak{R}(\mathbb{C}^n)$, $X \subset\subset Y$ be hyperconvex (cf. Definition 2.5.1), and let $K \subset\subset X$ be compact. Then*

(a) $\lim_{z\to z_0} \boldsymbol{h}_{K,X}(z) = 1$ *for every $z_0 \in \partial X$.*

(b) $\mu_{K,X}(X \setminus K) = 0$.

(c) *If $P \subset K$ is such that $\mu_{K,X}(P) = 0$, then $\boldsymbol{h}^*_{K\setminus P,X} \equiv \boldsymbol{h}^*_{K,X}$.*

(d) *If $K = \bigcup_{i\in I} \overline{\widehat{\mathbb{P}}(a_i, r_i)}$, where $0 < r_i < d_X(a_i)$, $i \in I$, then $\boldsymbol{h}_{K,X} = \boldsymbol{h}^*_{K,X}$ is continuous.*

Theorem* 3.2.32 (Domination principle; [Bed-Tay 1982], [Kli 1991]). *Let $Y \in \mathfrak{R}(\mathbb{C}^n)$, $X \subset\subset Y$ be open, and let $u_+, u_- \in PSH(X) \cap L^\infty(X)$ be such that*

$$(dd^c u_+)^n \le (dd^c u_-)^n \text{ in } X \text{ and } \liminf_{X\ni z\to z_0} (u_+(z) - u_-(z)) \ge 0 \text{ for all } z_0 \in \partial X.$$

Then $u_+ \ge u_-$ on X.

Corollary 3.2.33. *Let $Y \in \mathfrak{R}(\mathbb{C}^n)$, $X \subset\subset Y$ be hyperconvex, $K \subset\subset X$ be compact, $U \subset X \setminus K$ be open, and $u \in \mathcal{PSH}(U) \cap L^\infty(U)$, $u \le 1$, be such that*

$$\liminf_{U \ni z \to z_0} (\boldsymbol{h}^*_{K,X}(z) - u(z)) \ge 0 \textit{ for all } z_0 \in (\partial U) \cap X.$$

Then $u \le \boldsymbol{h}^*_{K,X}$ in U.

Proof. We take $u_+ := h^*_{K,X}$, $u_- := u$. By Theorem 3.2.31 (b) we have $(dd^c u_-)^n = 0$ on $X \setminus K$. In particular, $(dd^c u_+)^n \le (dd^c u_-)^n$ in U. Moreover, by Theorem 3.2.31 (a), $\lim_{z \to z_0} u_+(z) = 1$ for every $z_0 \in \partial X$. Now we may apply Theorem 3.2.32 for U. □

3.3 Balanced case

Proposition 3.3.1. *Let $h\colon \mathbb{C}^n \to \mathbb{R}_+$ be a plurisubharmonic function with $h(\lambda z) = |\lambda| h(z)$, $\lambda \in \mathbb{C}$, $z \in \mathbb{C}^n$ (cf. Proposition 2.3.18 (b)), e.g. a complex seminorm (cf. Proposition 2.3.18 (a)). Put*

$$B(r) := \{z \in \mathbb{C}^n : h(z) < r\}, \quad B[r] := \{z \in \mathbb{C}^n : h(z) \le r\}.$$

Then for arbitrary $0 < r < R$ we have

$$\boldsymbol{h}^*_{B(r),B(R)} = \boldsymbol{h}_{B(r),B(R)} = \boldsymbol{h}^*_{B[r],B(R)} = \boldsymbol{h}_{B[r],B(R)} = \max\left\{0, \frac{\log \frac{h}{r}}{\log \frac{R}{r}}\right\}.$$

Proof. It is clear that

$$\boldsymbol{h}^*_{B(r),B(R)} = \boldsymbol{h}_{B(r),B(R)} \ge \boldsymbol{h}^*_{B[r],B(R)} \ge \boldsymbol{h}_{B[r],B(R)} \ge \max\left\{0, \frac{\log \frac{h}{r}}{\log \frac{R}{r}}\right\} =: \Phi.$$

Moreover, $\boldsymbol{h}^*_{B(r),B(R)} = \Phi = 0$ on $B[r]$ – use Oka's theorem: if $h(a) = r$, then $\boldsymbol{h}^*_{B(r),B(R)}(a) = \lim_{t \to 1-} \boldsymbol{h}^*_{B(r),B(R)}(ta) = 0$ (cf. [Vla 1966], § II.9.16). Take an $a \in B(R) \setminus B[r]$ and let

$$A := \mathbb{A}(r/h(a), R/h(a)) \ni \lambda \overset{v}{\mapsto} \boldsymbol{h}^*_{B(r),B(R)}(\lambda a) - \Phi(\lambda a).$$

Observe that v is subharmonic and $v \ge 0$. Moreover, $\limsup_{\lambda \to \partial A} v(\lambda) \le 0$. Thus, by the maximum principle, $v \le 0$. In particular, $v(1) = \boldsymbol{h}^*_{B(r),B(R)}(a) - \Phi(a) \le 0$. □

3.4 Reinhardt case

Proposition 3.4.1. *Let $\varnothing \neq A \subset D$, where D is a Reinhardt domain and A is a locally pluriregular Reinhardt set. Then*

$$\boldsymbol{h}^*_{A,D}(z) = \Phi_{\log A, \log D}(\log|z_1|, \dots, \log|z_n|), \quad z = (z_1, \dots, z_n) \in D \setminus V_0$$

(*cf. Definitions* 3.1.1, 3.2.1). *In particular, the function $\boldsymbol{h}^*_{A,D}$ is continuous on $D \setminus V_0$.*

Notice that in the case where D is log-convex and $A \subset\subset D$ is a domain the result had been proved (using different methods) in [Thor 1989].

[?] We do not know how to characterize $\boldsymbol{h}^*_{A,D}$ in the case where A is not Reinhardt [?]

Proof. Since A and D are invariant under rotations, we easily conclude that $\boldsymbol{h}^*_{A,D}(z) = \boldsymbol{h}^*_{A,D}(|z_1|, \dots, |z_n|)$, $z = (z_1, \dots, z_n) \in D$. Thus, by Proposition 2.9.5, $\boldsymbol{h}^*_{A,D}(z) = \varphi(\log|z_1|, \dots, \log|z_n|)$, $z = (z_1, \dots, z_n) \in D \setminus V_0$, where $\varphi \in \mathcal{CVX}(\log D)$. Since $\boldsymbol{h}^*_{A,D} = 0$ on A, we get $\varphi = 0$ on $\log A$. Finally, $\varphi \le \Phi_{\log A, \log D}$.

To prove the opposite inequality, observe that by Proposition 2.9.5, the function

$$D \setminus V_0 \ni z \overset{u}{\mapsto} \Phi_{\log A, \log D}(\log|z_1|, \dots, \log|z_n|)$$

is plurisubharmonic, $u \le 1$, and $u = 0$ on $A \setminus V_0$. By Proposition 2.3.29 (a), u extends to a $\tilde{u} \in \mathcal{PSH}(D)$. Clearly, $\tilde{u} \le 1$ (Proposition 2.3.29 (b)). Thus $\tilde{u} \le \boldsymbol{h}^*_{A \setminus V_0, D} = \boldsymbol{h}^*_{(A \setminus V_0)^*, D} = \boldsymbol{h}^*_{A,D}$ (cf. Remark 3.2.9 and Corollary 3.2.13). □

Remark 3.4.2. Let $U \subset \mathbb{R}^n$ be an *arbitrary domain*, let $\varphi\colon U \to \mathbb{R}$, and let $V := \operatorname{conv}(U)$. For $x \in V$ define

$$\psi(x) = \inf\{t_1\varphi(a_1) + \dots + t_k\varphi(a_k) : x = t_1a_1 + \dots + t_ka_k,\ a_1, \dots, a_k \in U,\\ t_1, \dots, t_k \in [0,1],\ t_1 + \dots + t_k = 1\}.$$

Observe that $\psi \in \mathcal{CVX}(V)$, $\psi \le \varphi$ on U, and $\sup_V \psi \le \sup_U \varphi$. One can easily prove (Exercise) that the following conditions are equivalent:

- $\psi = \varphi$ on U,
- φ has an extension to a convex function on V,
- $\varphi(t_1a_1 + \dots + t_ka_k) \le t_1\varphi(a_1) + \dots + t_k\varphi(a_k)$ for any $a_1, \dots, a_k \in U$ and $t_1, \dots, t_k \in [0,1]$ with $t_1 + \dots + t_k = 1$ such that $t_1a_1 + \dots + t_ka_k \in U$.

Corollary 3.4.3. *Let $\varnothing \neq A \subset D$, where D is a Reinhardt domain and A is a locally pluriregular Reinhardt set. Let $\widehat{D}$ be the envelope of holomorphy of D (cf. Theorem* 2.9.3). *Then $\boldsymbol{h}^*_{A,\widehat{D}} = \boldsymbol{h}^*_{A,D}$ on D iff $\Phi_{\log A, \log D}$ extends to a convex function on* $\log \widehat{D}$.

3.5 An example

The aim of this section is to get an effective formula for $\boldsymbol{h}^*_{A,D}$ for certain Reinhardt domains $D \subset \mathbb{C}^2$ and open Reinhardt sets $\varnothing \neq A \subset D$. Obviously in this case we have $\boldsymbol{h}^*_{A,D} \equiv \boldsymbol{h}_{A,D}$. Let $\varphi \in \mathcal{CVX}(\log D)$ be such that $\boldsymbol{h}_{A,D}(z) = \varphi(\log|z_1|, \log|z_2|)$, $z = (z_1, z_2) \in D \setminus V_0$ (Proposition 3.4.1). Recall that $\boldsymbol{h}_{A,D}$ is continuous on $D \setminus V_0$. It is clear that $\boldsymbol{h}_{A,D} \geq \boldsymbol{h}_{A,\widehat{D}}$ on D, where $\widehat{D}$ denotes the envelope of holomorphy of D. We are interested in

- explicit formulas for $\boldsymbol{h}_{A,D}$,
- relations between $\boldsymbol{h}_{A,D}$ and $\boldsymbol{h}_{A,\widehat{D}}$ (cf. Corollary 3.4.3).

First, to get an intuition of how the problem may be difficult, let us recall a result from [Jar-Pfl 1993] (Example 4.2.8) related to the *Green function* $\boldsymbol{g}_D$ defined as

$$\boldsymbol{g}_D(a,z) := \sup\Big\{u : u\colon D \to [0,1),\ \log u \in \mathcal{PSH}(D),\ \sup_{w \in D \setminus \{a\}} \frac{u(w)}{\|w - a\|} < +\infty\Big\}, \quad a, z \in D.$$

Given $0 < \alpha, \beta < 1$, we define Reinhardt domains:

$$D_- := \{(z_1, z_2) \in \mathbb{D}^2 : |z_2| \geq \beta \implies |z_1| > \alpha\} = \mathbb{D}^2 \setminus (\overline{\mathbb{D}}(\alpha) \times \overline{\mathbb{A}}(\beta, 1)),$$
$$D_+ := \{(z_1, z_2) \in \mathbb{D}^2 : |z_1| < \alpha \text{ or } |z_2| < \beta\} = \mathbb{D}^2 \setminus (\overline{\mathbb{A}}(\alpha, 1) \times \overline{\mathbb{A}}(\beta, 1));$$

$D_\pm$ is not a domain of holomorphy, $\widehat{D}_- = \mathbb{D}^2$,

$$\widehat{D}_+ = \{(z_1, z_2) \in \mathbb{D}^2 : |z_1|^{-\log\beta} |z_2|^{-\log\alpha} < e^{-\log\alpha \log\beta}\}.$$

We have the following effective formulas for $\boldsymbol{g}_{D_\pm}(0, \cdot)$:
(a) if $\beta \geq \alpha$, then

$$\boldsymbol{g}_{D_-}(0,z) = \max\{|z_1|, |z_2|\} = \boldsymbol{g}_{\widehat{D}_-}(0,z), \quad z \in D_-;$$

(b) if $\beta < \alpha$, then

$$\boldsymbol{g}_{D_-}(0,z) = \left\{\begin{matrix} \max\{|z_1|, \frac{\alpha}{\beta}|z_2|\}, & \text{if } |z_2| < \beta \\ \max\{|z_1|, |z_2|^{\frac{\log\alpha}{\log\beta}}\}, & \text{if } |z_1| > \alpha \end{matrix}\right\} \mathrel{\substack{\geq \\ \not\equiv}} \boldsymbol{g}_{\widehat{D}_-}(0,z), \quad z \in D_-;$$

(c)

$$\boldsymbol{g}_{D_+}(0,z) = \max\left\{|z_1|, |z_2|, \left(e^{\log\alpha\log\beta}|z_1|^{-\log\beta}|z_2|^{-\log\alpha}\right)^{-\frac{1}{\log\alpha\beta}}\right\} = \boldsymbol{g}_{\widehat{D}_+}(0,z), \quad z \in D_+.$$

The Green function $\boldsymbol{g}_D$ may be considered as a "prototype" of the relative extremal function $\boldsymbol{h}_{A,D}$. Consider $A := \mathbb{D}(\varepsilon) \times \mathbb{D}(\delta)$, $0 < \varepsilon, \delta < 1$. We are going to determine $\boldsymbol{h}_{A,D}$ with $D \in \{D_-, D_+\}$ and $A \subset D$. Define the auxiliary function

$$H_r(\lambda) := \boldsymbol{h}_{\mathbb{D}(r),\mathbb{D}}(\lambda) = \max\left\{0,\ 1 - \frac{\log|\lambda|}{\log r}\right\}, \quad 0 < r < 1,\ \lambda \in \mathbb{D}.$$

We consider the following five cases:

(A) (Figure 3.5.1) If $\frac{\log\delta}{\log\varepsilon} \geq \frac{\log\beta}{\log\alpha}$, then

$$\boldsymbol{h}_{A,D_-}(z) = \max\{H_\varepsilon(z_1), H_\delta(z_2)\} = \boldsymbol{h}_{A,\widehat{D}_-}(z), \quad z \in D_-.$$

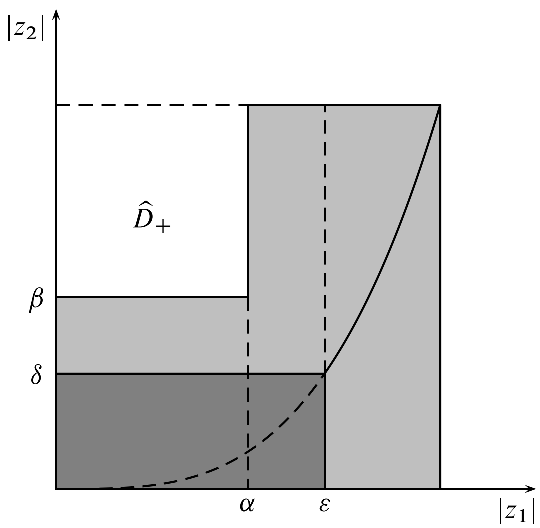


Figure 3.5.1

Proof. The formula for $\boldsymbol{h}_{A,\widehat{D}_-}$ follows directly from the product property of the relative extremal function (Proposition 3.2.17). Namely,

$$\begin{aligned}\boldsymbol{h}_{A,\widehat{D}_-}(z) = \boldsymbol{h}_{A,\mathbb{D}^2}(z) &= \boldsymbol{h}_{\mathbb{D}(\varepsilon)\times\mathbb{D}(\delta),\,\mathbb{D}\times\mathbb{D}}(z_1, z_2)\\ &= \max\{\boldsymbol{h}_{\mathbb{D}(\varepsilon),\mathbb{D}}(z_1), \boldsymbol{h}_{\mathbb{D}(\delta),\mathbb{D}}(z_2)\}\\ &= \max\{H_\varepsilon(z_1), H_\delta(z_2)\}, \quad z = (z_1, z_2) \in \mathbb{D}^2.\end{aligned}$$

Obviously, $\boldsymbol{h}_{A,D_-}(z_1, 0) \leq \boldsymbol{h}_{\mathbb{D}(\varepsilon),\mathbb{D}}(z_1) = H_\varepsilon(z_1)$, $z_1 \in \mathbb{D}$. For $r, s \in \mathbb{N}$ with $\frac{s}{r} \geq \frac{\log\delta}{\log\varepsilon}$, consider the subharmonic function

$$\mathbb{D} \ni \lambda \overset{u}{\mapsto} \boldsymbol{h}_{A,D_-}(\lambda^r, \lambda^s).$$

Then

$$u(\lambda) \le \boldsymbol{h}_{\mathbb{D}(\varepsilon^{1/r}),\,\mathbb{D}}(\lambda) = H_{\varepsilon^{1/r}}(\lambda) = H_\varepsilon(\lambda^r), \quad \lambda \in \mathbb{D}.$$

This implies (using the continuity of $\boldsymbol{h}_{A,D_-}$ on $D_- \setminus \boldsymbol{V}_0$) that

$$\boldsymbol{h}_{A,D_-}(z) \le H_\varepsilon(z_1), \quad |z_2| \le |z_1|^{\frac{\log\delta}{\log\varepsilon}}.$$

In particular, using the convexity of the function $\varphi(\log|z_1|,\cdot)$, we get

$$\varphi(\log|z_1|,t) \le 1 - \frac{t}{\log\delta}$$

for (z_1,t) such that $\big(\max\{\alpha,\varepsilon\} < |z_1| < 1,\ \frac{\log\delta}{\log\varepsilon}\log|z_1| \le t < 0\big)$ or $(\alpha < |z_1| \le \varepsilon,\ \log\delta \le t < 0$ (provided that $\alpha < \varepsilon$)). Finally, if $\delta < \beta$ then the maximum principle, applied to the subharmonic function $\boldsymbol{h}_{A,D_-}(\cdot, z_2)$, gives

$$\boldsymbol{h}_{A,D_-}(z_1,z_2) \le H_\delta(z_2), \quad |z_1| \le \alpha. \qquad \square$$

(B) (Figure 3.5.2) If $\frac{\log\delta}{\log\varepsilon} < \frac{\log\beta}{\log\alpha}$ (in particular, $\varepsilon < \alpha$), then

$$\begin{aligned}\boldsymbol{h}_{A,D_-}(z) &= \left\{\begin{array}{ll} H_\varepsilon(z_1) & \text{if } |z_2| < \beta = \delta, \\ \max\left\{H_\varepsilon(z_1),\ -\dfrac{\frac{\log\alpha/\varepsilon}{\log\beta/\delta}\log\frac{|z_2|}{\delta}}{\log\varepsilon}\right\} & \text{if } |z_2| < \beta > \delta, \\ \max\left\{H_\varepsilon(z_1),\ 1 - \dfrac{\frac{\log\alpha}{\log\beta}\log|z_2|}{\log\varepsilon}\right\} & \text{if } |z_1| > \alpha \end{array}\right\} \\ &\mathrel{\substack{\ge\\ \not\equiv}} \boldsymbol{h}_{A,\hat{D}_-}(z), \quad z \in D_-.\end{aligned}$$

Proof. First observe that the function R defined by the right-hand side of the above formula is well defined, $R \in \mathcal{PSH}(D_-)$, $0 \le R \le 1$, $R = 0$ on A (Exercise). Thus $\boldsymbol{h}_{A,D_-} \ge R$. Using the same methods as in (A), one may easily prove (Exercise) that

$$\boldsymbol{h}_{A,D_-}(z) \le \begin{cases} H_\varepsilon(z_1), & |z_2| \le |z_1|^{\frac{\log\beta}{\log\alpha}}, \\ 1 - \dfrac{\frac{\log\alpha}{\log\beta}\log|z_2|}{\log\varepsilon}, & \alpha < |z_1| < 1,\ |z_2| \ge |z_1|^{\frac{\log\beta}{\log\alpha}}. \end{cases} \tag{3.5.1}$$

This finishes the proof in the case where $\delta = \beta$. If $\delta < \beta$, then put

$$\psi(t) := -\frac{\frac{\log\alpha/\varepsilon}{\log\beta/\delta}(t_2 - \log\delta)}{\log\varepsilon}, \quad t = (t_1,t_2) \in \mathbb{R}^2.$$

Then $\psi(t_1,\log\delta) = 0$ and $\psi(\log\alpha,\log\beta) = 1 - \frac{\log\alpha}{\log\varepsilon}$. Consequently, using the convexity of φ and (3.5.1), we get

$$\boldsymbol{h}_{A,D_-}(z) \le -\frac{\frac{\log\alpha/\varepsilon}{\log\beta/\delta}\log\frac{|z_2|}{\delta}}{\log\varepsilon}, \quad \varepsilon \le |z_1| \le \alpha,\ \delta \le |z_2| \le \delta\Big(\frac{|z_1|}{\varepsilon}\Big)^{\frac{\log\beta/\delta}{\log\alpha/\epsilon}}.$$

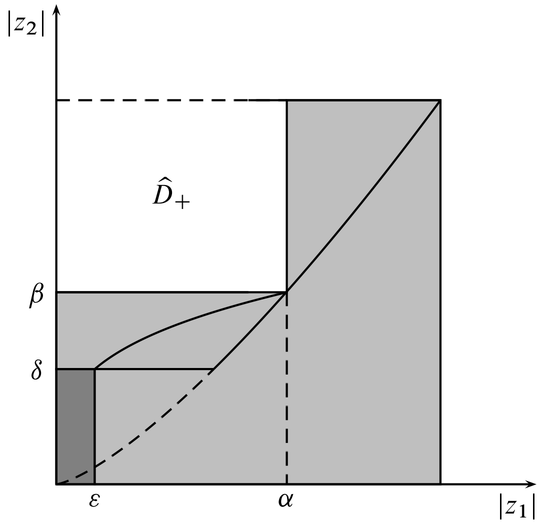


Figure 3.5.2

The maximum principle applied to the subharmonic function $\boldsymbol{h}_{A,D_-}(\cdot, z_2)$, implies that

$$\boldsymbol{h}_{A,D_-}(z) \le -\frac{\frac{\log \alpha/\varepsilon}{\log \beta/\delta} \log \frac{|z_2|}{\delta}}{\log \varepsilon}, \quad |z_1| \le \varepsilon\left(\frac{|z_2|}{\delta}\right)^{\frac{\log \alpha/\varepsilon}{\log \beta/\delta}}, \ \delta < |z_2| < \beta. \qquad \square$$

(C) (Figure 3.5.3) If $\varepsilon \le \alpha$ and $\delta \le \beta$, $(\varepsilon, \delta) \ne (\alpha, \beta)$, then

$$\boldsymbol{h}_{A,D_+}(z) = \max\left\{H_\varepsilon(z_1),\ H_\delta(z_2),\ 1 - \frac{\frac{\log|z_1|}{\log \alpha} + \frac{\log|z_2|}{\log \beta} - 1}{\frac{\log \varepsilon}{\log \alpha} + \frac{\log \delta}{\log \beta} - 1}\right\} = \boldsymbol{h}_{A,\widehat{D}_+}(z),$$
$$z \in D_+.$$

Proof. We may assume that $\varepsilon < \alpha$ and $\delta < \beta$ (cf. Proposition 3.2.23). The function R defined by the right-hand side of the above formula is well defined on $\widehat{D}_+$, $R \in \mathcal{PSH}(\widehat{D}_+)$, $0 \le R \le 1$, $R = 0$ on A (Exercise). Thus $\boldsymbol{h}_{A,D_-} \ge \boldsymbol{h}_{A,\widehat{D}_+} \ge R$.

Using the function

$$\mathbb{D} \ni \lambda \mapsto \boldsymbol{h}_{A,D_+}(\lambda^r, \beta\lambda^s)$$

with $r, s \in \mathbb{N}$, $\frac{s}{r} \ge \frac{\log \delta/\beta}{\log \varepsilon}$, and arguing as in (A), we get

$$\boldsymbol{h}_{A,D_+}(z) \le H_\varepsilon(z_1), \quad |z_2| \le \beta|z_1|^{\frac{\log \delta/\beta}{\log \varepsilon}}.$$

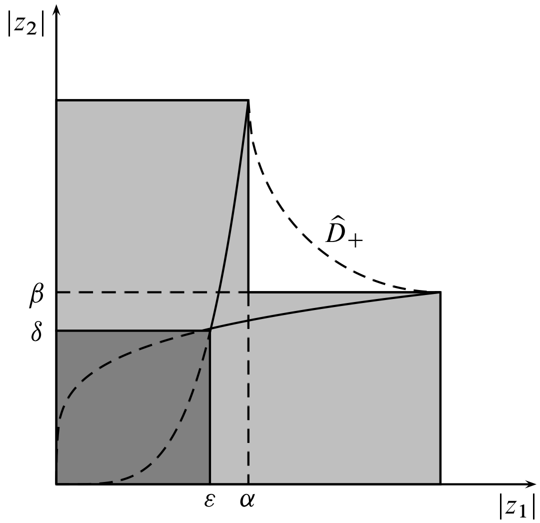


Figure 3.5.3

In the same way one shows (EXERCISE) that

$$\boldsymbol{h}_{A,D_+}(z) \le H_\delta(z_2), \quad |z_1| \le \alpha |z_2|^{\frac{\log \varepsilon/\alpha}{\log \delta}}.$$

Put

$$\psi(t) := 1 - \frac{\frac{t_1}{\log \alpha} + \frac{t_2}{\log \beta} - 1}{\frac{\log \varepsilon}{\log \alpha} + \frac{\log \delta}{\log \beta} - 1}, \quad t = (t_1, t_2) \in \mathbb{R}^2.$$

Observe that

$$\psi(t, \log \beta + \frac{\log \delta/\beta}{\log \varepsilon} t) = 1 - \frac{t}{\log \varepsilon}, \quad \log \varepsilon \le t < 0,$$
$$\psi(\log \alpha + \frac{\log \varepsilon/\alpha}{\log \delta} t, t) = 1 - \frac{t}{\log \delta}, \quad \log \delta \le t < 0.$$

Hence, using the convexity of φ, we conclude (EXERCISE) that

$$\boldsymbol{h}_{A,D_+}(z) \le 1 - \frac{\frac{\log |z_1|}{\log \alpha} + \frac{\log |z_2|}{\log \beta} - 1}{\frac{\log \varepsilon}{\log \alpha} + \frac{\log \delta}{\log \beta} - 1}$$

for $(z_1, z_2) \in D_+$ with $|z_1| \ge \varepsilon$, $|z_2| \ge \delta$, $|z_1| \ge \alpha |z_2|^{\frac{\log \varepsilon/\alpha}{\log \delta}}$, $|z_2| \ge \beta |z_1|^{\frac{\log \delta/\beta}{\log \varepsilon}}$.
□

(D) (Figure 3.5.4) Assume that $\varepsilon > \alpha$ and $\delta < \beta$. Define

$$T_1 := \Big\{(z_1, z_2) : \varepsilon \le |z_1| < 1, \delta \le |z_2| \le \beta |z_1|^{\frac{\log \delta/\beta}{\log \varepsilon}}\Big\},$$

$$T_2 := \Big\{(z_1, z_2) : \alpha \le |z_1| \le \varepsilon,\ \delta \le |z_2| \le \delta(|z_1|/\varepsilon)^{\frac{\log \delta/\beta}{\log \varepsilon/\alpha}}\Big\},$$

$$T_3 := \Big\{(z_1, z_2) : \alpha \le |z_1| < 1,\ \max\{\beta |z_1|^{\frac{\log \delta/\beta}{\log \varepsilon}},\ \delta(|z_1|/\varepsilon)^{\frac{\log \delta/\beta}{\log \varepsilon/\alpha}}\} \le |z_2| < \beta\}\Big\},$$

$$A := -\frac{\log \beta}{\log \alpha \log \delta},\ B := \frac{1 + A \log \varepsilon}{\log \beta/\delta},\ C := 1 - B \log \beta.$$

Then

$$\boldsymbol{h}_{A,D_+}(z) = \begin{cases} \max\big\{H_\varepsilon(z_1),\ H_\delta(z_2)\big\} & \text{if } |z_1| < \alpha \text{ or } |z_2| \le \delta \\ & \text{or } (z_1, z_2) \in T_1 \cup T_2, \\ A \log |z_1| + B \log |z_2| + C & \text{if } (z_1, z_2) \in T_3, \end{cases}$$

$$\gneqq \boldsymbol{h}_{A,\widehat{D}_+}(z), \quad z \in D_+.$$

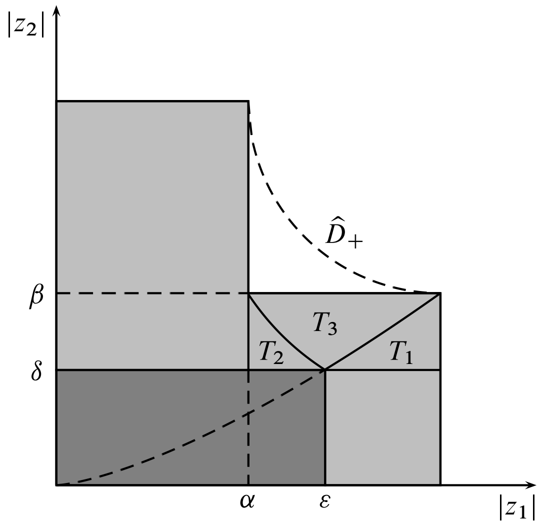


Figure 3.5.4

Proof. The function R defined by the right-hand side of the above formula is well defined, $R \in \mathcal{PSH}(D_+)$, $0 \le R \le 1$, $R = 0$ on A (EXERCISE). Thus $\boldsymbol{h}_{A,D_+} \ge R$. Using the standard methods, one may easily prove (EXERCISE) that

$$\boldsymbol{h}_{A,D_+}(z) \le \max\big\{H_\varepsilon(z_1),\ H_\delta(z_2)\big\}, \quad |z_1| < \alpha \text{ or } |z_2| \le \delta \text{ or } (z_1, z_2) \in T_1.$$

In $T_2 \cup T_3$ the formula follows from convexity arguments (EXERCISE). □

(E) If $\varepsilon \geq \alpha$ and $\delta = \beta$, then

$$\boldsymbol{h}_{A,D_+}(z) = \max\{H_\varepsilon(z_1),\ H_\delta(z_2)\} = \boldsymbol{h}_{A,\mathbb{D}^2}(z), \quad z \in D_+.$$

Proof. EXERCISE. □

Exercise 3.5.1 (Figure 3.5.5). Find a formula for $\boldsymbol{h}_{A,D}$, where $A := \mathbb{D}(\varepsilon) \times \mathbb{D}(\delta)$,

$$D := \mathbb{D}^2 \setminus \{(z_1, z_2) \in \mathbb{D}^2 : \alpha_1 \leq |z_1| \leq \alpha_2,\ \beta_1 \leq |z_2| \leq \beta_2\},$$

with $0 < \varepsilon, \delta < 1$, $0 \leq \alpha_1 < \alpha_2 \leq 1$, $0 \leq \beta_1 < \beta_2 \leq 1$, $A \subset D$.

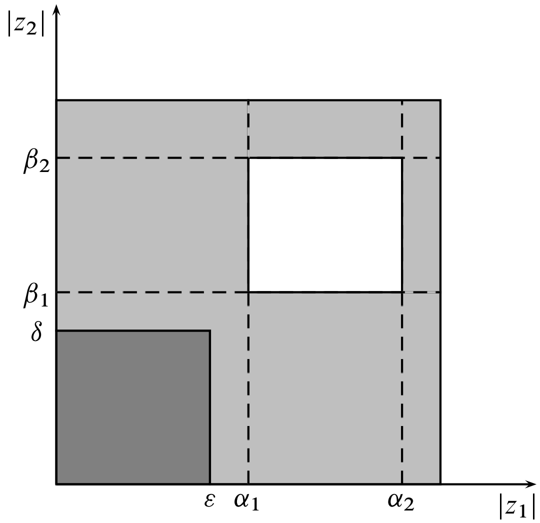


Figure 3.5.5

3.6 Plurithin sets

> § 3.2.

Let Ω be a Riemann region over $\mathbb{C}^n$.

Definition 3.6.1. We say that a set $A \subset \Omega$ is *plurithin at a point* $a \in \Omega$ if either $a \notin \overline{A}$ or $a \in \overline{A}$ and $\limsup_{A\setminus\{a\}\ni z \to a} u(z) < u(a)$ for some function u plurisubharmonic in a neighborhood of a.

To avoid a collision with Definition 2.3.20 we will use the notion *plurithin* also for $n = 1$.

Remark 3.6.2. (a) ([Kli 1991], Corollary 4.8.4) If A, B are plurithin at a, then $A \cup B$ is plurithin at a.

(b) ([Arm-Gar 2001], Theorem 7.2.2) Every polar set $P \subset \mathbb{C}$ is plurithin at any point $a \in \mathbb{C}$.

(c) If $A \subset \mathbb{C}$ is not plurithin at a point $a \in \overline{A}$, then for any polar set $P \subset \mathbb{C}$, the set $A \setminus P$ is not thin at a ((c) follows directly from (a) and (b)).

(d) If $A \subset \Omega$ is locally pluriregular at a point $a \in \overline{A}$, then A is not plurithin at a. If $A \subset \mathbb{C}$ is not plurithin at a point $a \in \overline{A}$, then A is locally regular at a.

Indeed, suppose that $A \subset \Omega$ is locally pluriregular at a and

$$\limsup_{A \setminus \{a\} \ni z \to a} u(z) < c < u(a)$$

for some $u \in \mathcal{PSH}(V)$, where V is an open neighborhood of a. We may assume that $u \le 0$ on V. Take an open neighborhood $U \subset V$ of a such that $u < c$ on $(A \setminus \{a\}) \cap U$. Put $v := \frac{u}{-c} + 1$. Then $v \le 1$ on U and $v \le 0$ on $(A \setminus \{a\}) \cap U$. Hence $v \le \boldsymbol{h}^*_{(A \setminus \{a\}) \cap U, U} = \boldsymbol{h}^*_{A \cap U, U}$ on U. In particular, $0 < v(a) = \frac{u(a)}{-c} + 1 \le 0$; a contradiction.

Now suppose that $A \subset \mathbb{C}$ is not plurithin at a and $\boldsymbol{h}^*_{A \cap U, U}(a) > 0$ for some neighborhood U of a. Let $P \subset U$ be a polar set such that $\boldsymbol{h}^*_{A \cap U, U} = \boldsymbol{h}_{A \cap U, U}$ on $U \setminus P$. In particular, $\boldsymbol{h}^*_{A \cap U, U} = 0$ on $A \setminus P$. By (c), the set $A \setminus P$ is not plurithin at a. Hence $0 < \boldsymbol{h}^*_{A \cap U, U}(a) = \limsup_{A \setminus P \ni z \to a} \boldsymbol{h}^*_{A \cap U, U}(z) = 0$; a contradiction.

(e) ([Arm-Gar 2001], Theorem 7.3.9) If $A \subset \mathbb{C}$ is plurithin at a point $a \in \overline{A}$, then there is a sequence $r_k \searrow 0$ such that $\{z \in A : |z - a| = r_k\} = \varnothing$, $k = 1, 2, \dots$ (cf. Proposition 2.3.21).

3.7 Relative boundary extremal function

$\boxed{>}$ § 3.2.

While in Section 3.2 the “thickness” of a subset A of a domain D has been studied via all bounded psh functions, we turn here to discuss boundary sets $A \subset \partial D$, see [Pfl-NVA 2004], [NVA 2005], [NVA 2008], [NVA 2009].

Definition 3.7.1. Let $D \subset \mathbb{C}^n$ be a domain and $A \subset \partial D$. A *system of approach regions for* (A, D) is given by a family $\mathfrak{A} = \big((\mathcal{A}_\alpha(a))_{\alpha \in I_a}\big)_{a \in A}$ (I_a a non-empty index set depending on a) of open subsets $\mathcal{A}_\alpha(a)$ of D with $a \in \overline{\mathcal{A}_\alpha(a)}$, $a \in A$ and $\alpha \in I_a$.

Example 3.7.2. (a) Let D, A be as above. Denote by $\big(U_\alpha(a)\big)_{\alpha \in I_a}$ a neighborhood basis of $a \in A$. Put $\mathcal{K}_\alpha(a) := D \cap U_\alpha(a)$. Then

$$\mathfrak{K} = \mathfrak{K}_{A,D} := \big((\mathcal{K}_\alpha(a))_{\alpha \in I_a}\big)_{a \in A}$$

is a system of approach regions for (A, D). It is called a *canonical system of approach regions for* (A, D).

(b) Let $D = \mathbb{D}$ and $A \subset \mathbb{T}$. Fix an $a \in A$. Put

$$\mathcal{S}_\alpha(a) := \left\{ z \in \mathbb{D} : \left| \operatorname{Arg}\left(\frac{a-z}{z}\right)\right| < \alpha \right\}, \quad 0 < \alpha < \pi/2.$$

Then $\mathfrak{S} = \mathfrak{S}_{A,\mathbb{D}} := \big((\mathcal{S}_\alpha(a))_{\alpha\in I_a}\big)_{a\in A}$ with $I_a = (0, \pi/2)$ is a system of approach regions for $(A, \mathbb{D})$. It is called the system of *Stolz regions* or *non-tangential approach regions*.

Let us fix D, A and $\mathfrak{A}$ as above. If $u\colon D \to \overline{\mathbb{R}}$, then we put

$$u^{*,\mathfrak{A}}(a) := \begin{cases} \limsup\limits_{D\ni z\to a} u(z) & \text{if } a \in \overline{D} \setminus A, \\ \sup_{\alpha\in I_a} \limsup\limits_{\mathcal{A}_\alpha(a)\ni z\to a} u(z) & \text{if } a \in A, \end{cases} \qquad a \in \overline{D}.$$

Note that $u^{*,\mathfrak{A}}|_D$ is nothing other than the upper semicontinuous regularization u^* of u.

Lemma 3.7.3. $(u^*)^{*,\mathfrak{A}} = u^{*,\mathfrak{A}}$.

Proof. Obviously, $(u^*)^{*,\mathfrak{A}} \ge u^{*,\mathfrak{A}}$ on $\overline{D}$ and $(u^*)^{*,\mathfrak{A}} = u^* = u^{*,\mathfrak{A}}$ on D. Suppose that $(u^*)^{*,\mathfrak{A}}(a) > C > u^{*,\mathfrak{A}}(a)$ for an $a \in \partial D$. Then there exists an $\alpha \in I_a$ such that $\limsup_{\mathcal{A}_\alpha(a)\ni z\to a} u^*(z) > C$. Let $(z_s)_{s=1}^\infty \subset \mathcal{A}_\alpha(a)$ be such that $z_s \to a$ and $u^*(z_s) > C$, $s \in \mathbb{N}$. Consequently, for each $s \in \mathbb{N}$ there exists a $z'_s \in \mathcal{A}_\alpha(a)$ such that $\|z'_s - z_s\| \le 1/s$ and $u(z'_s) > C$. Hence $C > u^{*,\mathfrak{A}}(a) \ge \limsup_{\mathcal{A}_\alpha(a)\ni z\to a} u(z) \ge \limsup_{s\to+\infty} u(z'_s) \ge C$; a contradiction. □

Define

$$\boldsymbol{h}_{\mathfrak{A},A,D} := \sup\{u \in \mathcal{PSH}(D) : u \le 1,\ u^{*,\mathfrak{A}}|_A \le 0\}.$$

With this preparation we have the new relative extremal function, namely:

Definition 3.7.4. Let $D, A, \mathfrak{A}$ be as above. The *relative boundary extremal function for* $(\mathfrak{A}, A, D)$ is defined as $\boldsymbol{h}^*_{\mathfrak{A},A,D} := h^{*,\mathfrak{A}}_{\mathfrak{A},A,D}$ on $\overline{D}$.

Note that $\boldsymbol{h}^*_{\mathfrak{A},A,D}|_D \in \mathcal{PSH}(D)$.

Example 3.7.5. Let $A \subset \mathbb{T}$ be relatively open. Then $\boldsymbol{h}^*_{\mathfrak{K},A,\mathbb{D}}$ is the solution of the Dirichlet problem with boundary values given by $\chi_{\mathbb{T}\setminus A}$. It is continuous on $\overline{\mathbb{D}}\setminus(\overline{A}\setminus A)$, harmonic on $\mathbb{D}$, and $\boldsymbol{h}^*_{\mathfrak{K},A,\mathbb{D}} = 1$ on $\mathbb{T} \setminus \overline{A}$ and $= 0$ on A. $\boldsymbol{h}^*_{\mathfrak{K},A,\mathbb{D}}$ is the so-called *harmonic measure of* $\mathbb{T} \setminus A$. Moreover, one has the following explicit representation (see [Ran 1995], Theorem 4.3.3):

$$\boldsymbol{h}^*_{\mathfrak{K},A,\mathbb{D}}(z) = \frac{1}{2\pi}\int_0^{2\pi} \chi_{\mathbb{T}\setminus A}(e^{it})\frac{1-|z|^2}{|e^{it}-z|^2}\,dt, \quad z \in \mathbb{D}.$$

Lemma 3.7.6. *Let $A \subset \mathbb{T}$ be relatively open and let $u \in \mathcal{SH}(\mathbb{D})$, $u \le 1$, such that, for each $\alpha \in (0, \pi/2)$,*

$$\limsup_{\mathcal{S}_\alpha(a) \ni \zeta \to z} u(\zeta) \le 0 \quad \textit{for almost all } z \in A.$$

Then $u \le \boldsymbol{h}^*_{\mathfrak{K},A,\mathbb{D}}$. In particular, $\boldsymbol{h}^*_{\mathfrak{K},A,\mathbb{D}} = \boldsymbol{h}^*_{\mathfrak{S},A,\mathbb{D}}$.

Proof. Fix a $z \in \mathbb{D}$. Then for $R \in (|z|, 1)$ the mean value inequality gives

$$u(z) \le \frac{1}{2\pi} \int_0^{2\pi} u(Re^{it}) \frac{R^2 - |z|^2}{|Re^{it} - z|^2} dt.$$

Applying Fatou's lemma and the above representation we end up with $u(z) \le \boldsymbol{h}^*_{\mathfrak{K},A,\mathbb{D}}(z)$. □

Similarly as in Section 3.2 we define the notion of local pluriregularity.

Definition 3.7.7. Let $D, A, \mathfrak{A}$ be as above. A is said to be $\mathfrak{A}$*-pluriregular at a point* $a \in \overline{A}$ if $\boldsymbol{h}^*_{\mathfrak{A} \cap U, A \cap U, D \cap U}(a) = 0$ for any open neighborhood U of a.

A is called *locally* $\mathfrak{A}$*-pluriregular*, if $A \neq \varnothing$ and A is $\mathfrak{A}$-pluriregular at each of its points.

Remark 3.7.8. Let $D, A, \mathfrak{A}$ be as above and assume that A is locally $\mathfrak{A}$-pluriregular. For a $\delta \in (0, 1)$ put $\widetilde{A}_\delta := \{z \in D : \boldsymbol{h}^*_{\mathfrak{A},A,D}(z) < \delta\}$. Then $\widetilde{A}_\delta$ is a non-empty open subset of D and so it is locally pluriregular.

(a) Fix $a \in A$, $\alpha \in I(a)$. Then there exists an open neighborhood U of a such that $\mathcal{A}_\alpha(a) \cap U \subset \widetilde{A}_\delta$ (Exercise).

(b) Moreover, we have

$$\boldsymbol{h}^*_{\widetilde{A}_\delta, D} \le \boldsymbol{h}^*_{\mathfrak{A},A,D} \le \boldsymbol{h}^*_{\widetilde{A}_\delta, D} + \delta \quad \text{on } D.$$

Indeed, take a $u \in \mathcal{PSH}(D)$, $u \le 1$ and $u|_{\widetilde{A}_\delta} \le 0$. Now fix an arbitrary point b in A. By the $\mathfrak{A}$-local pluriregularity we know that $\boldsymbol{h}^*_{\mathfrak{A},A,D}(b) = 0$. Hence, for all approach regions $\mathcal{A}_\alpha(b)$, we have $\limsup_{\mathcal{A}_\alpha(b) \ni z \to b} \boldsymbol{h}_{\mathfrak{A},A,D}(z) = 0$. Thus, there exists a neighborhood U of b such that $\boldsymbol{h}^*_{\mathfrak{A},A,D} < \delta$ on $U \cap \mathcal{A}_\alpha(b)$. Hence, $U \cap \mathcal{A}_\alpha(b) \subset \widetilde{A}_\delta$ and so, by assumption, $u \le 0$ on $U \cap \mathcal{A}_\alpha(b)$. Then $\limsup_{\mathcal{A}_\alpha(b) \ni z \to b} u(z) \le 0$. Therefore, $u \le \boldsymbol{h}_{\mathfrak{A},A,D} \le \boldsymbol{h}^*_{\mathfrak{A},A,D}$. Since u was arbitrarily chosen, the left-hand inequality is proved.

To verify the right-hand inequality one starts with a competitor $u \in \mathcal{PSH}(D)$ for the relative boundary extremal function $\boldsymbol{h}^*_{\mathfrak{A},A,D}$. Then $u \le \boldsymbol{h}^*_{\mathfrak{A},A,D} < \delta$ on $\widetilde{A}_\delta$ implying that $u - \delta \le 0$ on $\widetilde{A}_\delta$. Therefore, $u - \delta \le \boldsymbol{h}^*_{\widetilde{A}_\delta, D}$ and so this inequality follows.

The following property of the relative boundary extremal function under exhaustions will be used later.

Lemma 3.7.9. *Let $D_k \subset \mathbb{C}^n$ be domains with $D_k \nearrow D := \bigcup_{k=1}^{\infty} D_k$ and let $A_k \subset \partial D_k$ and $A \subset \partial D$ be non-empty relatively open subsets with $A_k \nearrow A$. Assume that for any $z \in A$ there exist a neighborhood U_z of z and an index k_z such that $D \cap U_z = D_{k_z} \cap U_z$. If $\boldsymbol{h}^*_{\mathfrak{K},A_k,D_k}|_{A_k} = 0$ (e.g. A_k is locally $\mathfrak{K}$-pluriregular), $k \in \mathbb{N}$, then $\boldsymbol{h}^*_{\mathfrak{K},A_k,D_k}|_{D_k \cup A_k} \searrow \boldsymbol{h}^*_{\mathfrak{K},A,D}|_{D \cup A}$.*

Proof. Note that $\boldsymbol{h}^*_{\mathfrak{K},A_{k+1},D_{k+1}} \le \boldsymbol{h}^*_{\mathfrak{K},A_k,D_k}$ on $D_k \cup A_k$. Then $\boldsymbol{h}^*_{\mathfrak{K},A_k,D_k}|_{D_k} \searrow u \in \mathcal{PSH}(D) \ge \boldsymbol{h}^*_{\mathfrak{K},A,D}|_D$. Fix a point $z \in A$ and take U_z and k_z as in the lemma. Then $\limsup_{D \ni \zeta \to z} u(\zeta) \le \limsup_{D_{k_z} \ni \zeta \to z} \boldsymbol{h}^*_{\mathfrak{K},A_{k_z},D_{k_z}}(\zeta) = 0$. Hence, $u = \boldsymbol{h}^*_{\mathfrak{K},A,D}$ on D. It remains to mention that $\boldsymbol{h}^*_{\mathfrak{K},A,D} = 0$ on A. □

Now we turn to discuss the relative boundary extremal function $\boldsymbol{h}^*_{\mathfrak{K},A,D}$ for special configurations of (A, D).

Proposition 3.7.10. *Let $D \subset \mathbb{C}^n$ be a domain and $\varnothing \ne A \subset \partial D$ relatively open. Assume that D is locally $\mathcal{C}^1$-smooth at each point of A (cf. Definition 2.5.3). Then A is locally $\mathfrak{K}$-pluriregular.*

Proof. Fix an $a \in A$ and a neighborhood U of a. There exists a $\mathcal{C}^1$ smooth bounded domain $G \subset D \cap U$ such that $G \cap V = D \cap V$, $\partial G \cap V \subset A$ for an open neighborhood $V \subset U$ of a (Exercise). Let $V_0 \subset\subset V_1 \subset\subset V$ be domains with $a \in V_0$. Let $\varphi \in \mathcal{C}(\mathbb{C}^n, [0,1])$ be such that $\varphi = 0$ on V_0 and $\varphi = 1$ on $\mathbb{C}^n \setminus V_1$. Then there exists the solution $H \in \mathcal{H}(G) \cap \mathcal{C}(\overline{G})$ of the Dirichlet problem with $H = \varphi$ on ∂G (cf. [Arm-Gar 2001], Theorem 1.3.10). Take an arbitrary $u \in \mathcal{PSH}(D \cap U)$ with $0 \le u \le 1$ and $\lim_{z \to \zeta} u(z) = 0$ for all $\zeta \in A \cap U$. Observe that $\limsup_{G \ni z \to \zeta}(u(z) - H(z)) \le 0$ for all $\zeta \in \partial G$. Hence, by the maximum principle for subharmonic functions, $u \le H$ in G. Thus $\boldsymbol{h}^*_{\mathfrak{K},A \cap U, D \cap U} \le H$ on G. In particular, $\boldsymbol{h}^*_{\mathfrak{K},A \cap U, D \cap U}(a) \le H(a) = 0$. □

Finally we give an example which we will need later.

Example 3.7.11. Let $D = \mathbb{D}$ and $A := \{z \in \mathbb{T} : \operatorname{Re} z > 0\}$. Then A is locally $\mathfrak{K}$-pluriregular and we have

$$\boldsymbol{h}^*_{\mathfrak{K},A,\mathbb{D}}(z) = \frac{1}{\pi} \operatorname{arccot} \frac{2 \operatorname{Re} z}{1 - |z|^2}, \quad z \in \mathbb{D}.$$

Use a conformal mapping from the disc to the strip $\{z \in \mathbb{C} : \operatorname{Im} z \in (0, 1)\}$ such that A is sent to $\mathbb{R}$ (Exercise).

Chapter 4
Classical results II

Summary. In the first section we complete our discussion of the Tuichiev problem from § 1.1.4. In § 4.2 we present the full solution of the Hukuhara problem from § 1.4. The main results are Terada's Theorems 4.2.2 and 4.2.5. The proof of Theorem 4.2.2 is based on Proposition 4.2.1, which may be considered as a generalization of the Hartogs lemma. The result is of independent interest and is applied for instance in the proof of Proposition 4.3.1.

4.1 Tuichiev theorem

$\boxed{>}$ § 2.3.

Let us come back to the Tuichiev problem (cf. § 1.1.4). We are going to consider the following more general situation.

We are given a domain $D \subset \mathbb{C}^p$ and an upper semicontinuous function $h\colon \mathbb{C}^q \to \mathbb{R}_+$ such that $h(\lambda w) = |\lambda| h(w)$, $\lambda \in \mathbb{C}$, $w \in \mathbb{C}^q$. Put

$$B(\rho) := \{w \in \mathbb{C}^q : h(w) < \rho\}.$$

For $0 < R \le +\infty$, let $f : D \times B(R) \to \mathbb{C}$ be such that $f(z, \cdot) \in \mathcal{O}(B(R))$ for every $z \in D$. Write

$$f(z, w) = \sum_{k=0}^{\infty} f_k(z, w), \quad (z, w) \in D \times B(R),$$

where $f_k(z, \cdot)$ is a homogeneous polynomial of degree k,

$$f_k(z, w) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(z, \lambda w)}{\lambda^{k+1}} d\lambda, \quad (z, w) \in D \times B(R).$$

Recall that if $f \in \mathcal{O}(D \times \mathbb{P}_q(\delta))$ for certain $\mathbb{P}_q(\delta) \subset B(R)$, then $f_k \in \mathcal{O}(D \times \mathbb{C}^q)$ and, by Proposition 1.1.10 (a), $f \in \mathcal{O}(D \times B(R))$.

In the general situation we *assume* that $f_k \in \mathcal{O}(D \times \mathbb{C}^q)$, $k \in \mathbb{N}$. Using Proposition 1.1.10 (a), we easily conclude that $\mathcal{S}_{\mathcal{O}}(f) = S \times B(R)$. The problem is to characterize the set S. Observe that the case where $q = 1$, $h(w) := |w|$ reduces to the original Tuichiev problem from § 1.1.4.

Theorem 4.1.1. *Under the above assumptions, the set S is nowhere dense in D.*

Proof. First observe that the standard formula for the radius of convergence of the power series $\mathbb{D}(R/h(w)) \ni \lambda \mapsto f(z, \lambda w)$ gives

$$\limsup_{k\to+\infty} |f_k(z,w)|^{1/k} \le h(w)/R, \quad (z,w) \in D \times \mathbb{C}^q. \tag{4.1.1}$$

Let U be the set of all $a \in D$ such that the sequence $(|f_k|^{1/k})_{k=1}^{\infty}$ is locally bounded in $\mathbb{P}_p(a,\varepsilon) \times \mathbb{C}^q$ for some $\mathbb{P}_p(a,\varepsilon) \subset D$. Take a compact set $K \subset\subset U \times B(R)$ and let $\theta \in (0,1)$ be such that $\sup_{(z,w)\in K} h(w)/R < \theta$. By the Hartogs lemma (Proposition 2.3.13) there exists a k_0 such that $|f_k|^{1/k} \le \theta$ on K for $k \ge k_0$. This means that the series $\sum_{k=0}^{\infty} f_k(z,w)$ converges uniformly on K. Thus, the series converges locally uniformly in $U \times B(R)$ and, therefore $f \in \mathcal{O}(U \times B(R))$.

It remains to show that U is dense in D. Let $\varnothing \ne \Omega \subset D$ be an arbitrary open set and let

$$A_s := \{z \in \Omega : |f_k(z,w)| \le (s\|w\|)^k, \; w \in \mathbb{C}^q, \; k \in \mathbb{N}\}.$$

Then A_s is closed in Ω and (by (4.1.1)) $\Omega = \bigcup_{s=1}^{\infty} A_s$. Consequently, by the standard Baire argument, there exists an s_0 such that $\varnothing \ne \operatorname{int} A_{s_0} \subset \Omega \cap U$. □

Remark 4.1.2. The proof of Theorem 4.1.1 shows that if f is locally bounded in $D \times \mathbb{P}_q(\delta)$ (for some $\mathbb{P}_q(\delta) \subset B(R)$), then $\mathcal{S}_{\mathcal{O}}(f) = \varnothing$, i.e. $f \in \mathcal{O}(D \times B(R))$.

4.2 Terada theorem

> § 3.2.

Recall the Hukuhara problem (§ 1.4):

(S-$\mathcal{O}_{\mathrm{H}}$) Given two domains $D \subset \mathbb{C}^p$, $G \subset \mathbb{C}^q$, a non-empty set $B \subset G$, and a function $f \in \mathcal{O}_s(\boldsymbol{X})$, where $\boldsymbol{X} := (D \times G) \cup (D \times B)$, we ask whether $f \in \mathcal{O}(D \times G)$.

After Theorems 1.4.4 and 1.4.7, the next important step was the one by T. Terada ([Ter 1967]) who finally was able to answer the question raised by Hukuhara. We are going to present a proof of Terada's theorem, based on the relative extremal function – cf. § 3.2.

We begin with the following auxiliary result, which may be regarded as a generalization of Proposition 1.1.10.

Proposition 4.2.1. *Let $h\colon \mathbb{C}^p \to \mathbb{R}_+$ be a plurisubharmonic function satisfying $h(\lambda z) = |\lambda| h(z)$, $\lambda \in \mathbb{C}$, $z \in \mathbb{C}^p$. Put $B(\rho) := \{z \in \mathbb{C}^p : h(z) < \rho\}$. Let $0 < r < R$, let $G \subset \mathbb{C}^q$ be a domain, and let $B \subset G$ be locally pluriregular (cf. Definition 3.2.8). Assume that $f \in \mathcal{O}(B(r) \times G)$ is such that for each $b \in B$ the function $f(\cdot, b)$ extends to a function $\tilde{f}_b \in \mathcal{O}(B(R))$. Then f extends holomorphically to the domain*

$$\widehat{\boldsymbol{X}} := \big\{(z,w) \in B(R) \times G : \boldsymbol{h}^*_{B(r),B(R)}(z) + \boldsymbol{h}^*_{B,G}(w) < 1\big\}.$$

Moreover, if G is a domain of holomorphy, then $\widehat{X}$ is a domain of holomorphy and the maximal domain with the above extension property.

Proof. Let $h_k := \max\{h, \frac{1}{k}\|\ \|\}$, $k \in \mathbb{N}$. Then h_k satisfies all the assumptions of the proposition and moreover $h_k^{-1}(0) = \{0\}$. Observe that

$$B_k(\rho) := \{z \in \mathbb{C}^p : h_k(z) < \rho\} = B(\rho) \cap \mathbb{B}_p(k\rho).$$

In particular, $\boldsymbol{h}^*_{B_k(r),B_k(R)} \searrow \boldsymbol{h}^*_{B(r),B(R)}$ (cf. Proposition 3.2.25) and therefore

$$\widehat{X}_k := \big\{(z,w) \in B_k(R) \times G : \boldsymbol{h}^*_{B_k(r),B_k(R)}(z) + \boldsymbol{h}^*_{B,G}(w) < 1\big\} \nearrow \widehat{X}.$$

Thus we may additionally assume that $h^{-1}(0) = \{0\}$. Since $B(r)$ and $B(R)$ are balanced, we may write

$$f(z,w) = \sum_{k=0}^{\infty} f_k(z,w), \quad (z,w) \in B(r) \times G, \tag{†}$$

$$\tilde{f}_w(z) = \sum_{k=0}^{\infty} f_k(z,w), \quad (z,w) \in B(R) \times B,$$

where

- $f_k \in \mathcal{O}(\mathbb{C}^p \times G)$,
- for each $w \in G$, the function $f_k(\cdot, w)$ is a homogeneous polynomial of degree k.

We have

$$u(z,w) := \limsup_{k\to+\infty} |f_k(z,w)|^{1/k} \le \begin{cases} h(z)/r, & (z,w) \in \mathbb{C}^p \times G, \\ h(z)/R, & (z,w) \in \mathbb{C}^p \times B. \end{cases}$$

Moreover, if $\mathbb{P}_p(\delta) \times K \subset\subset B(r) \times G$, then using the Cauchy inequalities, we get

$$|f_k(z,w)| \le \|f\|_{\mathbb{P}_p(\delta)\times K} \left(\frac{\|z\|}{\delta}\right)^k, \quad (z,w) \in \mathbb{C}^p \times K,\ k \in \mathbb{N},$$

which implies that the sequence $(|f_k|^{1/k})_{k=1}^{\infty}$ is locally bounded in $\mathbb{C}^p \times G$. In particular, $\log u^* \in \mathcal{PSH}(\mathbb{C}^p \times G)$ – cf. Proposition 2.3.12. We are going to show that

$$v(z,w) := \frac{\log \frac{u^*(z,w)}{h(z)} + \log r}{\log R/r} + 1 \le \boldsymbol{h}^*_{B,G}(w), \quad (z,w) \in (\mathbb{C}^p)_* \times G.$$

It is clear that $v \le 1$. Let $P \subset \mathbb{C}^p \times G$ be pluripolar such that $u^* = u$ on $(\mathbb{C}^p \times G) \setminus P$ (cf. Theorem 2.3.33 (b)). Let $Q \subset \mathbb{C}^p$ be pluripolar such that $P_{(z,\cdot)}$ is pluripolar for all $z \in \mathbb{C}^p \setminus Q$ (cf. Proposition 2.3.31 (a)). Then $u^*(z,w) = u(z,w) \le h(z)/R$, $z \in \mathbb{C}^p \setminus Q$, $w \in B \setminus P_{(z,\cdot)}$. Hence

$$v(z,\cdot) \le \boldsymbol{h}^*_{B\setminus P_{(z,\cdot)},G} \le \boldsymbol{h}^*_{(B\setminus P_{(z,\cdot)})^*,G} = \boldsymbol{h}^*_{B^*,G} = \boldsymbol{h}^*_{B,G} \text{ on } G$$

for all $z \in (\mathbb{C}^p)_* \setminus Q$ (cf. Remark 3.2.9 (d), Corollary 3.2.13). Equivalently,

$$\log u^*(z,w) \le \log \frac{h(z)}{R} + \Big(\log \frac{R}{r}\Big)\boldsymbol{h}^*_{B,G}(w), \quad (z,w) \in ((\mathbb{C}^p)_* \setminus Q) \times G.$$

Consequently, by Proposition 2.3.9, we get the same inequality for all $(z,w) \in \mathbb{C}^p \times G$, which implies that $u^*(z,w) < 1$, $(z,w) \in \widehat{X}$. In particular, by the Hartogs lemma for plurisubharmonic functions (cf. Proposition 2.3.13), for every compact $K \subset\subset \widehat{X}$ there exist $\theta \in (0,1)$ and $k_0 \in \mathbb{N}$ such that $|f_k(z,w)|^{1/k} \le \theta$, $(z,w) \in K$, $k \ge k_0$. This implies that the series (†) converges locally uniformly in $\widehat{X}$ and defines there a holomorphic extension of f.

If G is a domain of holomorphy, then by Theorem 2.5.5 (e), $\widehat{X}$ is also a domain of holomorphy. In particular, by Proposition 2.1.27, there exists a $g \in \mathcal{O}(\widehat{X})$ that is not holomorphically continuable beyond $\widehat{X}$. The function $f := g|_{B(r)\times G}$ satisfies all the assumptions of the proposition. Consequently, if G is a domain of holomorphy, then $\widehat{X}$ is maximal. □

Theorem 4.2.2 (Terada). *Under the assumptions of the Hukuhara problem, if $B \notin \mathcal{PLP}$, then $\mathcal{O}_s(X) = \mathcal{O}(D \times G)$.*

Proof. We may assume that B is locally pluriregular (cf. Corollary 3.2.13). Fix an $f \in \mathcal{O}_s(X)$. By Theorem 1.4.7, $f \in \mathcal{O}(U_0 \times G)$, where U_0 is an open dense subset of D. To prove that $U_0 = D$ we only need to show that if $\mathbb{P}_p(a,r) \subset\subset U_0$ and $\mathbb{P}_p(a,R) \subset D$ for $0 < r < R$, then $\mathbb{P}_p(a,R) \subset U_0$. Using Proposition 4.2.1 we conclude that f extends holomorphically to the domain

$$\widehat{X} := \Big\{(z,w) \in \mathbb{P}_p(a,R) \times G : \frac{\log \frac{\|z-a\|_\infty}{r}}{\log \frac{R}{r}} + \boldsymbol{h}^*_{B,G}(w) < 1\Big\}.$$

Hence, by Proposition 1.1.10, $\mathbb{P}_p(a,R) = \mathrm{pr}_{\mathbb{C}^p}\widehat{X} \subset U_0$. □

Remark 4.2.3. Some particular cases of the Terada theorem (Theorem 4.2.2) had been studied by several authors. Let us mention for instance [Fil 1973] for $B := G \cap \mathbb{R}^q \ne \varnothing$ (cf. Example 3.2.20).

Exercise 4.2.4. Prove the Hartogs, Hukuhara, Shimoda, and Terada theorems in the case where D and G are Riemann domains over $\mathbb{C}^p$ and $\mathbb{C}^q$, respectively.

The following result shows that Theorem 4.2.2 is almost optimal.

Theorem 4.2.5 (Cf. [Ter 1972]). *Let $B \subset \mathbb{C}^q$ be an $\mathcal{F}_\sigma$ pluripolar set. Then there exist a point $w_0 \in \mathbb{C}^q$ and a function $f \in \mathcal{O}_s(X)$ with $\boldsymbol{X} := (\mathbb{C}^p \times \mathbb{C}^q) \cup (\mathbb{C}^p \times B)$ such that* $\limsup_{(z,w)\to(0,w_0)} |f(0,w_0)| = +\infty$. *In particular, $f \notin \mathcal{O}(\mathbb{C}^p \times \mathbb{C}^q)$.*

Remark 4.2.6. (a) Observe there are pluripolar sets that are not $\mathcal{F}_\sigma$, e.g. $B := \{0\}\times\widetilde{B} \subset \mathbb{C}^2$, where $\widetilde{B} \subset \mathbb{C}$ is an arbitrary set not in $\mathcal{F}_\sigma$.

(b) [?] We do not know whether the assumption that $B \in \mathcal{F}_\sigma$ is essential [?]

(c) [?] We do not know any example of a pluripolar set $B \subset \mathbb{C}^q$ such that there exists a locally bounded function $f \in \mathcal{O}_s(X) \setminus \mathcal{O}(\mathbb{C}^p \times \mathbb{C}^q)$ [?]

Proof of Theorem 4.2.5. We may assume that $p = 1$. Let

$$\Omega_k := \left\{x + iy \in \mathbb{D}(k) : x < \tfrac{1}{2^k} - \tfrac{1}{2^{k+2}} \text{ or } x > \tfrac{1}{2^k} + \tfrac{1}{2^{k+2}}\right\}, \quad z_k := \frac{1}{2^k}, \quad k \in \mathbb{N}.$$

Then (Exercise)

(∗) $\mathbb{C} = \bigcup_{k=1}^\infty \bigcap_{s=k}^\infty \Omega_s$, $z_k \in (\bigcap_{s=k+1}^\infty \Omega_s) \setminus \Omega_k$.

Using Runge's theorem we easily conclude (Exercise) that

(∗∗) $\forall_{k\in\mathbb{N}}\ \forall_{M>0}\ \forall_{\varepsilon>0}\ \exists_{\varphi\in\mathcal{P}(\mathbb{C})} : |\varphi(z_k)| \geq M,\ |\varphi| \leq \varepsilon$ on Ω_k.

Let $u \in \mathcal{PSH}(\mathbb{C}^q)$, $u \not\equiv -\infty$, be such that $B \subset u^{-1}(-\infty)$ (cf. Josefson's Theorem 2.3.23). Then $u = v^*$, where

$$v = \limsup_{m\to+\infty} \frac{1}{m} \log |g_m|$$

and $(g_m)_{m=1}^\infty \subset \mathcal{O}(\mathbb{C}^q)$ is such that the sequence $(|g_m|^{1/m})_{m=1}^\infty$ is locally bounded in $\mathbb{C}^q$ (cf. Proposition 2.5.11). Let $w_0 \in \mathbb{C}^q$ be such that $u(w_0) = v(w_0) > -\infty$ (cf. Propositions 2.3.10, 2.3.22). Fix a sequence $(G_k)_{k=1}^\infty$ of subdomains of $\mathbb{C}^q$ such that $w_0 \in G_k \subset\subset G_{k+1} \subset\subset \mathbb{C}^q$, $\mathbb{C}^q = \bigcup_{k=1}^\infty G_k$ and $B = \bigcup_{k=1}^\infty B_k$, where B_k is compact, $\varnothing \neq B_k \subset B \cap G_k$, $B_k \subset B_{k+1}$.

Let $c_k := \sup_{G_{k+1}} u$. Observe that

$$\limsup_{m\to+\infty} \frac{1}{m} \log |e^{-mc_k} g_m(w)| = v - c_k \leq u - c_k \begin{cases} < 0 & \text{on } G_{k+1}, \\ = -\infty & \text{on } B_k. \end{cases}$$

Put $\varrho_k := c_k - u(w_0) > 0$. Using the Hartogs lemma for plurisubharmonic functions (cf. Proposition 2.3.13), for every $n_k > 0$, we choose an $m_k \in \mathbb{N}$ such that with $\psi_k := e^{-m_k c_k} g_{m_k} \in \mathcal{O}(\mathbb{C}^q)$ we have

(1) $|\psi_k| \leq 1$ on G_k,

(2) $|\psi_k(w_0)| \geq e^{-2m_k \varrho_k}$,

(3) $|\psi_k| \leq e^{-m_k n_k \varrho_k}$ on B_k.

Take an arbitrary exhaustion sequence $(D_k)_{k=1}^\infty$ for $\mathbb{C}$. Using (∗∗), we construct inductively $M_k > 0$, $\varphi_k \in \mathcal{P}(\mathbb{C})$, and $n_k > 0$ such that

(4) $M_k e^{-m_k \varrho_k} \geq k + 1 + \sum_{s=1}^{k-1} |\varphi_s(z_k)\psi_s(w_0)|$ (we choose $M_k > 0$),

(5) $|\varphi_k(z_k)| \geq M_k$, $|\varphi_k| \leq 1/2^k$ on Ω_k (we choose φ_k),

(6) $|\varphi_k|e^{-m_k n_k Q_k} \le 1/2^k$ on D_k (we choose $n_k > 0$).

Define

$$f(z,w) := \sum_{k=1}^{\infty} \varphi_k(z)\psi_k(w), \quad (z,w) \in \mathbb{C} \times \mathbb{C}^q.$$

Take an arbitrary $a \in \mathbb{C}$, say $a \in \Omega_k$ for $k \ge k_0$ (cf. $(*)$). Then we get

$$|\varphi_k(a)\psi_k(w)| \overset{(1),(5)}{\le} \frac{1}{2^k}, \quad w \in G_k,\ k \ge k_0.$$

Hence $f(a,\cdot) \in \mathcal{O}(\mathbb{C}^q)$.

Take an arbitrary $b \in B$, say $b \in B_k$ for $k \ge k_0$. Then we get

$$|\varphi_k(z)\psi_k(b)| \overset{(3)}{\le} |\varphi_k(z)|e^{-m_k n_k Q_k} \overset{(6)}{\le} \frac{1}{2^k}, \quad z \in D_k,\ k \ge k_0.$$

Thus $f(\cdot,b) \in \mathcal{O}(\mathbb{C})$. Consequently, $f \in \mathcal{O}_s(\boldsymbol{X})$, $\boldsymbol{X} := (\mathbb{C} \times \mathbb{C}^q) \cup (\mathbb{C} \times B)$.

On the other hand, we have

$$|f(z_k,w_0)| \ge |\varphi_k(z_k)\psi_k(w_0)| - \sum_{s=1}^{k-1} |\varphi_s(z_k)\psi_s(w_0)| - \sum_{s=k+1}^{\infty} |\varphi_s(z_k)\psi_s(w_0)|$$

$$\overset{(1),(2),(4),(5)}{\ge} M_k e^{-2m_k Q_k} - (M_k e^{-2m_k Q_k} - k - 1) - \sum_{s=k+1}^{\infty} \frac{1}{2^s} \ge k \underset{k\to+\infty}{\longrightarrow} +\infty.$$

□

4.3 Separately harmonic functions I

Proposition 4.2.1 permits us to find an analogue of Proposition 1.1.10 for harmonic functions.

Proposition 4.3.1. *Let $0 < r < R$ and let $V \subset \mathbb{R}^q$ be a domain. Let $u\colon \mathbb{B}_p^{\mathbb{R}}(R) \times V \to \mathbb{R}$, where $\mathbb{B}_p^{\mathbb{R}}(R)$ stands for the real Euclidean ball, be such that*

- *$u \in \mathcal{H}(\mathbb{B}_p^{\mathbb{R}}(r) \times V)$,*
- *for each $b \in V$ we have $u(\cdot,b) \in \mathcal{H}(\mathbb{B}_p^{\mathbb{R}}(R))$.*

Then $u \in \mathcal{H}(\mathbb{B}_p^{\mathbb{R}}(R) \times V)$.

Proposition 4.3.1 will be generalized in Proposition 5.7.4.

Definition 4.3.2. We say that a function $u\colon \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^p \times \mathbb{R}^q$ is open, is *separately harmonic* on Ω ($u \in \mathcal{H}_{(p,q)}(\Omega)$) if for each $(a,b) \in \Omega$ the functions $z \mapsto u(z,b)$, $w \mapsto u(a,w)$ are harmonic.

In the sequel, we will see that $\mathscr{H}_{(p,q)}(\Omega) \subset \mathscr{H}(\Omega)$ (cf. Theorem 5.6.5 (b)). Notice that in fact the function u in Proposition 4.3.1 will be of the class $\mathscr{H}(\mathbb{B}_p^{\mathbb{R}}(R) \times V) \cap \mathscr{H}_{(p,q)}(\mathbb{B}_p^{\mathbb{R}}(R) \times V)$.

Proof of Proposition 4.3.1. It suffices to prove that u is real analytic (and then use the identity principle for real analytic functions to show that u is harmonic).

It is known (cf. [Sic 1974]) that every function $v \in \mathscr{H}(\mathbb{B}_k^{\mathbb{R}}(\rho))$ extends to a function $\widehat{v} \in \mathscr{O}(\mathbb{L}_k(\rho))$, where $\mathbb{L}_k(\rho)$ stands for the Lie ball

$$\mathbb{L}_k(\rho) := \{z \in \mathbb{C}^k : L_k(z) < \rho\}, \quad L_k(z) := (\|z\|^2 + (\|z\|^4 - |\langle z, \bar{z}\rangle|^2)^{1/2})^{1/2}.$$

Let $\widehat{u}_b \in \mathscr{O}(\mathbb{L}_p(R))$ be the extension of $u(\cdot, b)$, $b \in V$. Moreover, there exists a domain $\Omega \subset \mathbb{C}^p \times \mathbb{C}^q$ such that $\mathbb{B}_p^{\mathbb{R}}(r) \times V \subset \Omega$ and u extends to a $\widehat{u} \in \mathscr{O}(\Omega)$.

Fix a $y_0 \in V$ and let $\rho \in (0, r)$ be such that $\mathbb{L}_p(\rho) \times \mathbb{B}_q(y_0, \rho) \subset \Omega$. In particular, $\widehat{u} \in \mathscr{O}(\mathbb{L}_p(\rho) \times \mathbb{B}_q(y_0, \rho))$. Observe that for every $b \in \mathbb{B}_q^{\mathbb{R}}(y_0, \rho)$ we have $\widehat{u}_b = \widehat{u}(\cdot, b)$ on $\mathbb{L}_p(\rho)$. Using Proposition 4.2.1 (with $G := \mathbb{B}_q(y_0, \rho)$, $B := \mathbb{B}_q^{\mathbb{R}}(y_0, \rho)$, $h := L_p$), we conclude that $\widehat{u}$ extends to a $\tilde{\widehat{u}} \in \mathscr{O}(\widehat{\boldsymbol{X}})$, where

$$\widehat{\boldsymbol{X}} := \Big\{(z, w) \in \mathbb{L}_p(R) \times \mathbb{B}_q(y_0, \rho) : \boldsymbol{h}^*_{\mathbb{L}_p(\rho), \mathbb{L}_p(R)}(z) + \boldsymbol{h}^*_{\mathbb{B}_q^{\mathbb{R}}(y_0,\rho), \mathbb{B}_q(y_0,\rho)}(w) < 1\Big\}.$$

It remains to observe that:

- $\mathbb{B}_p^{\mathbb{R}}(R) \times \mathbb{B}_q^{\mathbb{R}}(y_0, \rho) = \widehat{\boldsymbol{X}} \cap \mathbb{R}^{p+q}$,
- $\tilde{\widehat{u}} = u$ on $\mathbb{B}_p^{\mathbb{R}}(R) \times \mathbb{B}_q^{\mathbb{R}}(y_0, \rho)$. □

4.4 Miscellanea

4.4.1 Hartogs' problem for "non-linear" fibers

A possible generalization of the problem of separate holomorphy is to consider "non-linear" fibers. Let us illustrate the main idea by the following particular case.

(S-$\mathscr{O}_{\mathrm{F}}$) Let $D \subset \mathbb{C}^p$, $G \subset \mathbb{C}^q$, $\Omega \subset D \times \mathbb{C}^q$ be domains and let

$$D \times G \ni (z, w) \mapsto (z, \varphi(z, w)) \in \Omega$$

be a homeomorphic mapping such that $\varphi(\cdot, w)$ is holomorphic for every $w \in G$. Suppose that $f : \Omega \to \mathbb{C}$ is such that

- $f(a, \cdot)$ is holomorphic on the fiber domain $\varphi(\{a\} \times G)$ for every $a \in D$,
- $f(\cdot, \varphi(\cdot, b)) \in \mathscr{O}(D)$ for every $b \in G$.

We ask whether $f \in \mathscr{O}(\Omega)$. The problem is discussed e.g. in [Chi 2006]. Note that the classical Hartogs theorem is just the case where $\varphi(z, w) := w$ and $\Omega := D \times G$.

4.4.2 Boundary analogues of the Hartogs theorem

Let $\mathbb{B} = \mathbb{B}_2$. Observe that the function $f : \partial\mathbb{B} \to \mathbb{C}$, $z \mapsto |z_1|^2$, satisfies the following conditions:

- $f(a, \cdot)$ is identically constant on $(\partial\mathbb{B})_{(a,\cdot)}$,
- $f(\cdot, b)$ is identically constant on $(\partial\mathbb{B})_{(\cdot,b)}$;

in particular, the functions $f(a, \cdot)$ and $f(\cdot, b)$ have a holomorphic extension to $\mathbb{B}_{(a,\cdot)}$ and $\mathbb{B}_{(\cdot,b)}$, respectively. But f itself cannot be extended to a holomorphic function in $\mathbb{B}$.

So one may study the following question:

(S-$\mathcal{O}_{\mathcal{L}}$) Let $D \subset \mathbb{C}^n$ be a bounded domain and let $\mathcal{L}_D$ be a family of complex lines intersecting D. Under which condition on $\mathcal{L}_D$ the following result is true: If $f \in \mathcal{C}(\partial D)$ such that $f|_{L\cap\partial D}$ extends to an $\tilde{f}_L \in \mathcal{C}(L \cap \partial D) \cap \mathcal{O}(L \cap D)$, $L \in \mathcal{L}_D$, then there exists an $\hat{f} \in \mathcal{C}(\bar{D}) \cap \mathcal{O}(D)$ with $\hat{f}|_{\partial D} = f$.

There exists a series of positive results for different families of complex lines, e.g.

(a) $D \subset \mathbb{C}^n$ has C^2-smooth boundary and $\mathcal{L}_D$ is the set of all complex lines intersecting D (cf. [Sto 1977]).

(b) D is as in (a), $\varnothing \neq V \subset D$ open, and $\mathcal{L}_D$ is the set of all complex lines passing to a point from V (cf. [Agr-Sem 1991]).

(c) $D = \mathbb{B}_n$, $a_1, \dots, a_{n+1} \in \mathbb{B}_n$ affinely independent, and $\mathcal{L}_D$ is the set of all complex lines containing at least one of the points a_j (cf. [Agr 2010]).

(d) In case that the function f is, in addition, $\mathcal{C}^\infty$-smooth on $\partial\mathbb{B}_n$, then there are even stronger results – see for example [Glo 2009].

4.4.3 Forelli type results

Finally, let us recall the following result due to F. Forelli (cf. [For 1977]). Let D be a balanced domain in $\mathbb{C}^n$ and let $f : D \to \mathbb{C}$ be such that $f \in \mathcal{C}^\infty(U)$ for a certain open neighborhood of the origin. If for all $a \in \partial\mathbb{B}_n$ the function $\lambda \mapsto f(\lambda a)$ is holomorphic on $\{\lambda \in \mathbb{C} : \lambda a \in D\}$, then $f \in \mathcal{O}(D)$. Note that in this situation the test-lines are the ones passing through the origin.

Chapter 5

Classical cross theorem

Summary. Section 5.1 contains the formulation of the main extension problem for separately holomorphic functions defined on N-fold crosses. For $N = 2$ this is a generalization of the Hukuhara problem to the case where the holomorphicity is controlled only along some "horizontal" and "vertical" directions (recall that in the Hukuhara problem all the "vertical" directions were under control). First, to get some intuition of the problem, we consider the case of Reinhardt domains (§ 5.2) and the case of separately polynomial functions (§ 5.3). Section 5.4 presents the main cross theorem (Theorems 5.4.1, 5.4.2) and various reduction procedures which permit us to simplify proofs (Remark 5.4.4). Subsection 5.4.1 discusses a geometric approach proposed by J. Siciak in the papers [Sic 1969a], [Sic 1969b], which together with [Cam-Sto 1966] initiated modern theory of separately holomorphic functions on crosses. Although Siciak's methods require special additional assumptions, his approach may help the reader to get an intuition how to prove the general cross theorem. The proof of the main cross theorem is presented in Subsection 5.4.2, which is the central part of the chapter. Sections 5.6 and 5.7 contain applications of the main cross theorem to the Bochner tube theorem (Theorem 5.6.1), Browder's theorem on separately real analytic functions (Theorem 5.6.5 (a)), Lelong's theorem on separately harmonic functions (Theorem 5.6.5 (b)), and a cross theorem for separately harmonic functions (Proposition 5.7.3).

5.1 N-fold crosses

$\boxed{>}$ §§ 2.3, 3.2.

After Terada's results (cf. Theorems 4.2.2, 4.2.5) it was clear that the next step should be a solution of the following general problem (cf. Introduction).

(S-$\mathcal{O}_{\mathrm{C}}$) We are given two domains $D \subset \mathbb{C}^p$, $G \subset \mathbb{C}^q$, two non-empty sets $A \subset D$, $B \subset G$. Define the *cross*

$$X = \mathbb{X}(A, B; D, G) := (A \times G) \cup (D \times B).$$

We say that a function $f : X \to \mathbb{C}$ is *separately holomorphic on* X ($f \in \mathcal{O}_s(X)$) if

- $f(a, \cdot) \in \mathcal{O}(G)$ for every $a \in A$,
- $f(\cdot, b) \in \mathcal{O}(D)$ for every $b \in B$.

We ask whether there exists an open neighborhood $\widehat{X} \subset D \times G$ of X such that every function $f \in \mathcal{O}_s(X)$ extends holomorphically to $\widehat{X}$. Observe that the Hukuhara problem was just the case where $A = D$ and $\widehat{X} = D \times G$.

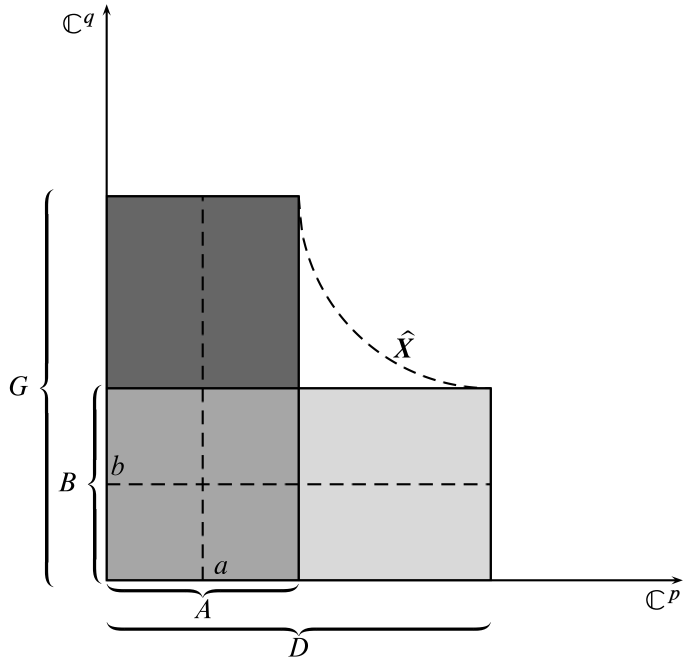


Figure 5.1.1. $X = (A \times G) \cup (D \times B) \subset \widehat{X}$.

The extension problem (S-$\mathcal{O}_C$) may be formulated in a more general context (see Chapters 6, 7).

Let D_j be a Riemann domain over $\mathbb{C}^{n_j}$ and let $\varnothing \neq A_j \subset D_j$, $j = 1, \dots, N$, $N \geq 2$. Let

$$A'_j := A_1 \times \cdots \times A_{j-1}, \ j = 2, \dots, N, \quad A''_j := A_{j+1} \times \cdots \times A_N, \ j = 1, \dots, N-1.$$

Similarly, for $a = (a_1, \dots, a_N) \in A_1 \times \cdots \times A_N$, we write $a'_j := (a_1, \dots, a_{j-1})$, $a''_j := (a_{j+1}, \dots, a_N)$. Moreover, let

$$\mathfrak{X}_j := A'_j \times D_j \times A''_j,$$

where $\mathfrak{X}_1 = A'_1 \times D_1 \times A''_1 := D_1 \times A''_1$ and $\mathfrak{X}_N = A'_N \times D_N \times A''_N := A'_N \times D_N$.

Definition 5.1.1. Define the *N-fold cross*

$$X = \mathbb{X}(A_1, \dots, A_N; D_1, \dots, D_N) = \mathbb{X}((A_j, D_j)_{j=1}^N) := \bigcup_{j=1}^N \mathfrak{X}_j.$$

We say that $\mathfrak{X}_j(X) := \mathfrak{X}_j$, $j = 1, \dots, N$, are *branches* of X. The set

$$\boldsymbol{c}(X) := A_1 \times \cdots \times A_N.$$

is called the *center of the cross* $\boldsymbol{X}$.

Remark 5.1.2. (a) $\boldsymbol{X} = D_1 \times \cdots \times D_N \iff \#\{j : A_j = D_j\} \geq N - 1$. In particular, if $A_j = D_j$, $j = 1, \ldots, N-1$, then $\boldsymbol{X} = D_1 \times \cdots \times D_N$ independently of A_N.

(b) If $\#\{j : A_j = D_j\} \leq N - 2$, then $A_1, \ldots, A_N$ are uniquely determined by $\boldsymbol{X}$.

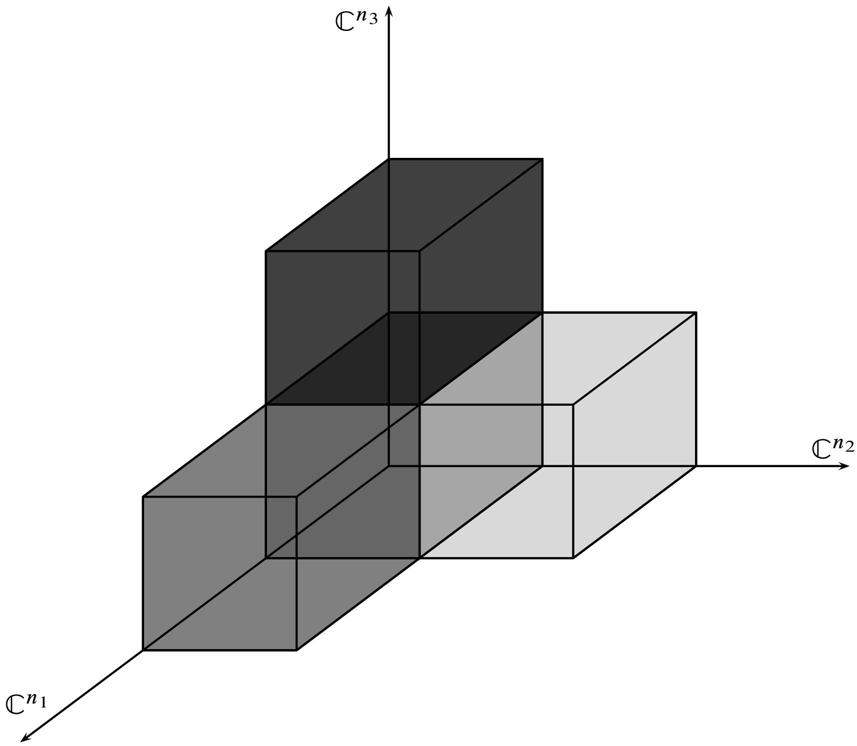


Figure 5.1.2. $\boldsymbol{X} = (D_1 \times A_2 \times A_3) \cup (A_1 \times D_2 \times A_3) \cup (A_1 \times A_2 \times D_3)$.

Definition 5.1.3. We say that a function $f : \boldsymbol{X} \to \mathbb{C}$ is *separately holomorphic on* $\boldsymbol{X}$ ($f \in \mathcal{O}_s(\boldsymbol{X})$) if for any $(a_1, \ldots, a_N) \in A_1 \times \cdots \times A_N$ and $j \in \{1, \ldots, N\}$, the function

$$D_j \ni z_j \mapsto f(a'_j, z_j, a''_j) \in \mathbb{C}$$

is holomorphic in D_j.

Remark 5.1.4. Observe that if $\boldsymbol{X} = D_1 \times \cdots \times D_N$, then $f : \boldsymbol{X} \to \mathbb{C}$ is separately holomorphic (in the above sense) iff $f : D_1 \times \cdots \times D_N \to \mathbb{C}$ is separately holomorphic in the sense of the Hukuhara problem (S-$\mathcal{O}_{\mathrm{H}}$) (cf. § 4.2).

In fact, suppose that $A_j = D_j$, $j = 1, \dots, N-1$ (cf. Remark 5.1.2 (a)). It is clear that if $f\colon (D_1 \times \dots \times D_{N-1}) \times D_N \to \mathbb{C}$ is separately holomorphic in the sense of (S-$\mathcal{O}_{\mathrm{H}}$) with $D := D_1 \times \dots \times D_{N-1}$, $G := D_N$, $B := A_N$, then f is separately holomorphic on $\boldsymbol{X}$ in the sense of Definition 5.1.3.

Conversely, if $f\colon \boldsymbol{X} \to \mathbb{C}$ is separately holomorphic in the sense of Definition 5.1.3, then clearly $f(a, \cdot) \in \mathcal{O}(D_N)$ for every $a \in D_1 \times \dots \times D_{N-1}$. Moreover, for each $b \in A_N$ the function $f(\cdot, b)$ is separately holomorphic on $D_1 \times \dots \times D_{N-1}$ in the sense of Definition 1.1.1. Thus, by Hartogs' theorem (Theorem 1.1.7), $f(\cdot, b) \in \mathcal{O}(D_1 \times \dots \times D_{n-1})$ for every $b \in A_N$.

Discussing examples like in § 5.2, the following definition appears in a natural way.

Definition 5.1.5. Put

$$\begin{aligned}\widehat{\boldsymbol{X}} &= \widehat{\mathbb{X}}(A_1, \dots, A_N; D_1, \dots, D_N) = \widehat{\mathbb{X}}((A_j, D_j)_{j=1}^N) \\ &:= \Big\{(z_1, \dots, z_N) \in D_1 \times \dots \times D_N : \textstyle\sum_{j=1}^N \boldsymbol{h}^*_{A_j^*, D_j}(z_j) < 1\Big\}.\end{aligned}$$

Remark 5.1.6. (a) If A_{j_0} is pluripolar for a $j_0 \in \{1, \dots, N\}$, then $\widehat{\boldsymbol{X}} = \varnothing$.

(b) If $A_1, \dots, A_N$ are not pluripolar, then

$$\widehat{\boldsymbol{X}} = D_1 \times \dots \times D_N \iff \#\{j : \boldsymbol{h}^*_{A_j^*, D_j} \equiv 0\} \geq N - 1.$$

Indeed, first recall that if A_j is not pluripolar, then $A_j \cap A_j^*$ is locally pluriregular (Corollary 3.2.13). Hence, $\boldsymbol{h}^*_{A_j^*, D_j} = 0$ on $A_j \cap A_j^*$, and so $\boldsymbol{h}^*_{A_j^*, D_j}(z_j) < 1$, $z_j \in D_j$. In particular, we get the implication "$\Longleftarrow$". To prove the converse implication, suppose that e.g. $\boldsymbol{h}^*_{A_{N-1}^*, D_{N-1}}(a_{N-1}) > 0$, $\boldsymbol{h}^*_{A_N^*, D_N} \not\equiv 0$. Then, by Proposition 3.2.2 (b), there exists an $a_N \in D_N$ such that

$$\boldsymbol{h}^*_{A_N^*, D_N}(a_N) > 1 - \boldsymbol{h}^*_{A_{N-1}^*, D_{N-1}}(a_{N-1}).$$

Take arbitrary $a_j \in A_j \cap A_j^*$, $j = 1, \dots, N-2$. Then $(a_1, \dots, a_{N-2}, a_{N-1}, a_N) \notin \widehat{\boldsymbol{X}}$; a contradiction.

(c) If $A_1, \dots, A_N$ are not pluripolar and $\#\{j : \boldsymbol{h}^*_{A_j^*, D_j} \equiv 0\} \leq N - 2$, then

$$\boldsymbol{X} \subset \widehat{\boldsymbol{X}} \iff \boldsymbol{h}^*_{A_j^*, D_j} = 0 \quad \text{on } A_j,\ j = 1, \dots, N.$$

Indeed, the implication "$\Longleftarrow$" is obvious. To prove the converse implication, suppose that e.g. $\boldsymbol{h}^*_{A_{N-1}^*, D_{N-1}}(a_{N-1}) > 0$ for an $a_{N-1} \in A_{N-1}$, and $\boldsymbol{h}^*_{A_N^*, D_N} \not\equiv 0$. Next, we argue as above.

(d) If $D_j \in \mathfrak{R}_b(\mathbb{C}^{n_j})$, $j = 1, \dots, N$, or $A_1, \dots, A_N$ are locally pluriregular, then

$$\widehat{\boldsymbol{X}} = \Big\{(z_1, \dots, z_N) \in D_1 \times \dots \times D_N : \textstyle\sum_{j=1}^N \boldsymbol{h}^*_{A_j, D_j}(z_j) < 1\Big\}$$

(cf. Remark 3.2.26).

(e) If $A_1, \dots, A_N$ are locally pluriregular, then $\boldsymbol{X} \subset \widehat{\boldsymbol{X}}$.

The main extension problem for separately holomorphic functions defined on an N-fold cross $\boldsymbol{X}$ is to answer the following question.

(S-$\mathcal{O}_{\mathrm{C}}^{N}$) When does $f \in \mathcal{O}_s(\boldsymbol{X})$ extend holomorphically to $\widehat{\boldsymbol{X}}$?

Remark 5.1.7. (a) In view of Remark 5.1.6 (a) we must assume that $A_1, \dots, A_N$ are not pluripolar.

(b) If $A_1, \dots, A_N$ are not pluripolar and $\#\{j : A_j = D_j\} \geq N-1$, then, by Remark 5.1.4, our extension problem reduces to the Hukuhara problem and, therefore, (cf. Theorem 4.2.2) it has the positive solution (with $\boldsymbol{X} = \widehat{\boldsymbol{X}} = D_1 \times \cdots \times D_N$).

Thus, from now on we assume $A_1, \dots, A_N$ are not pluripolar and $\#\{j : A_j = D_j\} \leq N-2$.

Remark 5.1.8 (Properties of N-fold crosses). The reader is asked to complete details.

(a) $\boldsymbol{X}$ is arcwise connected.

(b) $\mathfrak{X}_1, \dots, \mathfrak{X}_N$ are non-pluripolar. In particular, $\boldsymbol{X} \notin \mathcal{PLP}$. If $A_1, \dots, A_N$ are locally pluriregular, then $\mathfrak{X}_1, \dots, \mathfrak{X}_N$ are locally pluriregular and, consequently, $\boldsymbol{X}$ is locally pluriregular (cf. Exercise 3.2.19).

(c) If $(D_{j,k})_{k=1}^{\infty}$ is a sequence of subdomains of D_j and $(A_{j,k})_{k=1}^{\infty}$ is a sequence of subsets of A_j such that

$$D_{j,k} \nearrow D_j, \quad D_{j,k} \supset A_{j,k} \nearrow A_j, \quad j = 1, \dots, N,$$

then

$$\mathbb{X}((A_{j,k}, D_{j,k})_{j=1}^{N}) \nearrow \boldsymbol{X} \quad \text{and} \quad \widehat{\mathbb{X}}((A_{j,k}, D_{j,k})_{j=1}^{N}) \nearrow \widehat{\boldsymbol{X}}$$

(cf. Proposition 3.2.25).

(d) $\widehat{\boldsymbol{X}}$ is connected.

Indeed, using (c), we may assume that $D_j \in \mathfrak{R}_b(\mathbb{C}^{n_j})$, $j = 1, \dots, N$. Since $\boldsymbol{X}$ is connected, it suffices to show that every point $a = (a_1, \dots, a_N) \in \widehat{\boldsymbol{X}}$ may be connected in $\widehat{\boldsymbol{X}}$ with a point from $\boldsymbol{c}(\boldsymbol{X})$. Put $\varepsilon := \sum_{j=1}^{N-1} \boldsymbol{h}_{A_j,D_j}^{*}(a_j)$. If $\varepsilon = 0$, then $\{(a_1, \dots, a_{N-1})\} \times D_N \subset \widehat{\boldsymbol{X}}$. If $\varepsilon > 0$, then by Proposition 3.2.27, the connected component S of the open set $\{z_N \in D_N : \boldsymbol{h}_{A_N,D_N}^{*}(z_N) < 1 - \varepsilon\}$, that contains a_N, intersects A_N. Consequently, a may be connected inside of $\widehat{\boldsymbol{X}}$ with a point $(a_1, \dots, a_{N-1}, b_N)$, where $b_N \in A_N$. Repeating the above argument, we easily show that a may be connected inside of $\widehat{\boldsymbol{X}}$ with a point $b \in \boldsymbol{c}(\boldsymbol{X})$.

(e) If $D_1, \dots, D_N$ are domains of holomorphy, then $\widehat{\boldsymbol{X}}$ is a domain of holomorphy (cf. Theorem 2.5.5 (e)).

(f) If $P_j \in \mathcal{PLP}(D_j)$, $j = 1, \dots, N$, then

$$\widehat{\mathbb{X}}((A_j \setminus P_j, D_j)_{j=1}^{N}) = \widehat{\boldsymbol{X}}$$

(cf. Corollary 3.2.13). In particular,

$$\widehat{\mathbb{X}}((A_j \cap A_j^{*}, D_j)_{j=1}^{N}) = \widehat{\boldsymbol{X}}$$

(cf. Proposition 3.2.10).

(g) If $A_1, \dots, A_N$ are locally pluriregular and

$$\widehat{\boldsymbol{Y}} := \widehat{\mathbb{X}}((A_j, D_j)_{j=1}^{N-1}) \subset D_1 \times \cdots \times D_{N-1},$$

then

$$\widehat{\mathbb{X}}(A'_N, A_N; \widehat{\boldsymbol{Y}}, D_N) = \widehat{\boldsymbol{X}}$$

(cf. Proposition 3.2.28).

(h) Assume that $B_j \subset A_j$, $B_j \notin \mathcal{PLP}$, $j = 1, \dots, N$. Let $f \in \mathcal{O}_s(\boldsymbol{X})$ be such that $f = 0$ on $B_1 \times \cdots \times B_N$. Then $f = 0$ on $\boldsymbol{X}$.

Indeed, it suffices to show that $f = 0$ on $\boldsymbol{c}(\boldsymbol{X})$. Fix a point $(a_1, \dots, a_N) \in \boldsymbol{c}(\boldsymbol{X})$. We know that for any $b_j \in B_j$, $j = 1, \dots, N-1$, we have $f(b_1, \dots, b_{N-1}, \cdot) = 0$ on B_N. Since $B_N \notin \mathcal{PLP}$, we conclude that $f(b_1, \dots, b_{N-1}, \cdot) = 0$ on D_j and, therefore, $f(b_1, \dots, b_{N-1}, a_N) = 0$. Now we repeat the same procedure with respect to the $(N-1)$-th variable: $f(b_1, \dots, b_{N-2}, \cdot, a_N) = 0$ on B_{N-1} and hence $f(b_1, \dots, b_{N-2}, a_{N-1}, a_N) = 0$. Finite induction finishes the proof.

(i) If $D_1, \dots, D_N$ are pseudoconvex, $A_1, \dots, A_N$ are locally pluriregular, and the problem (S-$\mathcal{O}_{\mathrm{C}}^N$) has positive solution, then

$$\boldsymbol{h}^*_{\boldsymbol{X}, \widehat{\boldsymbol{X}}} \equiv 0.$$

Indeed, first recall that $\boldsymbol{X}$ is connected, $\widehat{\boldsymbol{X}}$ is a pseudoconvex domain, and $\boldsymbol{X} \subset \widehat{\boldsymbol{X}}$. Let $0 < \varepsilon < 1$ and let Ω_ε be the connected component of the open set $\{z \in \widehat{\boldsymbol{X}} : \boldsymbol{h}^*_{\boldsymbol{X}, \widehat{\boldsymbol{X}}}(z) < \varepsilon\}$ that contains $\boldsymbol{X}$. Observe that $\widehat{\boldsymbol{X}}$ is the envelope of holomorphy of Ω_ε. For, we only need to show that every function $g \in \mathcal{O}(\Omega_\varepsilon)$ has a holomorphic extension to $\widehat{\boldsymbol{X}}$. If $g \in \mathcal{O}(\Omega_\varepsilon)$, then $g|_{\boldsymbol{X}} \in \mathcal{O}_s(\boldsymbol{X})$. In view of our assumptions, there exists a $\widehat{g} \in \mathcal{O}(\widehat{\boldsymbol{X}})$ such that $\widehat{g} = g$ on $\boldsymbol{X}$. Now, since $\boldsymbol{X}$ is not pluripolar, we conclude that $\widehat{g} = g$ on Ω_ε.

Now, using Example 3.2.6, we get $\boldsymbol{h}^*_{\boldsymbol{X}, \widehat{\boldsymbol{X}}} \leq \varepsilon$ on $\widehat{\boldsymbol{X}}$.

[?] We do not know whether $\boldsymbol{h}^*_{\boldsymbol{X}, \widehat{\boldsymbol{X}}} \equiv 0$ for arbitrary Riemann domains [?]

5.2 Reinhardt case

[>] §§ 2.9, 3.1.

As a geometric motivation for the cross problem, consider a special situation where $D_j \subset \mathbb{C}^{n_j}$ is a Reinhardt domain and $A_j \subset D_j$ is a non-empty Reinhardt open set, $j = 1, \dots, N$, $N \geq 2$ (cf. § 2.9). Observe that in this case $\boldsymbol{X}$ is a Reinhardt domain in $\mathbb{C}^n$ with $n := n_1 + \cdots + n_N$. The Hartogs theorem implies that every $f \in \mathcal{O}_s(\boldsymbol{X})$ is holomorphic on $A_1 \times \cdots \times A_N$. Thus, by Proposition 1.1.10, f is holomorphic on $\boldsymbol{X}$ and consequently, on the envelope of holomorphy of $\boldsymbol{X}$ (cf. § 2.9).

Theorem 5.2.1. *Let* $\varnothing \neq A_j \subset D_j \subset \mathbb{C}^{n_j}$*, where* D_j *is a Reinhardt domain of holomorphy and* A_j *is a Reinhardt open set,* $j = 1, \dots, N$*,* $N \geq 2$*. Then* $\widehat{\boldsymbol{X}}$ *is the envelope of holomorphy of* $\boldsymbol{X}$*.*

Proof. Let $\widetilde{\boldsymbol{X}}$ be the envelope of holomorphy of $\boldsymbol{X}$. Since $\widehat{\boldsymbol{X}}$ is a domain of holomorphy containing $\boldsymbol{X}$ (cf. Remarks 5.1.6 (e), 5.1.8 (e), (d)), we must have $\widetilde{\boldsymbol{X}} \subset \widehat{\boldsymbol{X}}$.

On the other hand, by Propositions 3.1.3 and 3.4.1, $\log \widehat{\boldsymbol{X}} = \operatorname{conv}(\log \boldsymbol{X}) = \log \widetilde{\boldsymbol{X}}$. Thus, using Theorem 2.9.2 (for $\widehat{\boldsymbol{X}}$) and Corollary 2.9.4 (for $\widetilde{\boldsymbol{X}}$), we only need to show that if $\boldsymbol{V}_s \cap \widehat{\boldsymbol{X}} \neq \varnothing$, then $\boldsymbol{V}_s \cap \boldsymbol{X} \neq \varnothing$, $s = 1, \dots, n$, $n := n_1 + \dots + n_N$.

Indeed, let for example $a = (a_1, \dots, a_N) \in \boldsymbol{V}_n \cap \widehat{\boldsymbol{X}} \neq \varnothing$. Take arbitrary $b_j \in A_j$, $j = 1, \dots, N-1$. Then $(b_1, \dots, b_{N-1}, a_N) \in \boldsymbol{V}_n \cap \boldsymbol{X}$. $\square$

5.3 Separately polynomial functions

This section is entirely based on [Sic 1995].

Before we discuss the general case (in § 5.4), we consider the case where $D_j = \mathbb{C}^{n_j}$, $j = 1, \dots, N$, and we are interested only in extension of *separately polynomial* functions on $\boldsymbol{X} := \mathbb{X}((A_j, \mathbb{C}^{n_j})_{j=1}^N)$ ($N \geq 2$), i.e. of those functions $f : \boldsymbol{X} \to \mathbb{C}$ for which

$$\mathbb{C}^{n_j} \ni z_j \mapsto f(a'_j, z_j, a''_j) \in \mathbb{C}$$

is a polynomial mapping for any $(a_1, \dots, a_N) \in A_1 \times \dots \times A_N$ and $j \in \{1, \dots, N\}$. Observe that $\widehat{\boldsymbol{X}} = \mathbb{C}^n$ with $n := n_1 + \dots + n_N$ (cf. Proposition 3.2.3). We ask whether there exists a (unique) polynomial $\hat{f} \in \mathcal{P}(\mathbb{C}^n)$ such that $\hat{f} = f$ on $\boldsymbol{X}$. Let $\mathcal{SP}(\boldsymbol{X})$ denote the space of all separately polynomial functions.

Definition 5.3.1. We say that a set $A \subset \mathbb{C}^n$ is a *determining set for polynomials* if for every $f \in \mathcal{P}(\mathbb{C}^n)$ if $f = 0$ on A, then $f \equiv 0$. We say that a set $A \subset \mathbb{C}^n$ is a *strongly determining set for polynomials* if for every representation $A = \bigcup_{s=1}^\infty A_s$ with $A_s \subset A_{s+1}$, $s \in \mathbb{N}$, there exists an s_0 such that A_{s_0} is a determining set for polynomials.

Remark 5.3.2. (a) A set $A \subset \mathbb{C}^n$ is determining for polynomials iff there is no $f \in \mathcal{P}(\mathbb{C}^n)$, $f \not\equiv 0$, such that $A \subset f^{-1}(0)$.

(b) A set $A \subset \mathbb{C}^n$ is strongly determining for polynomials iff there is no $f_s \in \mathcal{P}(\mathbb{C}^n)$, $f_s \not\equiv 0$, $s \in \mathbb{N}$, such that $A \subset \bigcup_{s=1}^\infty f_s^{-1}(0)$.

In particular, if $A \subset \mathbb{C}^n$ is non-pluripolar, then A is strongly determining for polynomials.

A set $A \subset \mathbb{C}$ is strongly determining for polynomials iff A is uncountable. The set $\{1/k : k \in \mathbb{N}\} \subset \mathbb{C}$ is determining, but not strongly determining.

(c) A set $B \times C \subset \mathbb{C}^p \times \mathbb{C}^q$ is determining for polynomials iff B and C are determining for polynomials.

(d) A set $B \times C \subset \mathbb{C}^p \times \mathbb{C}^q$ is strongly determining for polynomials iff B and C are strongly determining for polynomials.

Indeed, it is clear that if $B \times C$ is strongly determining, then B and C are strongly determining. Conversely, suppose that $B \times C \subset \bigcup_{s=1}^{\infty} f_s^{-1}(0)$ with $f_s \in \mathcal{P}(\mathbb{C}^p \times \mathbb{C}^q)$, $f_s \not\equiv 0$, $s \in \mathbb{N}$. Then $V_s := \{z \in \mathbb{C}^p : f_s(z, \cdot) \equiv 0\}$ is a (proper) algebraic set, $s \in \mathbb{N}$. Consequently, there exists a point $z_0 \in B \setminus \bigcup_{s=1}^{\infty} V_s$. Since $C \subset \bigcup_{s=1}^{\infty}\{w \in \mathbb{C}^q : f_s(z_0, w) = 0\}$, we get a contradiction.

(e) $A \subset \mathbb{C}^n$ is determining for polynomials iff for every $s \in \mathbb{N}$ there exist $\zeta_{s,j} \in A$ and $L_{s,j} \in \mathcal{P}_s(\mathbb{C}^n)$, $j = 1, \dots, d(s) := \binom{s+n}{n} = \dim \mathcal{P}_s(\mathbb{C}^n)$ such that $L_{s,j}(\zeta_{s,k}) = \delta_{j,k}$, $j, k = 1, \dots, d(s)$, and $f = \sum_{j=1}^{d(s)} f(\zeta_{s,j}) L_{s,j}$ for every $f \in \mathcal{P}_s(\mathbb{C}^n)$.

Indeed, if f is as above and $f = 0$ on A, then $f \equiv 0$. Thus the above condition is sufficient for A to be determining.

Now suppose that A is determining, fix an s, and let $e_j \in \mathcal{P}_s(\mathbb{C}^n)$, $j = 1, \dots, d := d(s)$, be an arbitrary basis of the space $\mathcal{P}_s(\mathbb{C}^n)$. Define

$$V_r(z_1, \dots, z_r) := \det[e_j(z_k)]_{j,k=1,\dots,r}, \quad z_1, \dots, z_r \in \mathbb{C}^n,\ r = 1, \dots, d.$$

Since A is determining for polynomials, there exists a $\zeta_1 \in A$ with $e_1(\zeta_1) = V_1(\zeta_1) \neq 0$. Suppose that for some $t \in \{1, \dots, d-1\}$ we have already constructed $\zeta_1, \dots, \zeta_t \in A$ in such a way that $V_t(\zeta_1, \dots, \zeta_t) \neq 0$. Observe that the function

$$\mathbb{C}^n \ni z \overset{f}{\mapsto} V_{t+1}(\zeta_1, \dots, \zeta_t, z) = V_t(\zeta_1, \dots, \zeta_t) e_{t+1}(z) + c_t e_t(z) + \dots + c_1 e_1(z)$$

is a non-zero polynomial. Since A is determining for polynomials, there exists a $\zeta_{t+1} \in A$ such that $f(\zeta_{t+1}) \neq 0$.

Finally, we put

$$L_j(z) := \frac{V_d(\zeta_1, \dots, \zeta_{j-1}, z, \zeta_{j+1}, \zeta_d)}{V_d(\zeta_1, \dots, \zeta_d)}, \quad z \in \mathbb{C}^n,\ j = 1, \dots, d.$$

It is clear that $L_j(\zeta_k) = \delta_{j,k}$, $j, k = 1, \dots, d$. Take an $f \in \mathcal{P}_s(\mathbb{C}^n)$. Then f and $\sum_{j=1}^{d} f(\zeta_j) L_j$ are two polynomials from $\mathcal{P}_s(\mathbb{C}^n)$ with the same value at ζ_k for $k = 1, \dots, d$. Thus, in order, to prove that $f = \sum_{j=1}^{d} f(\zeta_j) L_j$, we only need to observe that if $g = \sum_{j=1}^{d} \lambda_j e_j \in \mathcal{P}_s(\mathbb{C}^n)$ and $g(\zeta_k) = 0$, $k = 1, \dots, d$, then $\lambda_j = 0$, $j = 1, \dots, d$ (which follows from Cramer's formulas because $0 = g(\zeta_k) = \sum_{j=1}^{d} \lambda_j e_j(\zeta_k)$, $k = 1, \dots, d$).

(f) If $A \subset \mathbb{C}^n$ is determining, then there exists a $g \in \mathcal{O}(\mathbb{C}^n)$ such that $g|_A$ can not be continued to a polynomial of n-variables.

Indeed, suppose that the result is not true. Let

$$F_s := \{g \in \mathcal{O}(\mathbb{C}^n) : g|_A \text{ can be extended to an } f_g \in \mathcal{P}_s(\mathbb{C}^n)\}$$

(note that f_g is uniquely determined). Obviously, F_s is a vector space, $F_s \subset F_{s+1}$, $s \in \mathbb{N}$, and $\mathcal{O}(\mathbb{C}^n) = \bigcup_{s=1}^{\infty} F_s$. Observe that F_s is closed in the Fréchet space

$\mathcal{O}(\mathbb{C}^n)$ (in the topology of locally uniform convergence). For, let $(g_k)_{k=1}^{\infty} \subset F_s$, $g_k \to g$ locally uniformly in $\mathbb{C}^p$. Then, using (e), one can easily conclude that $(f_{g_k})_{k=1}^{\infty} \subset \mathcal{P}_s(\mathbb{C}^n)$ is also locally uniformly convergent.

Thus, by Baire's theorem, there exists an s_0 such that $F_{s_0} = \mathcal{O}(\mathbb{C}^n)$. In particular, for every $g \in \mathcal{P}(\mathbb{C}^n)$ with $\deg g > s_0$, we have $h := g - f_g \in \mathcal{P}(\mathbb{C}^n)$, $h \not\equiv 0$, and $h = 0$ on A; a contradiction.

Theorem 5.3.3 ([Sic 1995]). *Let $A_j \subset \mathbb{C}^{n_j}$, $j = 1, \dots, N$. Then the following conditions are equivalent:*

(i) *there exists a $j_0 \in \{1, \dots, N\}$ such that $A_1, \dots, A_{j_0-1}, A_{j_0+1}, \dots, A_N$ are strongly determining and A_{j_0} is determining for polynomials (e.g. $A_1, \dots, A_N$ are non-pluripolar);*

(ii) *for every $f \in \mathcal{SP}(\boldsymbol{X})$ there exists exactly one $\hat{f} \in \mathcal{P}(\mathbb{C}^n)$, $n := n_1 + \dots + n_N$, such that $\hat{f} = f$ on $\boldsymbol{X}$.*

Proof. (i) $\Longrightarrow$ (ii): Since $A_1 \times \dots \times A_N$ is a determining set (Remark 5.3.2 (c)), the polynomial $\hat{f}$ must be uniquely determined.

We apply induction on N.

First consider the case $N = 2$. Write $p := n_1$, $q := n_2$, $A := A_1$, $B := A_2$. We may assume that $j_0 = 2$, i.e. A is strongly determining and B is determining for polynomials. Take an $f \in \mathcal{SP}(\boldsymbol{X})$. Let $A_s := \{z \in A : \deg f(z, \cdot) \le s\}$, $s \in \mathbb{N}$. Then $A_s \subset A_{s+1}$, $s \in \mathbb{N}$, and $A = \bigcup_{s=1}^{\infty} A_s$. Since A is strongly determining, there exists an s_0 such that A_{s_0} is determining. Put $d = d(s_0) = \binom{q+s_0}{q}$. Let $\zeta_j \in B$ and $L_j \in \mathcal{P}_{s_0}(\mathbb{C}^q)$, $j = 1, \dots, d$, be such that $L_j(\zeta_k) = \delta_{j,k}$, $j, k = 1, \dots, d$, and $g = \sum_{j=1}^{d} g(\zeta_j) L_j$ for every $g \in \mathcal{P}_{s_0}(\mathbb{C}^q)$ (cf. Remark 5.3.2 (e)). Define

$$\hat{f}(z, w) := \sum_{j=1}^{d} f(z, \zeta_j) L_j(w), \quad (z, w) \in \mathbb{C}^p \times \mathbb{C}^q.$$

Then $\hat{f} \in \mathcal{P}(\mathbb{C}^p \times \mathbb{C}^q)$ and $\hat{f} = f$ on $A_{s_0} \times \mathbb{C}^q$. Since A_{s_0} is determining, we conclude that also $\hat{f} = f$ on $\mathbb{C}^p \times B$, which finishes the proof in the case $N = 2$.

Now let $N \ge 3$ and suppose that the result is true for $N - 1$. We may assume that $j_0 = N$. Define $\boldsymbol{Y} := \mathbb{X}((A_j, \mathbb{C}^{n_j})_{j=1}^{N-1})$. Obviously

$$\boldsymbol{X} = (\boldsymbol{Y} \times A_N) \cup (A_N' \times \mathbb{C}^{n_N})$$

(recall that $A_N' := A_1 \times \dots \times A_{N-1}$). Fix an $f \in \mathcal{SP}(\boldsymbol{X})$. Then $f(\cdot, a_N) \in \mathcal{SP}(\boldsymbol{Y})$ for every $a_N \in A_N$. Consequently, by inductive assumption, for every $a_N \in A_N$, there exists a polynomial $\hat{f}_{a_N} \in \mathcal{P}(\mathbb{C}^p)$ with $p := n_1 + \dots + n_{N-1}$ such that $\hat{f}_{a_N} = f(\cdot, a_N)$ on $\boldsymbol{Y}$.

Let $q := n_N$, consider the 2-fold cross $\boldsymbol{Z} := (A'_N \times \mathbb{C}^q) \cup (\mathbb{C}^p \times A_N)$, and define

$$g : \boldsymbol{Z} \to \mathbb{C}, \quad g(z', z_N) := \begin{cases} f(z', z_N) & \text{for } (z', z_N) \in A'_N \times \mathbb{C}^q, \\ f_{z_N}(z') & \text{for } (z', z_N) \in \mathbb{C}^p \times A_N. \end{cases}$$

Observe that g is well defined and $g \in \mathcal{SP}(\boldsymbol{Z})$. Moreover, by Remark 5.3.2 (d), the set A'_N is strongly determining for polynomials. Thus, using the case $N = 2$, we get an $\hat{f} \in \mathcal{P}(\mathbb{C}^p \times \mathbb{C}^q)$ such that $\hat{f} = g$ on $\boldsymbol{Z}$. In particular, $\hat{f} = f$ on $\boldsymbol{X}$.

(ii) $\Longrightarrow$ (i): First observe that $\boldsymbol{X}$ must be determining for polynomials, because otherwise the zero function $0 \in \mathcal{SP}(\boldsymbol{X})$ would have two different polynomial continuations. Consequently, there exists a j such that $\mathfrak{X}_j = A'_j \times \mathbb{C}^{n_j} \times A''_j$ is determining. Hence, by Remark 5.3.2 (c), $A_1, \dots, A_{j-1}, A_{j+1}, \dots, A_N$ are determining. We may assume that $j = N$. Suppose that A_N is not determining and let $f_N \in \mathcal{P}(\mathbb{C}^{n_N})$, $f_N \not\equiv 0$, be such that $A_N \subset f_N^{-1}(0)$. Since A'_N is determining, there exists a $g \in \mathcal{O}(\mathbb{C}^p)$ with $p := n_1 + \dots + n_{N-1}$ such that $g|_{A'_N}$ cannot be extended to a global polynomial of p-variables (cf. Remark 5.3.2(f)). Put $f(z', z_N) := g(z') f_N(z_N)$, $(z', z_N) \in \mathbb{C}^p \times \mathbb{C}^{n_N}$. Then $f|_{\boldsymbol{X}} \in \mathcal{SP}(\boldsymbol{X})$. By (ii), there exists an $\hat{f} \in \mathcal{P}(\mathbb{C}^m)$ such that $\hat{f} = f$ on $\boldsymbol{X}$. In particular, if $b_N \in \mathbb{C}^{n_N}$ is such that $f_N(b_N) \neq 0$, then the function $\mathbb{C}^p \ni z' \mapsto \hat{f}(z', b_N)/f_N(b_N)$ gives a polynomial extension of g; a contradiction.

Now suppose that there are indices $j < k$ such that A_j and A_k are not strongly determining. We may assume that $j = N-1$, $k = N$. Let $j \in \{N-1, N\}$. There exists a representation $A_j = \bigcup_{s=1}^{\infty} A_{j,s}$ with $A_{j,s} \subset A_{j,s+1}$ such that each set $A_{j,s}$ is not determining. Let $f_{j,s} \in \mathcal{P}(\mathbb{C}^{n_j})$, $f_{j,s} \not\equiv 0$, be such that $A_{j,s} \subset f_{j,s}^{-1}(0)$. Put $g_{j,s} := f_{j,1} \cdots f_{j,s}$. Observe that $g_{j,s} \in \mathcal{P}(\mathbb{C}^{n_j})$, $g_{j,s} \not\equiv 0$, $\deg g_{j,s} \geq s$, and $A_{j,t} \subset g_{j,s}^{-1}(0)$ for $t \leq s$. Since A_j is determining, for each $s \in \mathbb{N}$, there exists a $b_{j,s} \in A_j$ such that $g_{j,s}(b_{j,s}) \neq 0$. Choose $\varepsilon_s > 0$ so that the series

$$\hat{f}(z_{N-1}, z_N) := \sum_{s=1}^{\infty} \varepsilon_s g_{N-1,s}(z_{N-1}) g_{N,s}(z_N)$$

converges locally uniformly in $\mathbb{C}^{n_{N-1}} \times \mathbb{C}^{n_N}$. The series defines an entire function $\hat{f} \in \mathcal{O}(\mathbb{C}^{n_{N-1}} \times \mathbb{C}^{n_N}) \subset \mathcal{O}(\mathbb{C}^n)$. Observe that $f := \hat{f}|_{\boldsymbol{X}} \in \mathcal{SP}(\boldsymbol{X})$.

Indeed, let e.g. $a_{N-1} \in A_{N-1}$ and put

$$\varkappa(a_{N-1}) := \sup\{s \in \mathbb{N} : g_{N-1,s}(a_{N-1}) \neq 0\}.$$

We know that if $a_{N-1} \in A_{N-1,t}$, then $\varkappa(a_{N-1}) < t$. This shows that

$$f(a_{N-1}, \cdot) = \sum_{s=1}^{\varkappa(a_{N-1})} \varepsilon_s g_{N-1,s}(a_{N-1}) g_{N,s}$$

is a polynomial and $\deg f(a_{N-1}, \cdot) = \deg g_{N,\varkappa(a_{N-1})} \geq \varkappa(a_{N-1})$.

Note that $\varkappa(b_{N-1,s}) \geq s$. Consequently, f cannot be extended to a global polynomial of n-variables; a contradiction. □

Corollary 5.3.4. *Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and let $f : \mathbb{K}^{n_1} \times \cdots \times \mathbb{K}^{n_N} \to \mathbb{C}$ be such that for any $(a_1, \dots, a_N) \in \mathbb{K}^{n_1} \times \cdots \times \mathbb{K}^{n_N}$ and $j \in \{1, \dots, N\}$, the function*

$$\mathbb{K}^{n_j} \ni x_j \mapsto f(a'_j, x_j, a''_j) \in \mathbb{C}$$

is a polynomial of n_j variables (i.e. f is a polynomial with respect to each group of variables separately). Then $f \in \mathcal{P}(\mathbb{K}^{n_1} \times \cdots \times \mathbb{K}^{n_N})$.

Proof. If $\mathbb{K} = \mathbb{C}$, then we may apply directly Theorem 5.3.3 to the cross $\boldsymbol{X} = \mathbb{X}((\mathbb{C}^{n_j}, \mathbb{C}^{n_j})_{j=1}^N) = \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_N}$. If $\mathbb{K} = \mathbb{R}$, then our assumption implies that f extends to an $\tilde{f} \in \mathcal{SP}(\boldsymbol{X})$ with $\boldsymbol{X} = \mathbb{X}((\mathbb{R}^{n_j}, \mathbb{C}^{n_j})_{j=1}^N)$ and we may once again apply Theorem 5.3.3. □

Remark 5.3.5. Instead of separately polynomial functions one may also study separately rational or algebraic functions.

Let $\Omega \subset \mathbb{K}^n$ be open and let $f : \Omega \to \mathbb{K}$.

We say that f is *rational* if there exist polynomials $P, Q : \mathbb{K}^n \to \mathbb{K}$, $Q \not\equiv 0$, such that $Qf - P \equiv 0$ on Ω.

The function f is said to be *algebraic* if there exist $k \in \mathbb{N}$ and polynomials $P_0, \dots, P_k : \mathbb{K}^n \to \mathbb{K}$, $P_k \not\equiv 0$, such that $P_k f^k + P_{k-1} f^{k-1} + \cdots + P_0 \equiv 0$ on Ω.

We say that f is *separately rational* (resp. *algebraic*) if for any $(a_1, \dots, a_n) \in \Omega$ and $j \in \{1, \dots, n\}$, the function $x_j \mapsto f(a_1, \dots, a_{j-1}, x_j, a_{j+1}, \dots, a_n)$ is rational (resp. algebraic) on the open set $\{x_j \in \mathbb{K} : (a_1, \dots, a_{j-1}, x_j, a_{j+1}, \dots, a_n) \in \Omega\}$.

One may ask *when a separately rational* (*resp. algebraic*) *function $f : \Omega \to \mathbb{K}$ is rational* (*resp. algebraic*).

The following results are known (cf. [Kno 1969], [Kno 1971], [Kno 1974]):

- If $\mathbb{K} = \mathbb{R}$, then every separately rational and separately locally bounded function is globally rational.
- If $\mathbb{K} = \mathbb{R}$, then every separately algebraic and separately $\mathcal{C}^\infty$ function is globally algebraic.
- For $r \in \mathbb{Z}_+$, the function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x, y) := \begin{cases} \big((x-k)(x-(k+1))(y-k)(y-(k+1))\big)^{k+r+1} \\ \qquad \text{if } k < x < k+1,\ k < y < k+1,\ k \in \mathbb{Z}_+, \\ 0 \qquad \text{otherwise}, \end{cases}$$

  is separately algebraic and globally $\mathcal{C}^r$, but not globally algebraic.
- If $\mathbb{K} = \mathbb{C}$, then every separately algebraic and separately continuous function is globally algebraic.

- The function $f : \mathbb{C}^2 \to \mathbb{C}$,

$$f(z, w) := \begin{cases} (zw)^k & \text{if } k \le |z| < k+1,\ k \le |w| < k+1,\ k \in \mathbb{Z}_+, \\ 0 & \text{otherwise,} \end{cases}$$

  is separately algebraic but not globally algebraic.

See also [Sic 1962], [STW 1990].

5.4 Main cross theorem

$\boxed{>}$ §§ 2.1.5, 2.1.7.

The problem of holomorphic continuation of separately holomorphic functions defined on N-fold crosses has been investigated in several papers, e.g. [Ber 1912], [Cam-Sto 1966], [Sic 1968], [Sic 1969a], [Sic 1969b], [Akh-Ron 1973], [Zah 1976], [Sic 1981a], [Shi 1989], [NTV-Sic 1991], [NTV-Zer 1991], [NTV-Zer 1995], [NTV 1997], [Ale-Zer 2001], [Zer 2002] and has led to the following result.

Theorem 5.4.1 (Main cross theorem). *Assume that D_j is a Riemann domain over $\mathbb{C}^{n_j}$ and $A_j \subset D_j$ is locally pluriregular, $j = 1, \dots, N$. Put $\boldsymbol{X} := \mathbb{X}((A_j, D_j)_{j=1}^N)$. Let $f \in \mathcal{O}_s(\boldsymbol{X})$. Then*

(∗) *there exists a uniquely determined $\hat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}})$ such that $\hat{f} = f$ on $\boldsymbol{X}$ and $\hat{f}(\widehat{\boldsymbol{X}}) \subset f(\boldsymbol{X})$; in particular*

- $\|\hat{f}\|_{\widehat{\boldsymbol{X}}} = \|f\|_{\boldsymbol{X}}$,
- $|\hat{f}(z)| \le \|f\|_{\boldsymbol{c}(\boldsymbol{X})}^{1-\sum_{j=1}^N \boldsymbol{h}^*_{A_j,D_j}(z_j)} \|f\|_{\boldsymbol{X}}^{\sum_{j=1}^N \boldsymbol{h}^*_{A_j,D_j}(z_j)}$, $z = (z_1, \dots, z_N) \in \widehat{\boldsymbol{X}}$.

A proof will be presented in § 5.4.2. Notice that the inclusion $\hat{f}(\widehat{\boldsymbol{X}}) \subset f(\boldsymbol{X})$ follows immediately from Lemma 2.1.14 with

$$(G, D, A_0, A, \mathcal{F}) = (\widehat{\boldsymbol{X}}, \widehat{\boldsymbol{X}}, \boldsymbol{c}(\boldsymbol{X}), \boldsymbol{X}, \mathcal{O}_s(\boldsymbol{X})).$$

To get the inequality we use Lemma 3.2.5 and Proposition 3.2.28 ($\boldsymbol{h}^*_{\boldsymbol{c}(\boldsymbol{X}),\widehat{\boldsymbol{X}}}(z) = \sum_{j=1}^N \boldsymbol{h}^*_{A_j,D_j}(z_j)$, $z = (z_1, \dots, z_N) \in \widehat{\boldsymbol{X}}$).

Sometimes the assumption that $A_1, \dots, A_N$ are locally pluriregular is too restrictive and it is better to consider the following equivalent form of Theorem 5.4.1.

Theorem 5.4.2 (Main cross theorem). *Let D_j be as in Theorem 5.4.1 and let $A_j \subset D_j$ be non-pluripolar, $j = 1, \dots, N$. Put*

$$\boldsymbol{X} := \mathbb{X}((A_j, D_j)_{j=1}^N), \quad \boldsymbol{Y} := \mathbb{X}((A_j \cap A_j^*, D_j)_{j=1}^N)$$

(*recall that $\widehat{\boldsymbol{Y}} = \widehat{\boldsymbol{X}}$ – cf. Remark* 5.1.8 (f)). *Let $f \in \mathcal{O}_s(\boldsymbol{X})$. Then*

(∗∗) *there exists a uniquely determined* $\hat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}})$ *such that* $\hat{f} = f$ *on* $\boldsymbol{Y}$ *and* $\hat{f}(\widehat{\boldsymbol{X}}) \subset f(\boldsymbol{X})$*; in particular*

- $\|\hat{f}\|_{\widehat{\boldsymbol{X}}} \le \|f\|_{\boldsymbol{X}}$,
- $|\hat{f}(z)| \le \|f\|_{\boldsymbol{c}(\boldsymbol{X})}^{1-\sum_{j=1}^{N} \boldsymbol{h}^*_{A_j^*,D_j}(z_j)} \|f\|_{\boldsymbol{X}}^{\sum_{j=1}^{N} \boldsymbol{h}^*_{A_j^*,D_j}(z_j)}$, $z = (z_1, \dots, z_N) \in \widehat{\boldsymbol{X}}$.

As above, the inclusion $\hat{f}(\widehat{\boldsymbol{X}}) \subset f(\boldsymbol{X})$ follows from Lemma 2.1.14 with

$$(G, D, A_0, A, \mathcal{F}) = (D_1 \times \cdots \times D_N, \widehat{\boldsymbol{X}}, \boldsymbol{c}(\boldsymbol{Y}), \boldsymbol{X}, \mathcal{O}_s(\boldsymbol{X})).$$

To get the inequality we use Lemma 3.2.5 and Proposition 3.2.28 ($\boldsymbol{h}^*_{\boldsymbol{c}(\boldsymbol{Y}),\widehat{\boldsymbol{X}}}(z) = \sum_{j=1}^{N} \boldsymbol{h}^*_{A_j \cap A_j^*,D_j}(z_j) = \sum_{j=1}^{N} \boldsymbol{h}^*_{A_j^*,D_j}(z_j)$).

It is obvious that Theorem 5.4.2 $\Longrightarrow$ Theorem 5.4.1. Conversely, since $A_j \cap A_j^*$ is locally pluriregular (cf. Corollary 3.2.13), $j = 1, \dots, N$, Theorem 5.4.1 implies that there exists an $f \in \mathcal{O}(\widehat{\boldsymbol{X}})$ such that $\hat{f} = f$ on $\boldsymbol{Y}$.

Exercise 5.4.3. Assuming that Theorems 5.4.1, 5.4.2 are true, prove analogous results for Riemann *regions* $D_j \in \mathfrak{R}_\infty(\mathbb{C}^{n_j})$, $j = 1, \dots, N$.

Remark 5.4.4. Let D_j, A_j, $j = 1, \dots, N$, $\boldsymbol{X}$, and $\widehat{\boldsymbol{X}}$ be as in Theorem 5.4.2.

We present below some procedures which allow us to reduce the proofs of Theorems 5.4.1 and 5.4.2 to some special configurations.

(P1) (Cf. Proposition 7.2.6.) Let $\varphi_j : D_j \to \widetilde{D}_j$ be the envelope of holomorphy (cf. Definition 2.1.18). Observe that $\widetilde{A}_j := \varphi_j(A_j) \subset \widetilde{D}_j$ is non-pluripolar (because φ_j is locally biholomorphic; cf. Remark 2.1.11 (b)), $j = 1, \dots, N$. Put

$$\varphi : D_1 \times \cdots \times D_N \to \widetilde{D}_1 \times \cdots \times \widetilde{D}_N, \quad \varphi(z_1, \dots, z_N) := (\varphi_1(z_1), \dots, \varphi_N(z_N)).$$

Let

$$\boldsymbol{Y} := \mathbb{X}((\widetilde{A}_j, \widetilde{D}_j)_{j=1}^N), \quad \widehat{\boldsymbol{Y}} := \widehat{\mathbb{X}}((\widetilde{A}_j, \widetilde{D}_j)_{j=1}^N).$$

Then

- $\varphi(\boldsymbol{X}) \subset \boldsymbol{Y}$, $\varphi(\widehat{\boldsymbol{X}}) \subset \widehat{\boldsymbol{Y}}$,
- for each function $f \in \mathcal{O}_s(\boldsymbol{X})$ there exists a function $\tilde{f} \in \mathcal{O}_s(\boldsymbol{Y})$ such that $\tilde{f} \circ \varphi \equiv f$.

Indeed, the inclusion $\varphi(\boldsymbol{X}) \subset \boldsymbol{Y}$ is trivial. The inclusion $\varphi(\widehat{\boldsymbol{X}}) \subset \widehat{\boldsymbol{Y}}$ follows immediately from the fact that $\boldsymbol{h}^*_{\varphi_j(A_j),\widetilde{D}_j} \circ \varphi_j \le \boldsymbol{h}^*_{A_j,D_j}$, $j = 1, \dots, N$.

Fix an $f \in \mathcal{O}_s(\boldsymbol{X})$. Take $a = (a_1, \dots, a_N), b = (b_1, \dots, b_N) \in A_1 \times \cdots \times A_N$. First observe that if $\varphi_j(a_j) = \varphi_j(b_j)$, $j = 1, \dots, N-1$, then $f(a'_N, \cdot) \equiv f(b'_N, \cdot)$ on D_N. Indeed, since $\varphi_j : D_j \to \widetilde{D}_j$ is the envelope of holomorphy, for each $g_j \in \mathcal{O}(D_j)$ there exists a $\tilde{g}_j \in \mathcal{O}(\widetilde{D}_j)$ such that $g_j \equiv \tilde{g}_j \circ \varphi_j$. In particular, if $\varphi(z_j) = \varphi(w_j)$, then

$g_j(z_j) = g_j(w_j)$. Take an arbitrary $c_N \in A_N$. We have: $f(a_1, \dots, a_{N-1}, c_N) = f(b_1, a_2, \dots, a_{N-1}, c_N) = \dots = f(b_1, \dots, b_{N-1}, c_N)$. Thus $f(a'_N, \cdot) = f(b'_N, \cdot)$ on A_N. It remains to use the identity principle.

Consequently, the formula

$$\tilde{f}_N(\varphi_1(a_1), \dots, \varphi_{N-1}(a_{N-1}), \cdot) := (\varphi_N^*)^{-1}(f(a_1, \dots, a_{N-1}, \cdot))$$

defines a function on $\widetilde{\mathfrak{X}}_N := \widetilde{A}_1 \times \dots \times \widetilde{A}_{N-1} \times \widetilde{D}_N$ with $\tilde{f}_N \circ \varphi = f$ on $\mathfrak{X}_N := A_1 \times \dots \times A_{N-1} \times D_N$.

The same construction may be repeated for other groups of variables and leads to the required function $\tilde{f}$.

Consequently, we may always additionally assume that $D_1, \dots, D_N$ are Riemann domains of holomorphy.

(P2) Assume that $D_{j,k} \nearrow D_j$, $D_{j,k} \supset A_{j,k} \nearrow A_j$, and each set $A_{j,k}$ is non-pluripolar, $j = 1, \dots, N$. Put

$$\boldsymbol{X}_k := \mathbb{X}((A_{j,k}, D_j)_{j=1}^N), \quad \boldsymbol{Y}_k := \mathbb{X}((A_{j,k} \cap A_{j,k}^*, D_j)_{j=1}^N).$$

Observe that $\boldsymbol{Y}_k \subset \boldsymbol{X}_k \nearrow \boldsymbol{X}$ and $\widehat{\boldsymbol{Y}}_k = \widehat{\boldsymbol{X}}_k \nearrow \widehat{\boldsymbol{X}}$ (cf. Remark 5.1.8 (c), (f)). Suppose that $(**)$ holds for each k, i.e. there exists an $\hat{f}_k \in \mathcal{O}(\widehat{\boldsymbol{X}}_k)$ with $\hat{f}_k = f$ on $\boldsymbol{Y}_k$. Then $(*)$ is true.

Indeed, since $\boldsymbol{Y}_k \notin \mathcal{PLP}$, we get $\hat{f}_{k+1} = \hat{f}_k$ on $\widehat{\boldsymbol{X}}_k$. Thus we get a function $\hat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}})$ such that $\hat{f} = f$ on each $\boldsymbol{Y}_k$. It remains to use Remark 5.1.8 (h) to show that $\hat{f} = f$ on every $\boldsymbol{X}_k$ and hence $\hat{f} = f$ on $\boldsymbol{X}$.

Consequently, we may always additionally assume that:

- *$D_1, \dots, D_N$ are strongly pseudoconvex (cf. § 2.5.4),*
- *$A_j \subset\subset D_j$, $j = 1, \dots, N$,*
- *the function $f \in \mathcal{O}_s(\boldsymbol{X})$ is such that for all $j \in \{1, \dots, N\}$ and $(a'_j, a''_j) \in A'_j \times A''_j$, the function $f(a'_j, \cdot, a''_j)$ is holomorphic in a neighborhood of $\overline{D}_j$.*

(P3) *We may additionally assume that $N = 2$.*

Indeed, we proceed by induction on $N \geq 2$. Suppose that $(*)$ holds for $N - 1 \geq 2$. Put $\boldsymbol{Y} := \mathbb{X}((A_j, D_j)_{j=1}^{N-1})$, $\boldsymbol{Z} := \mathbb{X}(A'_N, A_N; \widehat{\boldsymbol{Y}}, D_N)$. Observe that if $z_N \in A_N$, then $f(\cdot, z_N) \in \mathcal{O}_s(\boldsymbol{Y})$. By inductive assumption there exists an $\hat{f}_{z_N} \in \mathcal{O}(\widehat{\boldsymbol{Y}})$ with $\hat{f}_{z_N} = f(\cdot, z_N)$ on $\boldsymbol{Y}$. Define $g \colon \boldsymbol{Z} \to \mathbb{C}$ by

$$g(z', z_N) := \begin{cases} \hat{f}_{z_N}(z') & \text{if } (z', z_N) \in \widehat{\boldsymbol{Y}} \times A_N, \\ f(z', z_N) & \text{if } (z', z_N) \in A'_N \times D_N. \end{cases}$$

Obviously, g is well defined and $g \in \mathcal{O}_s(\boldsymbol{Z})$. It is clear that holomorphic functions on $\widehat{\boldsymbol{Y}}$ separate points and A'_N is locally pluriregular. Using the case $N = 2$, we find an $\hat{f} \in \mathcal{O}(\widehat{\boldsymbol{Z}})$ with $\hat{f} = g$ on $\boldsymbol{Z}$. It remains to recall that $\widehat{\boldsymbol{Z}} = \widehat{\boldsymbol{X}}$ (cf. Remark 5.1.8 (g)).

The above proof shows that if $(*)$ *is true for* $N = 2$ *and all bounded* $f \in \mathcal{O}_s(\boldsymbol{X})$, *then it holds for arbitrary* N *and all bounded separately holomorphic functions.*

In the case where $N = 2$ *we will always write* $D := D_1$, $p := n_1$, $G := D_2$, $q := n_2$, $A := A_1$, $B := A_2$.

(P4) *If* $N = 2$, *then we may additionally assume that* f *is bounded.*

Indeed, we already know (by **(P2)**) that we may assume that $D_j \in \mathfrak{R}_b(\mathbb{C}^{n_j})$, $j = 1, 2$, and that for arbitrary $(a_1, a_2) \in A_1 \times A_2$ we have $f(a_1, \cdot) \in \mathcal{O}(\overline{D}_2)$, $f(\cdot, a_2) \in \mathcal{O}(\overline{D}_1)$. Define

$$A_{1,k} := \{z_1 \in A_1 : |f(z_1, \cdot)| \le k \text{ on } D_2\},$$
$$A_{2,k} := \{z_2 \in A_2 : |f(\cdot, z_2)| \le k \text{ on } D_1\}, \quad k \in \mathbb{N}.$$

Observe that $A_{j,k} \nearrow A_j$. Hence $A_{j,k} \notin \mathcal{PLP}$, $k \gg 1$, $j = 1, 2$. Observe that $|f| \le k$ on $\mathbb{X}(A_{1,k}, A_{2,k}; D_1, D_2)$, $k \in \mathbb{N}$. Now we only need to use **(P2)**.

(P5) *If* $N = 2$ *and* $f \in \mathcal{O}_s(\boldsymbol{X})$ *is bounded, then* f *is continuous on* $\boldsymbol{X}$. *If moreover,* $A \subset\subset D$, $B \subset\subset G$, *then* f *extends to a continuous function* $\tilde{f} \in \mathcal{O}_s(\boldsymbol{Z})$, *with* $\boldsymbol{Z} := \mathbb{X}(\overline{A}, \overline{B}; D, G)$.

Indeed, let $\boldsymbol{X} \ni (z_s, w_s) \to (z_0, w_0) \in \boldsymbol{X}$, $f(z_s, w_s) \to \alpha$. The sequence of holomorphic functions $(f(z_s, \cdot))_{s=1}^{\infty}$ is bounded. Consequently, by Montel's theorem, we may assume that $f(z_s, \cdot) \to g$ locally uniformly in G with $g \in \mathcal{O}(G)$. In particular, $f(z_s, w_s) \to g(w_0) = \alpha$. On the other hand, $f(z_s, w) \to f(z_0, w)$ for $w \in B$. Hence, $g = f(z_0, \cdot)$ on B. If $w_0 \in B$, then $\alpha = g(w_0) = f(z_0, w_0)$ and the proof is finished. If $z_0 \in A$, then, by the identity principle, $g = f(z_0, \cdot)$ on G, which also implies that $\alpha = g(w_0) = f(z_0, w_0)$.

Now assume additionally that $A \subset\subset D$, $B \subset\subset G$. Let $A \ni z_k \to z_0 \in \overline{A} \subset D$. By Montel's argument there exists a subsequence $(k_s)_{s=1}^{\infty}$ such that $f(z_{k_s}, \cdot) \to g$ locally uniformly in G. Observe that $g(w) = f(z_0, w)$, $w \in B$. Consequently, the function g depends neither on the subsequence $(k_s)_{s=1}^{\infty}$, nor the sequence $(z_k)_{k=1}^{\infty} \subset A$ with $z_k \to z_0$.

Thus, we get an extension $\tilde{f}_1 : \overline{A} \times G \to \mathbb{C}$ of f, such that $\tilde{f}_1(a, \cdot) \in \mathcal{O}(G)$, $a \in \overline{A}$. In the same way we get an extension $\tilde{f}_2 : D \times \overline{B} \to \mathbb{C}$ of f, such that $\tilde{f}_2(\cdot, b) \in \mathcal{O}(D)$, $b \in \overline{B}$.

It remains to observe that $\tilde{f}_1 = \tilde{f}_2$ on $\overline{A} \times \overline{B}$. Indeed, let

$$A \times B \ni (z_s, w_s) \to (z_0, w_0) \in \overline{A} \times \overline{B}, \quad f(z_s, w_s) \to \alpha.$$

We may assume that $\varphi_s : = f(z_s, \cdot) \to \tilde{f}_1(z_0, \cdot)$ locally uniformly on G, $\psi_s : = f(\cdot, w_s) \to \tilde{f}_2(\cdot, w_0)$ locally uniformly on D. Hence,

$$\tilde{f}_1(z_0, w_0) = \lim_{s \to +\infty} \varphi_s(w_s) = \alpha = \lim_{s \to +\infty} \psi_s(z_s) = \tilde{f}_2(z_0, w_0).$$

(P6) Assume that $N = 2$, D, G are relatively compact, and $A \subset\subset D$, $B \subset\subset G$ are non-pluripolar. Put

$$\boldsymbol{Y} := \mathbb{X}(A \cap A^*, B \cap B^*; D, G), \quad \boldsymbol{Z} := \mathbb{X}(\bar{A}, \bar{B}; D, G),$$
$$\boldsymbol{W} := \mathbb{X}((\bar{A})^*, (\bar{B})^*; D, G).$$

Let $f \in \mathcal{O}_s(\boldsymbol{X})$ be bounded. We know by **(P5)** that f extends to an $\tilde{f} \in \mathcal{O}_s(\boldsymbol{Z})$. Suppose that $(**)$ holds for $\boldsymbol{Z}$, i.e. there exists an $\hat{f} \in \mathcal{O}(\widehat{\boldsymbol{Z}})$ such that $\hat{f} = \tilde{f}$ on $\boldsymbol{W}$. Observe that $\boldsymbol{X} \supset \boldsymbol{Y} \subset \boldsymbol{W} \subset \boldsymbol{Z}$ and $\widehat{\boldsymbol{X}} = \widehat{\boldsymbol{Y}} \subset \widehat{\boldsymbol{W}} = \widehat{\boldsymbol{Z}}$. Thus $\hat{f}|_{\widehat{\boldsymbol{X}}}$ solves $(**)$ for $\boldsymbol{X}$.

Summarizing, we get the following result.

Proposition 5.4.5. *To prove Theorems 5.4.1, 5.4.2 in their full generality, it suffices to prove Theorem 5.4.2 under the following additional assumptions:*

- $N = 2$,
- D, G *are strongly pseudoconvex domains,*
- A, B *are compact non-pluripolar,*
- $f(a, \cdot) \in \mathcal{O}(\bar{G})$, $a \in A$, $f(\cdot, b) \in \mathcal{O}(\bar{D})$, $b \in B$,
- $|f| \leq 1$ *on* $\boldsymbol{X}$,
- f *is continuous on* $\boldsymbol{X}$.

5.4.1 Siciak's approach

In the 60s, R. H. Cameron, D. A. Storvick ([Cam-Sto 1966]) and J. Siciak ([Sic 1969a], [Sic 1969b]) initiated a theory of separately holomorphic functions on crosses. To be historically correct, one should mention that already in 1911 Bernstein (cf. [Ber 1912]) discussed the following general 2-fold cross situation: $n_1 = n_2 = 1$, $D_1 = D_2 =$ an ellipse with foci $1, -1$, $A_1 = A_2 = [-1, 1]$, $f \in \mathcal{O}_s(\mathbb{X}(A_1, A_2; D_1 D_2))$ bounded. It seems that this result had not been recognized for a long time up to a paper by Akhiezer and Ronkin (cf. [Akh-Ron 1973], see also [Ron 1977]).

The aim of this section is to present some of Siciak's results from [Sic 1969a] (see also [Dru 1977]). Notice that, although these results require special additional assumptions, their proofs have a geometric interpretation and may help to get an intuition how to prove the general cross theorem.

Theorem 5.4.6. *Let $D \subset \mathbb{C}^p$ be a domain and let $G_1, \dots, G_q \subset \mathbb{C}$ be simply connected domains symmetric with respect to the real axis $\mathbb{R}$. Assume that $A \subset D$ is locally pluriregular and $B_j = [a_j, b_j] \subset G_j \cap \mathbb{R}$, $a_j < b_j$, $j = 1, \dots, q$. Put $\boldsymbol{X} := \mathbb{X}(A, B_1, \dots, B_q; D, G_1, \dots, G_q)$. Let $f \in \mathcal{O}_s(\boldsymbol{X})$ be bounded on $\boldsymbol{X}$. Then there exists an $\hat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}})$ such that $\hat{f} = f$ on $\boldsymbol{X}$ and $\sup_{\widehat{\boldsymbol{X}}} |\hat{f}| = \sup_{\boldsymbol{X}} |f|$. If A is additionally compact, then the result remains true for locally bounded $f \in \mathcal{O}_s(\boldsymbol{X})$.*

Remark 5.4.7. (a) Notice that the assumptions that f is bounded or locally bounded are, in fact, superfluous (cf. Theorem 5.4.1).

(b) Some particular cases of Theorem 5.4.6 have been studied using different methods by several authors. Let us mention for instance [Cam-Sto 1966], [Gór 1975], [Non 1977a], [Non 1977b], [Non 1979], where the case

$$\boldsymbol{X} := ((-R_1, R_1) \times \mathbb{D}(R_2)) \cup (\mathbb{D}(R_1) \times (-R_2, R_2))$$

was discussed. Recall (Example 3.2.20 (a)) that $\boldsymbol{h}^*_{(-R,R),\mathbb{D}(R)}$ may be given by an effective formula.

(c) Notice that in the case where $\boldsymbol{X} := \mathbb{X}(((-1,1), \mathbb{D})_{j=1}^n)$ we have

$$\widehat{\boldsymbol{X}} = \Big\{(z_1, \dots, z_n) \in \mathbb{D}^n : \sum_{j=1}^n \Big|\operatorname{Arg}\frac{1+z_j}{1-z_j}\Big| < \frac{\pi}{2}\Big\};$$

cf. [Cam-Sto 1966], Theorem 1.

We need some auxiliary results, whose proofs may be found e.g. in [Gol 1983].

Lemma* 5.4.8. *Let $D \subset \mathbb{H}^+ := \{x + iy : x \in \mathbb{R},\ y > 0\}$ be a simply connected domain and let $L \subset \mathbb{R} \cap \partial D$ be an open interval. Assume that $g\colon D \to \mathbb{H}^+$ is a biholomorphic mapping. Then g extends to a continuous injective mapping $\tilde{g}\colon D \cup L \to \overline{\mathbb{H}}^+$ with $\tilde{g}(L) \subset \mathbb{R}$.*

Lemma* 5.4.9. *For every $-\infty \le c < -1 < 1 < d \le +\infty$ there exist $0 < \rho \le +\infty$ and a biholomorphic mapping $h\colon \mathbb{H}^+ \to \mathfrak{R}$ with*

$$\mathfrak{R} = \mathfrak{R}(\rho) := \{u + iv : u \in (0, \rho),\ v \in (0, \pi)\},$$

such that $\tilde{h}(c) = \rho + i\pi$, $\tilde{h}(-1) = i\pi$, $\tilde{h}(1) = 0$, $\tilde{h}(d) = \rho$, where $\tilde{h}$ denotes the extension of h to a homeomorphic mapping $\tilde{h}\colon \overline{\mathbb{H}}^+ \to \overline{\mathfrak{R}}$ (which exists by the Carathéodory–Osgood theorem).

Corollary 5.4.10. *Let $D \subset \mathbb{H}^+$ be a simply connected domain such that $(c,d) \subset \mathbb{R} \cap \partial D$ with $-\infty \le c < -1 < 1 < d \le +\infty$. Then there exist $0 < \rho \le +\infty$ and a biholomorphic mapping $g\colon \mathbb{H}^+ \to \mathfrak{R}$ with $\mathfrak{R} = \mathfrak{R}(\rho)$ that extends to a continuous injective mapping $\tilde{g}\colon D \cup (c,d) \to \overline{\mathfrak{R}}$ such that $\tilde{g}((c,-1]) = (\rho + i\pi, i\pi]$, $\tilde{g}(-1) = i\pi$, $\tilde{g}([-1,1]) = [i\pi, 0]$, $\tilde{g}(1) = 0$, $\tilde{g}([1,d)) = [0,\rho)$.*

Lemma 5.4.11. *Let $D \subset \mathbb{C}$ be a simply connected domain symmetric with respect to a line L. Let $[a,b] \subset L \cap D$, $a \ne b$. Then there exist uniquely determined $R \in (1, +\infty]$ and*

$$g\colon D \to \mathcal{E} := \{w \in \mathbb{C} : |w + \sqrt{w^2 - 1}| < R\}, \quad g \text{ biholomorphic},$$

such that $g([a,b]) = [-1,1]$, $g(a) = -1$, $g(b) = 1$, and the branch of $\sqrt{w^2-1}$ is chosen so that $\sqrt{x^2-1} > 0$ for $x \in (1, +\infty)$.

Proof. Let

$$g_1(z) := \frac{2}{b-a}\Big(z - \frac{a+b}{2}\Big), \quad z \in \mathbb{C}.$$

Then g_1 maps biholomorphically D onto the simply connected domain $D_1 := g_1(D)$ that is symmetric with respect to the real axis, $g_1([a,b]) = [-1,1]$, $g_1(a) = -1$, $g_1(b) = 1$. If $D_1 = \mathbb{C}$, then we put $R := +\infty$, $g := g_1$.

Assume that $D_1 \neq \mathbb{C}$. Let $(c,d) := D_1 \cap \mathbb{R}$ (observe that $D_1 \cap \mathbb{R}$ must be connected because D_1 is symmetric and simply connected – EXERCISE). Put

$$D_1^+ := \{z \in D_1 : \operatorname{Im} z > 0\};$$

observe that D_1^+ is a simply connected domain. Then there exist $\rho \in (0,+\infty]$ and a biholomorphic mapping

$$g_2\colon D_1^+ \to D_2 := \{u + iv : u \in (0,\rho),\ v \in (0,\pi)\}$$

such that $\tilde{g}_2((c,-1]) = (\rho + i\pi, i\pi]$, $\tilde{g}_2(-1) = i\pi$, $\tilde{g}_2([-1,1]) = [i\pi, 0]$, $\tilde{g}_2(1) = 0$, $\tilde{g}_2([1,d)) = [0,\rho)$ (Corollary 5.4.10). The mapping $g_3 := \exp$ maps biholomorphically D_2 onto the domain

$$D_3 := \{w \in \mathbb{C} : 1 < |w| < R := e^{\rho},\ \operatorname{Im} w > 0\},$$

$g_3([\rho + i\pi, i\pi] = [-R,-1]$, $g_3([i\pi,0]) = C^+ := \{\zeta \in \mathbb{T} : \operatorname{Im}\zeta \geq 0\}$, $g_3([0,\rho]) = [1,R]$.

Next, the Zhukovski mapping $g_4(z) := \frac{1}{2}(z + 1/z)$ maps D_3 onto the domain $\mathcal{E}^+ := \{w \in \mathcal{E} : \operatorname{Im} w > 0\}$, $g_4([-R,-1]) = [-\frac{1}{2}(R + 1/R), -1]$, $g_4(C^+) = [-1,1]$, $g_4([1,R]) = [1, \frac{1}{2}(R + 1/R]$.

Let $g_5 := g_4 \circ g_3 \circ g_2\colon D_1^+ \to \mathcal{E}^+$,

$$g_6(w) := \begin{cases} g_5(w), & w \in D_1^+, \\ g_4 \circ g_3 \circ \tilde{g}_2(w), & w \in (c,d), \\ \overline{g_5(\overline{w})}, & \overline{w} \in D_1^+. \end{cases}$$

Finally, $g := g_6 \circ g_1$ satisfies all the required properties.

It remains to prove that R and g are uniquely determined. Suppose that $h\colon D \to \mathcal{E}' := \{w \in \mathbb{C} : |w + \sqrt{w^2-1}| < R'\}$ is another biholomorphic mapping with the above properties. Then the biholomorphic mapping $f := g_4^{-1} \circ h \circ g^{-1} \circ g_4\colon \mathbb{A}(1,R) \to \mathbb{A}(1,R')$ with $f(\pm 1) = \pm 1$, where

$$\mathbb{A}(r_-, r_+) := \{z \in \mathbb{C} : r_- < |z| < r_+\}.$$

Consequently, $R' = R$ and $g \equiv h$. □

Corollary 5.4.12. *Assume that D is a simply connected domain symmetric with respect to the real line $\mathbb{R}$ and $[a,b] \subset D \cap \mathbb{R}$, $a < b$. Let R and g be as in Lemma 5.4.11. Then the function $\Phi(z) := g(z) + \sqrt{g^2(z) - 1}$, $z \in D \setminus [a,b]$, is the unique biholomorphic mapping of $D \setminus [a,b]$ onto $\mathbb{A}(1,R)$ such that $\Phi(a) = -1$, $\Phi(b) = 1$. Moreover:*

- $\overline{\Phi(\bar{z})} = \Phi(z)$, $z \in D \setminus [a,b]$,
- *the limits* $\Phi(x+i0) := \lim_{y\to 0+} \Phi(x+iy)$ *and* $\Phi(x-i0) := \lim_{y\to 0-} \Phi(x+iy)$ *exist for* $x \in [a,b]$,
- $\Phi(x+i0) = \overline{\Phi(x-i0)} = 1/\Phi(x-i0)$, $x \in [a,b]$,
- *the functions* $(a,b) \ni x \mapsto \Phi(x+i0)$ *and* $(a,b) \ni x \mapsto \Phi(x-i0)$ *are real analytic,*
- *the function* $m(x) := g'(x)/\sqrt{1-g^2(x)} = i\Phi'(x-i0)/\Phi(x-i0)$, $x \in (a,b)$, *is Riemann integrable and* $\int_a^b m(x)dx = \arcsin g|_a^b = \pi$.

Put

$$\Phi_k(z) := \frac{1}{2}\begin{cases} \Phi^k(z) + \Phi^{-k}(z), & z \in D \setminus [a,b], \\ \Phi^k(x-i0) + \Phi^{-k}(x-i0), & z = x \in [a,b], \end{cases} \quad k \in \mathbb{Z}_+.$$

Then

- $\Phi_k \in \mathcal{O}(D)$,
- $|\Phi_k| \le |\Phi|^k$,
- $|\Phi_k| \le 1$ *on* $[a,b]$.

Corollary 5.4.13. *Let D, a, b, R, and Φ be as in Corollary 5.4.12. Then*

$$\boldsymbol{h}^*_{[a,b],D} = \frac{\log|\Phi|}{\log R}.$$

Proof. Let $u := \frac{\log|\Phi|}{\log R}$. It is clear that $u \in \mathcal{H}(D \setminus [a,b])$ and $0 < u < 1$ on $D \setminus [a,b]$. Moreover, u is continuous on D and $u = 0$ on $[a,b]$. Hence $u \in \mathcal{SH}(D)$ and, consequently, $\boldsymbol{h}_{[a,b],D} \ge u$. Applying the maximum principle to the subharmonic function $\boldsymbol{h}^*_{[a,b],D} - u$ on $D \setminus [a,b]$, gives the converse inequality. □

Lemma 5.4.14. *Let D, a, b, R, m, Φ, and $(\Phi_k)_{k=1}^\infty$ be as in Corollary 5.4.12. Let $f \in \mathcal{O}(D)$. Then*

$$f(z) = \sum_{k=0}^{\infty} c_k \Phi_k(z), \quad z \in D, \tag{5.4.1}$$

where

$$c_k := \frac{2^{\operatorname{sgn} k}}{\pi} \int_a^b m(x) f(x) \Phi_k(x) dx, \quad k \in \mathbb{Z}_+.$$

Moreover,

- *the series converges locally uniformly in* D,
- *if* $|f| \le M$, *then* $|c_k| \le 2M/R^k$, $k \in \mathbb{N}$.

Proof. Applying the Laurent expansion to the function $F := f \circ \Phi^{-1}$ in the annulus $\mathbb{A}(1, R)$ gives

$$F(w) = a_0 + \sum_{k=1}^{\infty}(a_k w^k + a_{-k} w^{-k}), \quad 1 < |w| < R,$$

where

$$\begin{aligned} a_k &= \frac{1}{2\pi i}\int_{|w|=r} \frac{F(\zeta)}{\zeta^{k+1}} d\zeta \\ &= \frac{1}{2\pi i}\int_{|\Phi(z)|=r} \frac{f(z)}{\Phi^{k+1}(z)}\Phi'(z)dz, \quad 1 < r < R,\ k \in \mathbb{Z}. \end{aligned}$$

Since F is continuous for $1 \le |w| < R$, we have

$$\begin{aligned} a_k &= \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{F(\zeta)}{\zeta^{k+1}} d\zeta \\ &= \frac{1}{2\pi i}\int_a^b \frac{f(x)}{\Phi^{k+1}(x-i0)}\Phi'(x-i0)dx - \frac{1}{2\pi i}\int_a^b \frac{f(x)}{\Phi^{k+1}(x+i0)}\Phi'(x+i0)dx \\ &= \frac{1}{2\pi}\int_a^b m(x)f(x)\big(\Phi^k(x-i0) + \Phi^{-k}(x-i0)\big)dx \\ &= \frac{1}{\pi}\int_a^b m(x)f(x)\Phi_k(x)dx. \end{aligned}$$

In particular, $c_0 = a_0$, $c_k = 2a_k = 2a_{-k}$, $k \in \mathbb{N}$. Consequently, we get (5.4.1) on $D \setminus [a, b]$, the series being locally uniformly convergent in $D \setminus [a, b]$. It remains to observe that series (5.4.1) is uniformly convergent in $\{z \in D : |\Phi(z)| \le \theta R\}$ for arbitrary $0 < \theta < 1$. In fact, if $|\Phi(z)| \le \theta R$, then

$$|c_k \Phi_k(z)| \le \frac{2M}{R^k}|\Phi(z)|^k \le 2M\theta^k, \quad k \in \mathbb{N}. \qquad \square$$

Proof of Theorem 5.4.6. First note that, using the same procedures as in Remark 5.4.4, we may reduce the proof to the case where D is bounded, $A \subset\subset D$, and $q = 1$. In the case where A is compact we easily reduce the proof to the case where f is bounded, $|f| \le M$ on $\boldsymbol{X}$. In particular, f is continuous on $\boldsymbol{X}$ (cf. Remark 5.4.4(P5)).

Write $G := G_1$, $[a, b] := [a_1, b_1]$. Let R, g be associated to $(G, [a, b])$ as in Lemma 5.4.11 and let m, Φ, $(\Phi_k)_{k=1}^{\infty}$ be associated by Corollary 5.4.12. Define

$$c_k(z) := \frac{2^{\operatorname{sgn} k}}{\pi}\int_a^b m(t)f(z,t)\Phi_k(t)dt, \quad z \in D,\ k \in \mathbb{Z}_+.$$

We have (cf. Lemma 5.4.14, Corollary 5.4.12):

$$f(z,w)=\sum_{k=0}^{\infty}c_k(z)\Phi_k(w),\quad (z,w)\in A\times G,$$

$$c_k\in\mathcal{O}(D),\quad |c_k|\le 2M,\quad |c_k(z)|\le\frac{2M}{R^k},\quad z\in A,\ k\in\mathbb{Z}_+.$$

Hence

$$|c_k|\le 2MR^{k(\boldsymbol{h}^*_{A,D}-1)}\text{ on }D,\quad k\in\mathbb{N}.$$

For $0<\theta<1$ define

$$\Omega_\theta:=\Big\{(z,w)\in D\times G:\boldsymbol{h}^*_{A,D}+\boldsymbol{h}^*_{B,G}(w)=\boldsymbol{h}^*_{A,D}+\frac{\log|\Phi(w)|}{\log R}<1+\frac{\log\theta}{\log R}\Big\}.$$

Observe that $\Omega_\theta\nearrow\widehat{\boldsymbol{X}}$ when $\theta\nearrow 1$. For $(z,w)\in\Omega_\theta$ we get

$$|c_k(z)\Phi_k(w)|\le 2MR^{k(\boldsymbol{h}^*_{A,D}(z)-1)}|\Phi(w)|^k\le 2M(R^{\boldsymbol{h}^*_{A,D}(z)-1}|\Phi(w)|)^k\le 2M\theta^k,\quad k\in\mathbb{N}.$$

Thus, the series is uniformly convergent on Ω_θ and its sum $\hat{f}$ satisfies the inequality

$$\sup_{\Omega_\theta}|\hat{f}|\le\frac{2M}{1-\theta}.$$

Using the same argument for the function f^m instead of f we conclude that

$$\sup_{\Omega_\theta}|\hat{f}^m|\le\frac{2M^m}{1-\theta},$$

which gives

$$\sup_{\Omega_\theta}|\hat{f}|\le M\Big(\frac{2}{1-\theta}\Big)^{1/m}.$$

Letting $m\to+\infty$ leads to the conclusion that $|\hat{f}|\le M$ on Ω_θ. □

The following theorem is one of the first results on extension of separately holomorphic functions on crosses without assuming that the extended function is bounded.

Theorem 5.4.15. *Let $D_1,\dots,D_n\subset\mathbb{C}$ be simply connected domains symmetric with respect to the real axis $\mathbb{R}$ and let $A_j=[a_j,b_j]\subset D_j\cap\mathbb{R}$, $a_j<b_j$, $j=1,\dots,n$. Put $\boldsymbol{X}:=\mathbb{X}((A_j,D_j)_{j=1}^n)$. Let $f\in\mathcal{O}_s(\boldsymbol{X})$. Then there exists an $\hat{f}\in\mathcal{O}(\widehat{\boldsymbol{X}})$ such that $\hat{f}=f$ on $\boldsymbol{X}$ and $\sup_{\widehat{\boldsymbol{X}}}|\hat{f}|\le\sup_{\boldsymbol{X}}|f|$.*

We need some auxiliary results.

Proposition 5.4.16 (Leja's polynomial lemma). *Let $K_1, \dots, K_n \subset \mathbb{C}$ be continua, $K := K_1 \times \dots \times K_n \subset \mathbb{C}^n$, and let $\mathcal{F} \subset \mathcal{P}(\mathbb{C}^n)$ be such that*

$$\forall_{z \in K}: \sup_{p \in \mathcal{F}} |p(z)| < +\infty,$$

i.e. $\mathcal{F}$ is pointwise bounded on K. Then

$$\forall_{a \in K}\ \forall_{\omega > 1}\ \exists_{M = M(K,a,\omega,\mathcal{F}) > 0}\ \exists_{\eta = \eta(K,a,\omega) > 0}\ \forall_{p \in \mathcal{F}}: \sup_{z \in \mathbb{P}(a,\eta)} |p(z)| \le M\omega^{\deg p},$$

equivalently,

$$\forall_{\omega > 1}\ \exists_{M = M(K,\omega,\mathcal{F}) > 0}\ \exists_{\substack{\Omega = \Omega(K,\omega) \text{ open} \\ K \subset \Omega}}\ \forall_{p \in \mathcal{F}}: \sup_{z \in \Omega} |p(z)| \le M\omega^{\deg p}.$$

Notice that η is independent of $\mathcal{F}$.

Proof. The case $n = 1$ is covered by Lemma 1.1.8. Assume that the result is true for $(n-1)$ variables and consider the case of n variables. Take a point $a' \in K' := K_1 \times \dots \times K_{n-1}$ and put $\mathcal{F}_{a'} := \{p(a', \cdot) : p \in \mathcal{F}\}$. By Lemma 1.1.8 there exists an open neighborhood $\Omega_n := \bigcup_{\zeta \in K_n} \mathbb{D}(\zeta, \eta(K_n, \zeta, \sqrt{\omega}))$ of K_n (Ω_n does not depend on a') and a constant $M(a')$ such that

$$|p(a', z_n)| \le M(a')\sqrt{\omega}^{\deg p}, \quad z_n \in \Omega_n,\ f \in \mathcal{F}.$$

Now let $\mathcal{F}' := \{\sqrt{\omega}^{-\deg p} p(\cdot, z_n) : z_n \in \Omega_n\}$. Observe that $\mathcal{F}'$ is pointwise bounded on K'. Thus, by the inductive assumption, there exist an open neighborhood Ω' of K' and a constant M such that

$$\sqrt{\omega}^{-\deg p} |p(z', z_n)| \le M\sqrt{\omega}^{\deg p}, \quad z' \in \Omega',\ z_n \in \Omega_n,\ p \in \mathcal{F}. \qquad \square$$

Proposition 5.4.17. *Let K be a compact subset of an open set $\Omega \subset \mathbb{C}^n$. Assume that for every point $a \in K$ there exist continua $K_1, \dots, K_n \subset \mathbb{C}$ such that $a \in K_1 \times \dots \times K_n \subset K$. Let a sequence $(f_\alpha)_{\alpha \in \mathbb{Z}_+^m} \subset \mathcal{O}(\Omega)$ be locally uniformly bounded in Ω such that*

$$\limsup_{|\alpha| \to +\infty} (|f_\alpha(z)| R^\alpha)^{1/|\alpha|} \le 1, \quad z \in K,$$

where $R \in \mathbb{R}^n_{>0}$. Then for every $\omega > 1$ there exist a constant $M = M(\omega) > 0$ and an open neighborhood Ω_ω of K, $\Omega_\omega \subset \Omega$, such that

$$|f_\alpha(z)| R^\alpha \le M\omega^{|\alpha|}, \quad z \in \Omega_\omega,\ \alpha \in \mathbb{Z}_+^m.$$

Proof. Take an arbitrary $\omega > 1$ and let $\widetilde{\omega} := \sqrt{\omega}$. It suffices to show that for every point $a \in K$ there exist $M, \eta > 0$ such that

$$|f_\alpha(z)| R^\alpha \le M\omega^{|\alpha|}, \quad z \in \mathbb{B}(a, \eta),\ \alpha \in \mathbb{Z}_+^m.$$

Fix a point $a \in K$ and let

$$\mathbb{B}(a, r_0) \subset \Omega, \quad 0 < \rho < r < r_0, \quad \rho/r \le \widetilde{\omega}/\max\{R_1, \dots, R_m\}.$$

Put

$$M_1 := \sup_{\alpha \in \mathbb{Z}_+^m} \max_{z \in \overline{\mathbb{B}}(a,r)} |f_\alpha(z)| < +\infty.$$

Write

$$f_\alpha(z) = \sum_{k=0}^{\infty} f_{\alpha,k}(z-a), \quad z \in \mathbb{B}(a, r_0),$$

where $f_{\alpha,k}$ is a homogeneous polynomial of degree k. Put

$$p_\alpha(z) := \sum_{k=0}^{|\alpha|} f_{\alpha,k}(z-a), \quad \mathcal{F} := \{(R/\widetilde{\omega})^\alpha p_\alpha : \alpha \in \mathbb{Z}_+^m\}.$$

The Cauchy inequalities imply that

$$|f_{\alpha,k}| \le \frac{M_1}{r^k}.$$

Consequently, if $z \in \overline{\mathbb{B}}(a, \rho)$, then

$$\begin{aligned}
|p_\alpha(z)| &\le |f_\alpha(z)| + \sum_{k=0}^{|\alpha|+1} |f_{\alpha,k}(z-a)| \le M_2(z)\frac{\widetilde{\omega}^{|\alpha|}}{R^\alpha} + M_1 \sum_{k=|\alpha|+1}^{\infty} \left(\frac{\rho}{r}\right)^k \\
&\le M_2(z)\frac{\widetilde{\omega}^{|\alpha|}}{R^\alpha} + M_1\left(\frac{\rho}{r}\right)^{|\alpha|+1} \frac{1}{1-\frac{\rho}{r}} \le M_2(z)\frac{\widetilde{\omega}^{|\alpha|}}{R^\alpha} + M_3\left(\frac{\rho}{r}\right)^{|\alpha|} \\
&\le M_4(z)\frac{\widetilde{\omega}^{|\alpha|}}{R^\alpha}.
\end{aligned}$$

Hence, the family $\mathcal{F}$ is pointwise bounded on $\overline{\mathbb{B}}(a, \rho)$. By Leja's polynomial lemma (Proposition 5.4.16) there exist $0 < \eta \le \rho$ and $M > 0$ such that

$$(R/\widetilde{\omega})^\alpha |p_\alpha(z)| \le M\widetilde{\omega}^{|\alpha|}, \quad z \in \mathbb{B}(a, \eta),\ \alpha \in \mathbb{Z}_+^m.$$

Finally, for $z \in \mathbb{B}(a, \eta)$, we get

$$R^\alpha |f_\alpha(z)| \le R^\alpha |p_\alpha(z)| + M_3\widetilde{\omega}^{|\alpha|} \le M\widetilde{\omega}^{2|\alpha|} + M_3\widetilde{\omega}^{|\alpha|} \le (M + M_3)\omega^{|\alpha|}. \qquad \square$$

Proof of Theorem 5.4.15. We use induction on n. Suppose that the result is true for $n - 1 \ge 1$. Fix an $f \in \mathcal{O}_s(\boldsymbol{X})$. In view of Theorem 5.4.6 we only need to show that f is locally bounded on $\boldsymbol{X}$, i.e. for any subdomains $D_j^0 \subset\subset D_j$ with $A_j \subset D_j^0$,

$j = 1, \dots, n$, the function f is bounded on $\mathbb{X}((A_j, D_j^0)_{j=1}^n)$. Fix an $j_0 \in \{1, \dots, n\}$. We are going to show that f is bounded on $A'_{j_0} \times D^0_{j_0} \times A''_{j_0}$. We may assume that $j_0 = n$. We want to prove that f is bounded on $A'_n \times D^0_n$. If $n = 2$, then put $\boldsymbol{Y} = \widehat{\boldsymbol{Y}} := D_1$, $\boldsymbol{Z} := \boldsymbol{X}$. If $n \geq 3$, then

$$\boldsymbol{Y} := \mathbb{X}((A_j, D_j)_{j=1}^{n-1}), \quad \boldsymbol{Z} := \mathbb{X}(A'_n, A_n; \widehat{\boldsymbol{Y}}, D_n).$$

Recall that $\widehat{\boldsymbol{Z}} = \widehat{\boldsymbol{X}}$. In view of the inductive assumption, for every $z_n \in A_n$, the function $f(\cdot, z_n)$ extends to a function $\hat{f}_{z_n}$ holomorphic on $\widehat{\boldsymbol{Y}}$ with $\hat{f}_{z_n} = f(\cdot, z_n)$ on $\boldsymbol{Y}$ (if $n = 2$, then $\hat{f}_{z_n} := f(\cdot, z_n)$). Define $g \colon Z \to \mathbb{C}$,

$$g(z', z_n) := \begin{cases} f(z', z_n), & (z', z_n) \in A'_n \times D_n, \\ \hat{f}_{z_n}(z'), & (z', z_n) \in \widehat{\boldsymbol{Y}} \times A_n. \end{cases}$$

Then $g \in \mathcal{O}_s(\boldsymbol{Z})$. For $0 < \varepsilon < 1$ put

$$\widehat{\boldsymbol{Y}}_\varepsilon := \big\{(z_1, \dots, z_{n-1}) \in \widehat{\boldsymbol{Y}} : \textstyle\sum_{j=1}^{n-1} \boldsymbol{h}^*_{A_j, D_j}(z_j) < 1 - \varepsilon\big\}.$$

Using a Baire argument, we show that there exist a constant $C > 0$ and a non-trivial interval $[a'_n, b'_n] \subset [a_n, b_n]$ such that $|g(z', z_n)| \leq C$ on $\widehat{\boldsymbol{Y}}_\varepsilon \times [a'_n, b'_n]$ and $g|_{\widehat{\boldsymbol{Y}}_\varepsilon \times [a'_n, b'_n]}$ is continuous (cf. Remark 5.4.4(P5)). Let R, g, m, Φ, $(\Phi_k)_{k=1}^\infty$ be associated to $(D_n, [a'_n, b'_n])$. Define

$$c_k(z') := \frac{2^{\operatorname{sgn} k}}{\pi} \int_{a'_n}^{b'_n} m(t) g(z', t) \Phi_k(t) dt, \quad z' \in \widehat{\boldsymbol{Y}}_\varepsilon, \ k \in \mathbb{Z}_+.$$

We have

$$g(z', z_n) = \sum_{k=0}^{\infty} c_k(z') \Phi_k(z_n), \quad (z', z_n) \in A'_n \times D_n,$$

$$c_k \in \mathcal{O}(\widehat{\boldsymbol{Y}}_\varepsilon), \quad |c_k| \leq 2C, \quad |c_k(z')| \leq \frac{2 \max_{t \in [a'_n, b'_n]} |g(z', t)|}{R^k}, \ z' \in A'_n, \ k \in \mathbb{Z}_+.$$

Hence, by Proposition 5.4.16,

$$|c_k| \leq \frac{C(\varepsilon)}{(Re^{-\varepsilon})^k} \text{ on } A'_n, \quad k \in \mathbb{Z}_+.$$

Finally, if

$$D_{n,\varepsilon} := \{z_n \in D_n : |\Phi(z_n)| \leq Re^{-2\varepsilon}\},$$

then

$$|g(z', z_n)| \leq C(\varepsilon) \sum_{k=0}^{\infty} \frac{(Re^{-2\varepsilon})^k}{(Re^{-\varepsilon})^k} = \frac{C(\varepsilon)}{1 - e^{-\varepsilon}}, \quad (z', z_n) \in A'_n \times D_{n,\varepsilon}.$$

In remains to observe that for sufficiently small ε we get $D^0_n \subset D_{n,\varepsilon}$. □

The above proofs are based on expanding f into a certain "concrete" series and checking the domain of convergence of this series. In the general case a similar idea, using an "abstract" series, has been first realized in [Zah 1976], as we will see in the next subsection.

5.4.2 Proof of the cross theorem

We begin with the following general theorem from functional analysis.

Theorem* 5.4.18 ([Mit 1961]). *Let $\mathcal{H}_0, \mathcal{H}_1$ be separable Hilbert spaces,* $\dim \mathcal{H}_0 = \dim \mathcal{H}_1 = \infty$*, and let $T : \mathcal{H}_0 \to \mathcal{H}_1$ be a linear injective compact operator such that $T(\mathcal{H}_0)$ is dense in $\mathcal{H}_1$. Then there exists an orthogonal basis $(b_k)_{k\in\mathbb{N}} \subset \mathcal{H}_0$ such that*

- *$(T(b_k))_{k=1}^{\infty}$ is an orthonormal basis in $\mathcal{H}_1$,*
- *$\|b_k\|_{\mathcal{H}_0} =: v_k \nearrow +\infty$ when $k \nearrow +\infty$.*

Moreover, if there exist a locally convex nuclear space $\mathcal{V}$ and linear continuous operators $\mathcal{H}_0 \overset{T_1}{\to} \mathcal{V} \overset{T_2}{\to} \mathcal{H}_1$ such that $T = T_2 \circ T_1$, then the above basis $(b_k)_{k\in\mathbb{N}}$ may be chosen in such a way that the series $\sum_{k=1}^{\infty} v_k^{-\varepsilon}$ is convergent for any $\varepsilon > 0$.

A proof may be found in [Jar-Pfl 2000], Lemma 3.5.9.

In the case of the cross theorem the above general result implies the following fundamental theorem.

Theorem 5.4.19 ([Zah 1976], [Zer 1982], [Zer 1986], [NTV-Zer 1991], [Zer 1991], [Ale-Zer 2001], [Zer 2002]). *Let $D \subset\subset X$ be a strongly pseudoconvex subdomain of a Riemann domain (X, p) over $\mathbb{C}^p$ and let $A \subset D$ be compact and non-pluripolar. Put*

- *$\mathcal{H}_0 := L_h^2(D)$ (cf. § 2.1.3),*
- *$\mathcal{H}_1 := \mathrm{cl}_{L^2(A,\mu)}(L_h^2(D)|_A) =$ the closure of $L_h^2(D)|_A$ in $L^2(A, \mu)$, where $\mu := \mu_{A,D}$ is the equilibrium measure for A (cf. Definition 3.2.30).*

Then the linear operators

$$\mathcal{H}_0 \ni f \overset{T_1}{\mapsto} f \in \mathcal{O}(D), \quad \mathcal{O}(D) \ni f \overset{T_2}{\mapsto} f|_A \in \mathcal{H}_1$$

are well defined, injective, and continuous (notice that $\mathcal{O}(D)$ is a nuclear space). Moreover, T_1 is compact. In particular, the operator $\mathcal{H}_0 \ni f \overset{T:=T_2\circ T_1}{\mapsto} f|_A \in \mathcal{H}_1$ is compact.

Let $(b_k)_{k=1}^{\infty} \subset \mathcal{H}_0$, $(v_k)_{k=1}^{\infty}$ be as in Theorem 5.4.18. Then for any $\alpha \in (0, 1)$ and for any compact

$$K \subset \{z \in D : \boldsymbol{h}_{A,D}^*(z) < \alpha\}$$

there exists a constant $C = C(\alpha, K) > 0$ such that

$$\|b_k\|_K \le C v_k^{\alpha}, \quad k \in \mathbb{N}. \tag{5.4.2}$$

Proof. (Cf. e.g. [Jar-Pfl 2000], Lemma 3.5.10.) Step 1^0: T_2 is well defined.

Every function $f \in \mathcal{O}(D)$ can be approximated uniformly on A by functions from $\mathcal{O}(\bar{D})$ (cf. Theorem 2.5.7 (b)). Moreover, $\mathcal{O}(\bar{D}) \subset \mathcal{H}_0$ and if $\mathcal{O}(\bar{D}) \ni f_s \to f$ uniformly on A, then $f_s|_A \to f|_A$ in $\mathcal{H}_1$.

Step 2^0: T_2 is injective.

Suppose that an $f \in \mathcal{O}(D)$ is such that $T_2(f) = 0$ in $\mathcal{H}_1$, i.e. $\mu(P) = 0$, where $P := \{z \in A : f(z) \neq 0\}$. We know that $\boldsymbol{h}^*_{A \setminus P, D} = \boldsymbol{h}^*_{A,D}$ (cf. Theorem 3.2.31 (c)). Hence $A \setminus P \notin \mathcal{PLP}$ (cf. Proposition 3.2.11), which implies that $f \equiv 0$.

Step 3^0: T_1 is continuous.

It is well known that for any compact $L \subset D$ there exists a constant $c(L) > 0$ such that

$$\|f\|_L \leq c(L)\|f\|_{L^2(D)}, \quad f \in L^2_h(D), \tag{5.4.3}$$

which is equivalent to the continuity of T_1.

The continuity of T_2 is trivial.

Step 4^0: T_1 is compact.

Let $(f_s)_{s=1}^{\infty} \subset \mathcal{H}_0$ be bounded. Then, by (5.4.3), it is locally uniformly bounded in D. Consequently, by the Montel theorem, there exists a subsequence that is locally uniformly convergent.

Step 5^0: Estimate (5.4.2). Let $k_0 \in \mathbb{N}$ be such that $\nu_k > 1$ for $k \geq k_0$. Put

$$u_k := \frac{\log |b_k|}{\log \nu_k}, \quad k \geq k_0.$$

By (5.4.3), for any compact $L \subset D$, we get

$$\|b_k\|_L \leq c(L)\nu_k, \quad k \in \mathbb{N}.$$

Thus the sequence $(u_k)_{k=k_0}^{\infty}$ is locally bounded from above in D. Let

$$u := \limsup_{k \to +\infty} u_k.$$

Since $\nu_k \to +\infty$, we get $u \leq 1$. Moreover, $u^* \in \mathcal{PSH}(D)$ and $P := \{z \in D : u(z) < u^*(z)\} \in \mathcal{PLP}$ (cf. Theorem 2.3.33 (b)).

We will show that there exists a Borel set $Q \subset A$ with $\mu(Q) = 0$ such that $u \leq 0$ on $A \setminus Q$. Put

$$Q_{k,r,s} := \{z \in A : |b_k(z)| \geq r\nu_k^{1/s}\}, \quad k, r, s \in \mathbb{N}$$

($Q_{k,r,s}$ is compact). Recall that $\|b_k\|_{\mathcal{H}_1} = 1$. Thus we have

$$\mu(Q_{k,r,s}) \leq \frac{1}{r^2\nu_k^{2/s}} \int_A |b_k|^2 d\mu = \frac{1}{r^2\nu_k^{2/s}}.$$

Let $Q_{r,s} := \bigcup_{k=1}^{\infty} Q_{k,r,s}$, $r, s \in \mathbb{N}$. Then, by Theorem 5.4.18,

$$\mu(Q_{r,s}) \le \frac{1}{r^2}\sum_{k=1}^{\infty} \nu_k^{-2/s} = \frac{\mathrm{const}(s)}{r^2}.$$

Consequently, $\mu(Q_s) = 0$, where $Q_s := \bigcap_{r=1}^{\infty} Q_{r,s}$, $s \in \mathbb{N}$. Finally, $\mu(Q) = 0$, where $Q := \bigcup_{s=1}^{\infty} Q_s$. Observe that if $z_0 \in A \setminus Q$, then $z_0 \notin Q_s$ for every s. Thus, for each s there exists an $r(s)$ such that $z_0 \notin Q_{r(s),s}$. This means that $z_0 \notin Q_{k,r(s),s}$ for any k, i.e. $|b_k(z_0)| < r(s)\nu_k^{1/s}$, $k \in \mathbb{N}$. Hence,

$$u_k(z_0) \le \frac{\log r(s)}{\log \nu_k} + \frac{1}{s}, \quad k \ge k_0.$$

Then $u(z_0) \le 1/s$. Since s was arbitrary, we conclude that $u(z_0) \le 0$.

Thus $u^* \le 0$ on $A \setminus (P \cup Q)$ and hence $u^* \le \boldsymbol{h}^*_{A\setminus(P\cup Q),D}$. Moreover,

$$\boldsymbol{h}^*_{A\setminus(P\cup Q),D} = \boldsymbol{h}^*_{A\setminus P,D} = \boldsymbol{h}^*_{A,D}$$

(cf. Theorem 3.2.31 (c)). Thus $u^* \le \boldsymbol{h}^*_{A,D}$. In particular, $\sup_K u^* < \alpha$. Hence, by the Hartogs lemma, $u_k \le \alpha$ on K for $k \gg 1$, which implies the required result. □

Proof of Theorem 5.4.2. We already know (cf. Proposition 5.4.5) that we may assume that $N = 2$, D, G are strongly pseudoconvex domains, $A \subset\subset D$, $B \subset\subset G$ are compact and non-pluripolar, $f(a,\cdot) \in \mathcal{O}(\bar{G})$, $a \in A$, $f(\cdot, b) \in \mathcal{O}(\bar{D})$, $b \in B$, $|f| \le 1$ on $\boldsymbol{X}$, and f is continuous on $\boldsymbol{X}$.

We keep notation from Theorem 5.4.19. For any $w \in B$ we have $f(\cdot, w) \in \mathcal{H}_0$ and $f(\cdot, w)|_A \in \mathcal{H}_1$. Hence, we get the following "abstract" series expansion of f

$$f(\cdot, w) = \sum_{k=1}^{\infty} c_k(w) b_k,$$

where

$$c_k(w) = \frac{1}{\nu_k^2}\int_D f(z,w)\bar{b}_k(z)d\mathcal{L}^D(z) = \int_A f(z,w)\bar{b}_k(z)d\mu(z), \quad k \in \mathbb{N};$$

cf. Theorem 5.4.18. The series is convergent in $L_h^2(D)$ (in particular, locally uniformly in D) and in $L^2(A, \mu)$. Since f is continuous, the formula

$$\hat{c}_k(w) := \int_A f(z,w)\bar{b}_k(z)d\mu(z), \quad w \in G,$$

defines a holomorphic function on G, $k \in \mathbb{N}$. We are going to prove that the series

$$\sum_{k=1}^{\infty} \hat{c}_k(w) b_k(z)$$

converges locally uniformly in $\widehat{\boldsymbol{X}}$.

Take a compact $K \times L \subset \widehat{\boldsymbol{X}}$ and let $\alpha > \max_K \boldsymbol{h}^*_{A,D}$, $\beta > \max_L \boldsymbol{h}^*_{B,G}$ be such that $\alpha + \beta < 1$. First, we will prove that there exists a constant $C'(L,\beta) > 0$ such that

$$\|\widehat{c}_k\|_L \le C'(L,\beta)\nu_k^{\beta-1}, \quad k \in \mathbb{N}. \tag{5.4.4}$$

Suppose for a moment that (5.4.4) is true. Then, using Theorems 5.4.19 and 5.4.18, we get

$$\begin{aligned}\sum_{k=1}^{\infty} \|\widehat{c}_k\|_L \|b_k\|_K &\le \sum_{k=1}^{\infty} C'(L,\beta)\nu_k^{\beta-1} C(K,\alpha)\nu_k^{\alpha} \\ &= C'(L,\beta)C(K,\alpha)\sum_{k=1}^{\infty} \nu_k^{\alpha+\beta-1} =: M(K,L) < +\infty,\end{aligned}$$

which gives the normal convergence on $K \times L$. Let

$$\hat{f}(z,w) := \sum_{k=1}^{\infty} \widehat{c}_k(w) b_k(z), \quad (z,w) \in \widehat{\boldsymbol{X}};$$

obviously $\hat{f}$ is holomorphic. Recall that $\hat{f} = f$ on $D \times B$. Hence $\hat{f} = f$ on $\boldsymbol{Y} := \mathbb{X}(A^*, B^*; D, G) \subset \boldsymbol{X} \cap \widehat{\boldsymbol{X}}$.

We move to the proof of (5.4.4). By the Hölder inequality, we get

$$|\widehat{c}_k(w)| \le \sqrt{\mu(A)}, \quad w \in G,\ k \in \mathbb{N}$$

(recall that $|f| \le 1$). On the other hand, if $w \in B$, then

$$|\widehat{c}_k(w)| = |c_k(w)| = \left|\frac{1}{\nu_k^2}\int_D f(z,w)\bar{b}_k(z)d\mathcal{L}^D(z)\right| \le \frac{1}{\nu_k}\sqrt{\mathcal{L}^D(D)}.$$

For $k \in \mathbb{N}$ such that $\nu_k > 1$, let

$$u_k := \frac{\log|\widehat{c}_k|}{\log \nu_k}.$$

The sequence $(u_k)_{k=1}^{\infty}$ is bounded from above in G, $u := \limsup_{k\to+\infty} u_k \le 0$, and $u \le -1$ on B. Let $P := \{w \in G : u(w) < u^*(w)\}$; $P \in \mathcal{PLP}$. Thus $u^* \in \mathcal{PSH}(G)$, $u^* \le 0$, and $u^* = u \le -1$ on $B \setminus P$. Consequently, $1 + u^* \le \boldsymbol{h}^*_{B\setminus P,G} = \boldsymbol{h}^*_{B,G}$ (cf. Proposition 3.2.21). Hence $1 + u^* < \beta$ on L. Now, by the Hartogs lemma, $u_k < \beta - 1$ on L for $k \gg 1$, which implies (5.4.4). □

5.5 A mixed cross theorem

In this section we shortly discuss a mixed cross theorem whose proof is very similar to the proof of Theorem 5.4.2. Let $D \subset \mathbb{C}^p$ and $G \subset \mathbb{C}^q$ be domains and let $A \subset D$ and $B \subset \partial G$ be non-empty. We define:

- $\boldsymbol{X}_m = \mathbb{X}_m(A, B; D, G) := (A \times (G \cup B)) \cup (D \times B)$ the *mixed cross*,
- $\boldsymbol{X}_m^o = \mathbb{X}_m^o(A, B; D, D) := (A \times G) \cup (D \times B)$ the *associated inner mixed cross*,
- $\widehat{\boldsymbol{X}}_m^* := \{(z, w) \in D \times (G \cup B) : \boldsymbol{h}_{A,D}^*(z) + \boldsymbol{h}_{\mathfrak{K},B,G}^*(w) < 1\}$,
- $\widehat{\boldsymbol{X}}_m := \{(z, w) \in D \times G : \boldsymbol{h}_{A,D}^*(z) + \boldsymbol{h}_{\mathfrak{K},B,G}^*(w) < 1\}$.

Remark 5.5.1. (a) If A is locally pluriregular and B locally $\mathfrak{K}$-pluriregular (see Definition 3.7.7), then $\widehat{\boldsymbol{X}}_m$ is connected.

Indeed, let $(z_j, w_j) \in \widehat{\boldsymbol{X}}_m$, $j = 1, 2$, be given. Then choose a positive δ with $\max\{\boldsymbol{h}_{A,D}^*(z_j), \boldsymbol{h}_{\mathfrak{K},B,G}(w_j)\} < 1 - 2\delta$, $j = 1, 2$. Put

$$\widetilde{A} := \{z \in D : \boldsymbol{h}_{A,D}(z) < \delta\}, \quad \widetilde{B} := \{w \in G : \boldsymbol{h}_{\mathfrak{K},B,G}(w) < \delta\}.$$

Note that $A \subset \widetilde{A}$, i.e. $\widetilde{A}$ is a non-empty set. Similarly, $\widetilde{B}$ is non-empty. Hence both sets are locally pluriregular.

Note that $\boldsymbol{h}_{\widetilde{A},D}^* \le \boldsymbol{h}_{A,D}^*$ and $\boldsymbol{h}_{\widetilde{B},G}^* \le \boldsymbol{h}_{\mathfrak{K},B,G}^*|_G$. Thus

$$(z_j, w_j) \in \mathbb{X}(\widetilde{A}, \widetilde{B}; D, G) =: \widehat{\boldsymbol{X}}, \quad j = 1, 2.$$

Using Remark 5.1.8 (e), it follows that both points can be connected along a curve inside of the classical $\widehat{\boldsymbol{X}}$. Now fix a point (z', w') on such a curve. Then (see Remark 3.7.8) $\boldsymbol{h}_{A,D}^*(z') + \boldsymbol{h}_{\mathfrak{K},B,G}^*(w') \le \boldsymbol{h}_{\widetilde{A},D}^*(z') + \boldsymbol{h}_{\widetilde{B},G}^*(w') + 2\delta < 1$, i.e. $(z', w') \in \widehat{\boldsymbol{X}}_m$.

(b) Moreover, if the assumptions of (a) are satisfied, then $\boldsymbol{X}_m \subset \widehat{\boldsymbol{X}}_m^* \cap \partial \widehat{\boldsymbol{X}}_m$.

Then there is the following result ([Pfl-NVA 2004], see also [Imo-Khu 2000], [Sad-Imo 2006a], [Sad-Imo 2006b]).

Theorem 5.5.2 (Mixed cross theorem). *Let $D \subset \mathbb{C}^p$ be a pseudoconvex domain, $G \subset \mathbb{C}^q$ a domain. Moreover, let $A \subset\subset D$ be locally pluriregular such that $A = \bigcup_{k=1}^{\infty} A_k$, where A_k is a compact, locally pluriregular set, $k \in \mathbb{N}$, and $B \subset \partial G$ relatively open, along which ∂G is locally $\mathcal{C}^1$. Put $\boldsymbol{X}_m = \mathbb{X}_m(A, B; D, G)$ and*

$$\mathcal{F} := \{f : \boldsymbol{X}_m \to \mathbb{C} : f \text{ locally bounded}, f(z, \cdot) \in \mathcal{C}(G \cup B) \cap \mathcal{O}(G), \\ z \in A, \ f(\cdot, w) \in \mathcal{O}(D), \ w \in B\}.$$

Then for an $f \in \mathcal{F}$ there exists a uniquely determined $\hat{f} \in \mathcal{C}(\widehat{\boldsymbol{X}}_m^*) \cap \mathcal{O}(\widehat{\boldsymbol{X}}_m)$ such that $f = \hat{f}$ on $\boldsymbol{X}_m$.

Recall that under the above assumption we know that B is $\mathfrak{K}$-pluriregular (cf. Proposition 3.7.10).

Proof. Fix points $a \in A$ and $b \in B$ and neighborhoods $U \subset\subset D$ and V of a and b, respectively, such that $V \cap G = \{w \in V : \rho(w) < 0\}$, where ρ is a local defining function of G, $V \cap \partial G \subset B$, and $U \times (V \cap G) \subset \widehat{\boldsymbol{X}}_m$. Assume that there are two functions f_1 and f_2 with the properties of $\hat{f}$, i.e. both functions are continuous on $\widehat{\boldsymbol{X}}_m^*$, holomorphic on $\widehat{\boldsymbol{X}}_m$, and coincide on $\boldsymbol{X}_m$. Then $f_1 - f_2 = 0$ on $\boldsymbol{X}_m$; in particular, on $U \times (V \cap \partial G)$. Put $\tilde{f}(z, w) := f_1 - f_2$ on $U \times (V \cap G)$ and $\tilde{f} = 0$ on $U \times (V \setminus G)$. Using Rado's theorem it follows that $f_1 = f_2$ on $U \times (V \cap G)$ and therefore by the identity theorem, on $\widehat{\boldsymbol{X}}_m$. Continuity leads to $f_1 = f_2$ on $\widehat{\boldsymbol{X}}_m^*$.

It remains to construct the extension $\hat{f}$.

Step 1^0: The case where, in addition, D is strongly pseudoconvex, A is compact, and $|f| \le 1$.

In this situation we proceed similar as in the proof of Theorem 5.4.1. Nevertheless, for the convenience of the reader, we repeat almost all details. Let (see Theorem 5.4.19)

- $\mathcal{H}_0 := L_h^2(D)$ (cf. § 2.1.3),
- $\mathcal{H}_1 := \mathrm{cl}_{L^2(A,\mu)}(L_h^2(D)|_A) =$ the closure of $L_h^2(D)|_A$ in $L^2(A, \mu)$, where $\mu := \mu_{A,D}$ is the equilibrium measure for A,
- $(b_k)_{k\in\mathbb{N}} \subset \mathcal{H}_0$ and $(\nu_k)_{k\in\mathbb{N}}$ with $\nu_k \nearrow +\infty$.

Take an $f \in \mathcal{F}$. Then $f(\cdot, w) \in \mathcal{H}_0$ and $f(\cdot, w)|_A \in \mathcal{H}_1$, if $w \in B$. Hence

$$f(\cdot, w) = \sum_{k=1}^{\infty} c_k(w) b_k(z), \quad w \in B,$$

where

$$c_k(w) = \frac{1}{\nu_k^2} \int_D f(z, w)\bar{b}_k(z) d\mathcal{L}^{2n} = \int_A f(z, w)\bar{b}_k(w) d\mu(z), \quad k \in \mathbb{N}.$$

Recall that the series converges in $L_h^2(D)$ and in $L_h^2(A, \mu)$. Put

$$\hat{c}_k(w) := \int_A f(z, w)\bar{b}_k(w) d\mu(z), \quad w \in G \cup B,\ k \in \mathbb{N}.$$

Then $c_k \in \mathcal{C}(G \cup B) \cap \mathcal{O}(G)$. We have to prove that the new series $\sum_{k=1}^{\infty} \hat{c}_k(w) b_k(z)$ is locally uniformly convergent on $\widehat{\boldsymbol{X}}_m^*$.

Fix a compact set $K \subset D$ and choose $\alpha \in (0, 1)$ such that $\boldsymbol{h}_{A,D}^* < \alpha$ on K. Then, using Theorem 5.4.19, we have $\|b_k\|_K \le C\nu_k^{\alpha}$, $k \in \mathbb{N}$, where $C = C(\alpha, K)$.

Now fix a positive ε with $\alpha + 2\varepsilon < 1$. Put

$$G_K := \{w \in G : \boldsymbol{h}_{\mathfrak{K},B,G}^*(w) < 1 - \alpha - 2\varepsilon\}.$$

Notice that $K \times G_K \subset \widehat{\boldsymbol{X}}_m$.

Recall that $|\widehat{c}_k(w)| \le \sqrt{\mu(A)}$ for $k \in \mathbb{N}$ and $w \in G \cup B$. Therefore,

$$u_k := \frac{\log|\widehat{c}_k|}{\log \nu_k} \in \mathcal{PSH}(G),\ k \in \mathbb{N},\ \text{and } u_k \le 1 \text{ on } G \text{ for } k \ge k_0.$$

Moreover,

$$\lim_{G \ni \eta \to w} |\widehat{c}_k(\eta)| = |\widehat{c}_k(w)| = |c_k(w)| \le \frac{1}{\nu_k}\sqrt{\mathcal{L}^{2n}(D)}, \quad k \in \mathbb{N},\ w \in B.$$

Hence, $\lim_{G \ni \eta \to w} u_k(\eta) \le 0, \quad w \in B,\ k \ge k_1 \ge k_0$, which means that u_k is a competitor in the definition of $\boldsymbol{h}_{\mathcal{K},B,G}$, $k \ge k_1$. Therefore we have

$$|\widehat{c}_k(w)| \le \nu_k^{-\alpha-\varepsilon}, \quad w \in G_K,\ k \ge k_1.$$

As in the proof of Theorem 5.4.2 we conclude that the series converges normally on $K \times G_K$; in particular $\hat{f}(z,w) := \sum_{k=1}^{\infty} \widehat{c}_k(w) b_k(z)$ is holomorphic on $\widehat{\boldsymbol{X}}_m$. Noting that $B \subset \overline{G}_K$ we even get that $\hat{f} \in \mathcal{C}(\boldsymbol{X}_m^*)$ and $\hat{f} = f$ on $D \times B$.

Finally, fix an $a \in A$. Then $\hat{f}(a,\cdot)$ and $f(a,\cdot)$ are holomorphic on G and continuous on $G \cup B$. Applying Rado's theorem it follows also that $\hat{f}(a,\cdot) = f(a,\cdot)$ on G. Hence, $\hat{f} = f$ on $\widehat{\boldsymbol{X}}_m^*$.

Step 2^0: The general case.

We may assume that $A_k \subset A_{k+1}$ for all k. Choose strongly pseudoconvex domains $D_k \subset\subset D$ with $A_k \subset D_k \subset\subset D_{k+1}$ and $D_k \nearrow D$. Then $\boldsymbol{h}^*_{A_k,D_k} \searrow \boldsymbol{h}^*_{A,D}$. Moreover, fix a point $b_0 \in B$ and a point b' on the inner normal at b_0 to ∂G such that the half open segment $S := [b', b_0) \subset G$. Fix $\mathbb{N} \ni r \ge \max\{\|b_0\|, \|b'\|\}$ and denote by G_k the connected component of $G \cap \mathbb{B}_n(r+k)$ which contains S. Moreover, put $\widetilde{B}_k := B \cap \partial G_k$ and $B_k := \{w \in \widetilde{B}_k : \operatorname{dist}(w, \partial_{\partial G}\widetilde{B}_k) > 1/k\}$. Then $\varnothing \ne B_k \subset B$ is relatively open in ∂G_k with $\overline{B}_k \subset B$, ∂G_k is $\mathcal{C}^1$ along B_k ($k \ge k_0$) and $\bigcup_{k=k_0}^{\infty} B_k = B$.

Let now $k \ge k_0$. Put $\boldsymbol{X}_{k,m} := \mathbb{X}_m(A_k, B_k; D_k, G_k) \subset \boldsymbol{X}_m$. Obviously, $\boldsymbol{X}_{k,m} \subset \boldsymbol{X}_{k+1,m} \subset \boldsymbol{X}_m$ and $\boldsymbol{X}_{k,m} \nearrow \boldsymbol{X}_m$. Therefore, $\widehat{\boldsymbol{X}}_{k,m} \subset \widehat{\boldsymbol{X}}_{k+1,m}$ and $\widehat{\boldsymbol{X}}^*_{k,m} \subset \widehat{\boldsymbol{X}}^*_{k+1,m}$. Finally, recall that $\boldsymbol{h}^*_{\mathfrak{K},B_k,G_k} \searrow \boldsymbol{h}^*_{\mathfrak{K},B,G}$ (see Lemma 3.7.9). Thus, $\widehat{\boldsymbol{X}}^*_{k,m} \nearrow \widehat{\boldsymbol{X}}^*_m$ and $\widehat{\boldsymbol{X}}_{k,m} \nearrow \widehat{\boldsymbol{X}}_m$.

Now let $f_k := f|_{\boldsymbol{X}_{k,m}}$. Note that f_k is bounded on $\boldsymbol{X}_{k,m}$ (EXERCISE). Hence we are in the situation of Step 1^0. Therefore there exist functions $\hat{f}_k \in \mathcal{C}(\widehat{\boldsymbol{X}}^*_{k,m}) \cap \mathcal{O}(\widehat{\boldsymbol{X}}_{k,m})$ such that $\hat{f}_k = f_k$ on $\boldsymbol{X}_{k,m}$. Hence, $\hat{f}_{k+1} = \hat{f}_k$ on $\widehat{\boldsymbol{X}}_{k,m}$. Using Rado's theorem we conclude that both functions coincide on $\widehat{\boldsymbol{X}}_{k,m}$ and then also on $\widehat{\boldsymbol{X}}^*_{k,m}$. What remains is to glue all these functions to obtain the desired extension $\hat{f}$. □

5.6 Bochner, edge of the wedge, Browder, and Lelong theorems

The aim of this section is to show how four classical theorems may be proved via the cross theorem.

Theorem 5.6.1 (Bochner tube theorem; [Boc 1938], [Boc-Mar 1948], [Hou 2009]). *Let $T = \omega + i\mathbb{R}^n \subset \mathbb{R}^n + i\mathbb{R}^n = \mathbb{C}^n$ be a **tube domain**. Then every $f \in \mathcal{O}(T)$ extends to an $\hat{f} \in \mathcal{O}(\hat{T})$, where $\hat{T} := \operatorname{conv}(\omega) + i\mathbb{R}^n$ ($\operatorname{conv}(\omega)$ stands for the convex hull of ω).*

Proof. Recall that by the Carathéodory theorem we have

$$\operatorname{conv}(\omega) = \bigcup_{a_0,\dots,a_n \in \omega} \operatorname{conv}(\{a_0,\dots,a_n\}).$$

Thus the main problem is to prove the following lemma (cf. [Hör 1973], Theorem 2.5.10).

Lemma 5.6.2. *Assume for each $j \in \{1,\dots,n\}$, the segment $[0, e_j]$ is contained in ω, where $e_j := (0,\dots,0,\underset{j\text{-th position}}{1},0,\dots,0)$. Then every $f \in \mathcal{O}(T)$ extends holomorphically to a neighborhood of $K + i\mathbb{R}^n$ with*

$$K := \{(x_1,\dots,x_n) \in \mathbb{R}^n_+ : x_1 + \dots + x_n \le 1\}.$$

Proof. Take an $\varepsilon \in (0,1)$ and let $D_j := \{x + iy \in \mathbb{C} : -\varepsilon < x < 1 + \varepsilon\}$, $A_j := i\mathbb{R} \subset D_j$; observe that A_j is locally regular, $j = 1,\dots,N$. Define $\boldsymbol{X} := \mathbb{X}((A_j, D_j)_{j=1}^n) \subset \mathbb{C}^n$. In view of our assumptions, $\boldsymbol{X} \subset T$ for $0 < \varepsilon \ll 1$. Take an $f \in \mathcal{O}(T)$. Obviously, $f|_{\boldsymbol{X}} \in \mathcal{O}_s(\boldsymbol{X})$. Thus, by the main cross theorem (Theorem 5.4.1), there exists an $\hat{f} \in \mathcal{O}(\hat{\boldsymbol{X}})$ such that $\hat{f} = f$ on $\boldsymbol{X}$. Since $\boldsymbol{X}$ is not pluripolar, we get $\hat{f} = f$ on the connected component of $T \cap \hat{\boldsymbol{X}}$ that contains $\boldsymbol{X}$. Now we only need to show that $K + i\mathbb{R}^n \subset \hat{\boldsymbol{X}}$. Observe that

$$\boldsymbol{h}^*_{A_j,D_j}(x + iy) = \max\left\{\frac{x}{-\varepsilon}, \frac{x}{1+\varepsilon}\right\}, \quad -\varepsilon < x < 1 + \varepsilon \quad (\text{EXERCISE});$$

in particular, $\boldsymbol{h}^*_{A_j,D_j}(x + iy) = \frac{x}{1+\varepsilon}$ for $0 \le x < 1 + \varepsilon$. Hence

$$\begin{aligned}\hat{\boldsymbol{X}} &= \{(z_1,\dots,z_n) \in D_1 \times \dots \times D_n : \boldsymbol{h}^*_{A_1,D_1}(z_1) + \dots + \boldsymbol{h}^*_{A_n,D_n}(z_n) < 1\} \\ &\supset K + i\mathbb{R}^n.\end{aligned}$$

□

Theorem 5.6.3 (Edge of the wedge type theorem; cf. [Sic 1981b]). *Let $I := (-1,1) \subset \mathbb{R}$,*

$$G := \{(x_0,\dots,x_n) + i(y_0,\dots,y_n) \in (I + iI)^{1+n} : y_0 \ne 0,\ |y_j| < |y_0|,\ j = 1,\dots,n\}.$$

Assume that $f \in \mathcal{O}(G)$ is such that

(a) *there exists an* $\varepsilon \in (0,1)$ *such that for each* $u \in \mathbb{R}^n$ *with* $\|u\|_\infty \le \varepsilon$, *the function*

$$\{\zeta \in \mathbb{D} : \operatorname{Im}\zeta \neq 0\} \ni \zeta \overset{f_u}{\mapsto} f(\zeta, u_1\zeta, \dots, u_n\zeta)$$

extends to an $\tilde{f}_u \in \mathcal{O}(\mathbb{D})$,

(b) *for every* $\alpha \in \mathbb{Z}_+^n$ *there exist constants* $M_\alpha > 0$, $r_\alpha \in (0,1)$ *such that*

$$|D^{(0,\alpha)} f(\zeta, 0)| \le M_\alpha, \quad \zeta \in \mathbb{D}(r_\alpha),\ \operatorname{Im}\zeta \neq 0.$$

Then f *extends holomorphically to a neighborhood of the origin.*

Proof. Define

$$D_0 := \mathbb{D}_*, \quad A_0 := \{it : t \in \mathbb{R},\ 1/4 \le |t| \le 3/4\},$$
$$D_j := \mathbb{D}, \quad A_j := [-\varepsilon, \varepsilon], \quad j = 1, \dots, n, \quad \boldsymbol{X} := \mathbb{X}((A_j, D_j)_{j=0}^n).$$

Let $g\colon \boldsymbol{X} \to \mathbb{C}$ be given by the formula

$$g(w) := \begin{cases} \tilde{f}_{(w_1,\dots,w_n)}(w_0) & \text{if } w \in \mathfrak{X}_0(\boldsymbol{X}), \\ f(w_0, w_0w_1, \dots, w_0w_n) & \text{if } w \in \mathfrak{X}_j(\boldsymbol{X}),\ j = 1, \dots, n, \end{cases}$$

$w = (w_0, \dots, w_n) \in \boldsymbol{X}$. Then g is well defined and separately holomorphic on $\boldsymbol{X}$ (EXERCISE). Consequently, by Theorem 5.4.1, g extends holomorphically to a $\hat{g} \in \mathcal{O}(\widehat{\boldsymbol{X}})$. Observe that there exists a $\delta \in (0,1)$ such that

$$V := \mathbb{D}_*(\delta) \times (\mathbb{D}(\delta))^n \subset \widehat{\boldsymbol{X}} \quad \text{(EXERCISE)}.$$

Define

$$C := \{(z_0, \dots, z_n) \in \mathbb{C}^{1+n} : 0 < |z_0| < \delta,\ |z_j| < \delta|z_0|,\ j = 1, \dots, n\},$$
$$\hat{f}(z_0, \dots, z_n) := \hat{g}\Big(z_0, \frac{z_1}{z_0}, \dots, \frac{z_n}{z_0}\Big), \quad (z_0, \dots, z_n) \in C.$$

Now we only need to show that $\hat{f}$ extends holomorphically to $\mathbb{D}(\delta) \times (\mathbb{D}(\delta^2))^n$. Write

$$\hat{f}(\zeta, z) = \sum_{\alpha \in \mathbb{Z}_+^n} f_\alpha(\zeta) z^\alpha, \quad (\zeta, z) \in C,$$

where $f_\alpha \in \mathcal{O}(\mathbb{D}_*(\delta))$, $\alpha \in \mathbb{Z}_+^n$. Then we have

$$f_\alpha(\zeta) = \frac{1}{\alpha!} D^{(0,\alpha)} \hat{f}(\zeta, 0) = \frac{1}{\alpha!} D^{(0,\alpha)} f(\zeta, 0), \quad \zeta \in \mathbb{D}(\delta),\ \operatorname{Im}\zeta \neq 0.$$

Consequently, in view of our assumptions, each function f_α extends holomorphically to an $\tilde{f}_\alpha \in \mathcal{O}(\mathbb{D}(\delta))$. Hence, the series

$$\tilde{f}(\zeta, z) = \sum_{\alpha \in \mathbb{Z}_+^n} \tilde{f}_\alpha(\zeta) z^\alpha, \quad (\zeta, z) \in \mathbb{D}(\delta) \times (\mathbb{D}(\delta^2))^n$$

defines the required extension (EXERCISE). □

Remark 5.6.4. Notice that, using a change of coordinates, one can extend Theorem 5.6.3 to more complicated geometric situations, where $G = U + i\Gamma$, $\Gamma = (\Gamma_+ \cup \Gamma_-) \cap B$, $\Gamma_- = -\Gamma_+$, $\Gamma_- \cap \Gamma_+ = \varnothing$, $\Gamma_+ \subset \mathbb{R}^{1+n}$ is an open connected cone, and $B \subset \mathbb{R}^{1+n}$ is an open ball centered at the origin.

Theorem 5.6.5 ([Bro 1961], [Lel 1961], [Sic 1969a], [Sic 1969b]). (a) *If $\Omega \subset \mathbb{R}^n \simeq \mathbb{R}^n + i0 \subset \mathbb{C}^n$ is open, then*

$$\mathcal{A}(\Omega) = \{f \in \mathcal{A}_s(\Omega) : \forall_{a\in\Omega}\ \exists_{r>0}\ \forall_{x\in\mathbb{R}^n\cap\mathbb{P}(a,r)\subset\Omega}\ \forall_{j\in\{1,\dots,n\}} : \text{the function } f(x_1,\dots,x_{j-1},\cdot,x_{j+1},\dots,x_n) \text{ extends holomorphically to } \mathbb{D}(a_j,r)\},$$

where $\mathcal{A}(\Omega)$ denotes the space of all real analytic functions $f : \Omega \to \mathbb{R}$.

(b) *If $\Omega \subset \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_N}$, then $\mathcal{H}_{(n_1,\dots,n_N)}(\Omega) \subset \mathcal{H}(\Omega)$, where $\mathcal{H}_{(n_1,\dots,n_N)}(\Omega)$ denotes the space of all functions $f : \Omega \to \mathbb{R}$ such that for every $(a_1,\dots,a_N) \in \Omega$, the function $x_j \mapsto f(a_1,\dots,a_{j-1},x_j,a_{j+1},\dots,a_N)$ is harmonic in a neighborhood of a_j (as a function of n_j variables), $j = 1,\dots,N$ – see Proposition 5.7.3 for a more general result.*

Notice that the function

$$f(x,y) := \begin{cases} xye^{-\frac{1}{x^2+y^2}}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0), \end{cases}$$

is separately real analytic and of class $\mathcal{C}^\infty(\mathbb{R}^2)$, but not real analytic (near $(0,0)$) (Exercise).

Proof. (a) Let $\mathcal{L}_\Omega$ denote the space on the right-hand side. Fix an $f \in \mathcal{L}_\Omega$ and an $a = (a_1,\dots,a_n) \in \Omega$. Let r be as in the definition of $\mathcal{L}_\Omega$. Take an arbitrary $0 < s < r$ and put

$$\boldsymbol{X} := \mathbb{X}(([a_j - s, a_j + s], \mathbb{D}(a_j,r))_{j=1}^n).$$

Directly from the definition of $\mathcal{L}_\Omega$ it follows that f extends to an $\tilde{f} \in \mathcal{O}_s(\boldsymbol{X})$. Now Theorem 5.4.15 implies that $\tilde{f}$ extends holomorphically to $\widehat{\boldsymbol{X}}$, which is a $\mathbb{C}^n$-neighborhood of a. In particular, f is real analytic in an $\mathbb{R}^n$-neighborhood of a.

(b) It suffices to show that $\mathcal{H}_{(n_1,\dots,n_N)}(\Omega) \subset \mathcal{L}_\Omega$. In fact, we only need to observe that if $f \in \mathcal{H}(\mathbb{B}(r) \cap \mathbb{R}^n)$, then f extends holomorphically to $\mathbb{P}_n(r/\sqrt{2n})$.

Indeed, recall (cf. the proof of Proposition 4.3.1) that f extends holomorphically to the Lie ball $\mathbb{L}_n(r)$ and $\mathbb{P}_n(r/\sqrt{2n}) \subset \mathbb{L}_n(r)$. □

5.7 Separately harmonic functions II

Having the cross theorem we can also substantially extend the result on separately harmonic functions from Proposition 4.3.1.

Definition 5.7.1. Let $U_j \subset \mathbb{R}^{n_j}$ be a domain and let $\varnothing \neq A_j \subset U_j$, $j = 1, \dots, N$. Define the *real cross* $\boldsymbol{X}^{\mathbb{R}} = \mathbb{X}((A_j, U_j)_{j=1}^N) \subset \mathbb{R}^{n_1+\dots+n_N} =: \mathbb{R}^n$ (here and in the sequel we adopt the "complex" notation to the "real" situation). We say that a function $u \colon \boldsymbol{X}^{\mathbb{R}} \to \mathbb{R}$ is *separately harmonic* ($u \in \mathcal{H}_s(\boldsymbol{X}^{\mathbb{R}})$) if for any $(a_1, \dots, a_N) \in A_1 \times \dots \times A_N$ and $j \in \{1, \dots, N\}$ the function

$$U_j \ni x_j \mapsto u(a'_j, x_j, a''_j)$$

is harmonic.

Exercise 5.7.2. Prove that $\boldsymbol{X}^{\mathbb{R}}$ is connected (cf. Remark 5.1.8 (a)).

Proposition 5.7.3. *Assume that A_j is locally pluripolar (as a subset of $\mathbb{C}^{n_j}$), $j = 1, \dots, N$. Then there exists an open connected neighborhood $\widehat{U} \subset \mathbb{R}^n$ of $\boldsymbol{X}^{\mathbb{R}}$ such that each function $u \in \mathcal{H}_s(\boldsymbol{X}^{\mathbb{R}})$ extends to a unique $\hat{u} \in \mathcal{H}_{(n_1,\dots,n_N)}(\widehat{U})$.*

See also [NTV 2000] and [Héc 2000] for another approaches to the problem.

Proof. Let $\widetilde{U}_j \subset \mathbb{C}^{n_j}$ be a domain such that $\widetilde{U}_j \cap \mathbb{R}^{n_j} = U_j$ and for each $\varphi \in \mathcal{H}(U_j)$ there exists a $\tilde{\varphi} = \mathcal{E}_j(\varphi) \in \mathcal{O}(\widetilde{U}_j)$ with $\tilde{\varphi} = \varphi$ on U_j, $j = 1, \dots, N$. Put $\boldsymbol{X} := \mathbb{X}((A_j, \widetilde{U}_j)_{j=1}^N)$,

$$\widehat{U} := \widehat{\boldsymbol{X}} \cap \mathbb{R}^n = \Big\{(x_1, \dots, x_N) \in U_1 \times \dots \times U_N : \textstyle\sum_{j=1}^N \boldsymbol{h}^*_{A_j, \widetilde{U}_j}(x_j) < 1\Big\}.$$

Observe that $\boldsymbol{X}^{\mathbb{R}} \subset \widehat{U}$.

Fix a $u \in \mathcal{H}_s(\boldsymbol{X}^{\mathbb{R}})$ and let $f : \boldsymbol{X} \to \mathbb{C}$ be given by the formula

$$f(a'_j, z_j, a''_j) := \mathcal{E}_j(u(a'_j, \cdot, a''_j))(z_j), \quad (a'_j, z_j, a''_j) \in A'_j \times \widetilde{U}_j \times A''_j, \quad j = 1, \dots, N.$$

Then $f \in \mathcal{O}_s(\boldsymbol{X})$ and $\hat{f}|_{\boldsymbol{X}^{\mathbb{R}}} = u$. Consequently, by Theorem 5.4.1, f extends to an $\hat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}})$. Put

$$\Delta^{\mathbb{C}}_{z_j} := \sum_{s=1}^{n_j} \frac{\partial^2}{\partial z_{j,s}^2}.$$

For any $(a'_j, x_j, a''_j) \in A'_j \times U_j \times A''_j$ we have

$$(\Delta^{\mathbb{C}}_{z_j} \hat{f})(a'_j, x_j, a''_j) = (\Delta u(a'_j, \cdot, a''_j))(x_j) = 0.$$

Thus, the identity principle for holomorphic functions implies that $\Delta^{\mathbb{C}}_{z_j} \hat{f} \equiv 0$ on $\widehat{\boldsymbol{X}}$, $j = 1, \dots, N$. In particular, $\hat{u} := \operatorname{Re} \hat{f}|_{\widehat{\boldsymbol{X}} \cap \mathbb{R}^n} \in \mathcal{H}_{(n_1,\dots,n_N)}(\widehat{U})$. □

As a direct corollary we get the following result.

Proposition 5.7.4 (Cf. [NTV 1997]). *Let $u \colon (A \times V) \cup (U \times B) \to \mathbb{R}$ be separately harmonic, where*

- *$U \subset \mathbb{R}^p$ is a domain, $A \subset U$, A is locally pluriregular at every point of U (as a subset of $\mathbb{C}^p$, cf. Definition 3.2.8),*
- *$V \subset \mathbb{R}^q$ is a domain, $B \subset V$, B is locally pluriregular.*

Then u extends to a $\hat{u} \in \mathcal{H}_{(p,q)}(U \times V)$.

In the case where $A = U$ the proposition generalizes Proposition 4.3.1 (cf. Example 3.2.20).

5.8 Miscellanea

5.8.1 *p*-separately analytic functions

In the context of Theorem 5.6.5 (a) one may ask what is the structure of the singular set $\mathcal{S}_{\mathcal{A}}(f) = \{a \in \Omega : \forall_{\substack{U \subset \Omega \\ a \in U \text{ open}}} : f \notin \mathcal{A}(U)\}$ for an arbitrary separately real analytic function $f : \Omega \to \mathbb{R}$.

Definition 5.8.1. Let $\Omega \subset \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_N}$ $(N \geq 2)$ be open, let $f : \Omega \to \mathbb{R}$, and let $1 \leq p \leq N-1$. We say that f is p-separately analytic in Ω ($f \in \mathcal{A}_{(n_1,\dots,n_N),p}(\Omega)$), if for any $a = (a_1, \dots, a_N) \in \Omega$ and $1 \leq i_1 < \cdots < i_p \leq N$ the function

$$(x_{i_1}, \dots, x_{i_p}) \mapsto f(a_1, \dots, a_{i_1-1}, x_{i_1}, a_{i_1+1}, \dots, a_{i_p-1}, x_{i_p}, a_{i_p+1}, \dots, a_N)$$

is analytic in an open neighborhood of $(a_{i_1}, \dots, a_{i_p})$.

Note that

- $\mathcal{A}_{(n_1,\dots,n_N),N}(\Omega) = \mathcal{A}(\Omega)$,
- $\mathcal{A}_{(n_1,\dots,n_N),1}(\Omega) = \mathcal{A}_{(n_1,\dots,n_N)}(\Omega) :=$ the space of all functions $f : \Omega \to \mathbb{R}$ such that for any $(a_1, \dots, a_N) \in \Omega$ and $j \in \{1, \dots, N\}$, the function $x_j \mapsto f(a_1, \dots, a_{j-1}, x_j, a_{j+1}, \dots, a_N)$ is analytic in a neighborhood of a_j (as a function of n_j variables),
- $\mathcal{A}_{\mathbb{1}}(\Omega) = \mathcal{A}_s(\Omega)$.

Theorem* 5.8.2. *Let $\Omega \subset \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_N}$ be open, $f : \Omega \to \mathbb{R}$, $1 \leq p \leq N-1$, and let*

$$\mathcal{S}_{\mathcal{A}}(f) = \{a \in \Omega : \forall_{\substack{U \subset \Omega \\ a \in U \text{ open}}} : f \notin \mathcal{A}(U)\}.$$

(a) *If $f \in \mathcal{A}_{(n_1,\dots,n_N),p}(\Omega)$, then for $S = \mathcal{S}_{\mathcal{A}}(f)$ we have*

(∗) $\operatorname{pr}_{\mathbb{R}^{n_{j_1}} \times \cdots \times \mathbb{R}^{n_{j_{N-p}}}}(S) \in \mathcal{PLP}(\mathbb{C}^{n_{j_1}} \times \cdots \times \mathbb{C}^{n_{j_{N-p}}})$ *for all* $1 \leq j_1 < \cdots < j_{N-p} \leq N$.

(b) *For every relatively closed set $S \subset \Omega$ with (∗), there exists a function $f \in \mathcal{A}_{(n_1,\dots,n_N),p}(\Omega)$ such that $S = \mathcal{S}_{\mathcal{A}}(f)$.*

In the case where $N = 2$, $n_1 = n_2 = 1$, the result was proved in [StR 1990]. Part (a) with $p \geq N/2$ and part (b) with an arbitrary p were proved in [Sic 1990]. Finally, part (a) with arbitrary p is due to Z. Błocki ([Bło 1992]).

5.8.2 Separate subharmonicity

Let $\Omega \subset \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_N}$ be open. A function $u\colon \Omega \to \mathbb{R}_{-\infty}$ is said to *separately subharmonic* ($u \in \mathcal{SH}_{(n_1,\dots,n_N)}(\Omega)$) if for every $(a_1, \dots, a_N) \in \Omega$, the function $x_j \mapsto f(a_1, \dots, a_{j-1}, x_j, a_{j+1}, \dots, a_N)$ is subharmonic in a neighborhood of a_j (as a function of n_j variables), $j = 1, \dots, N$.

In view of Theorem 5.6.5 (b), one could conjecture that every separately subharmonic functions is subharmonic.

In the case where $\Omega \subset \mathbb{C}^n$ is open, one could at least conjecture that a function $u\colon \Omega \to \mathbb{R}_{-\infty}$ is plurisubharmonic iff every $a \in X$ and $\xi \in \mathbb{C}^n$ the function $u_{a,\xi}$ is subharmonic in a neighborhood of zero, i.e. iff u is subharmonic on complex affine lines through Ω. Observe that every such a function is of class $\mathcal{SH}_{(2,\dots,2)}(\Omega)$. The above conjecture was formulated by P. Lelong, who proved ([Lel 1945]) that the answer is positive if we additionally assume that u is locally bounded from above in Ω. [?] The general answer is still open [?]

It is known that if $\Omega \subset \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_N}$ is open and $u \in \mathcal{SH}_{(n_1,\dots,n_N)}(\Omega)$ is such that for every point $a \in \Omega$ there exist an open neighborhood $U \subset \Omega$, a number $0 < r \leq +\infty$, and a function $v \in L^r(\Omega)$ such that $u \leq v$ in U, then $u \in \mathcal{SH}(\Omega)$ ([Rii 1989]). The case $r = +\infty$ is due to Avanissian [Ava 1961]. The case $r = 1$ is due to Arsove [Ars 1966].

In particular, if $\Omega \subset \mathbb{C}^n$ is open and $u\colon \Omega \to \mathbb{R}_{-\infty}$ is subharmonic on every complex affine line and such that for every point $a \in \Omega$ there exist an open neighborhood $U \subset \Omega$, a number $0 < r \leq +\infty$, and a function $v \in L^r(\Omega)$ such that $u \leq v$ in U, then $u \in \mathcal{PSH}(\Omega)$.

The examples constructed in [Wie 1988] and [Wie-Zei 1991] show that in general the answer is negative and we have $\mathcal{SH}_{(n_1,\dots,n_N)}(\Omega) \not\subset \mathcal{SH}(\Omega)$. More precisely, there exists a function $u\colon \mathbb{C}^2 \to \mathbb{R}_+$ such that for every $(z_0, w_0) \in \mathbb{C}^2$ the functions $u(z_0, \cdot)$ and $u(\cdot, w_0)$ are $\mathcal{C}^\infty$ subharmonic, but $u \notin \mathcal{SH}(\mathbb{C}^2)$.

Example 5.8.3 ([Wie-Zei 1991]). Let

$$\tilde{u}_k(z) := \begin{cases} k^k \operatorname{Re}(-iz^k) & \text{if } 0 < \operatorname{Arg} z < \pi/k, \\ 0 & \text{otherwise,} \end{cases} \qquad z \in \mathbb{C},$$

$$u_k := \tilde{u}_k * \Phi_{1/k^3}\Big(z - \frac{1}{k^2}\exp\Big(\frac{\pi i}{2k}\Big)\Big), \qquad z \in \mathbb{C},$$

$$u(z, w) := \sum_{k=1}^{\infty} u_k(z) u_k(w), \qquad (z, w) \in \mathbb{C}^2,$$

where $(\Phi_\varepsilon)_{\varepsilon>0}$ are regularization functions as in Definition 2.3.14. Observe that:

- if $z = re^{i\varphi}$, then $\operatorname{Re}(-iz^k) = r^k \sin(k\varphi)$;
- consequently, $\tilde{u}_k$ is a non-negative continuous function;
- $\tilde{u}_k$ is subharmonic on $\Delta_k := \{0 < \operatorname{Arg} z < \pi/k\}$ and, consequently, on $\mathbb{C}$;
- u_k is a non-negative subharmonic function on $\mathbb{C}$ (cf. Proposition 2.3.15);
- $\operatorname{supp} u_k \subset \Delta_k$ for $k \gg 1$;
- for every $z_0 \in \mathbb{C}$ we have $u_k(z_0) = 0$ for $k \gg 1$;
- consequently, $u(z_0, \cdot)$ is a well-defined, non-negative $\mathcal{C}^\infty$ subharmonic function on $\mathbb{C}$;
- u is unbounded in a neighborhood of the origin:

$$\begin{aligned}\sqrt{u\Big(\frac{2}{k}\exp\Big(\frac{\pi i}{2k}\Big), \frac{2}{k}\exp\Big(\frac{\pi i}{2k}\Big)\Big)} &\geq u_k\Big(\frac{2}{k}\exp\Big(\frac{\pi i}{2k}\Big)\Big)\\ &\geq \tilde{u}_k\Big(\frac{2}{k}\exp\Big(\frac{\pi i}{2k}\Big) - \frac{1}{k^2}\exp\Big(\frac{\pi i}{2k}\Big)\Big) = \tilde{u}_k\Big(\Big(\frac{2}{k} - \frac{1}{k^2}\Big)\exp\Big(\frac{\pi i}{2k}\Big)\Big)\\ &= k^k \operatorname{Re}\Big(-i\Big(\frac{2}{k} - \frac{1}{k^2}\Big)^k\Big(\exp\Big(\frac{\pi i}{2k}\Big)\Big)^k\Big) = \Big(2 - \frac{1}{k}\Big)^k \to +\infty.\end{aligned}$$

Notice that under some mixed "harmonic-subharmonic" assumptions, the answer is positive.

Proposition* 5.8.4 (Cf. [Kołodziej-Thomas 1996], see also [Rii 2007] for generalizations). *Let $u\colon D \times G \to \mathbb{R}$, where $D \subset \mathbb{R}^p$, $G \subset \mathbb{R}^q$ are domains, be such that*

- *$u(a, \cdot) \in \mathcal{H}(G)$ for every $a \in D$,*
- *$u(\cdot, b) \in \mathcal{SH}(D) \cap \mathcal{C}^2(D)$ for every $b \in G$.*

Then $u \in \mathcal{SH}(D \times G) \cap \mathcal{C}(D \times G)$.

Chapter 6

Discs method

Summary. So far, only crosses in Riemann domains over $\mathbb{C}^n$ have been discussed. We now pass to a cross theorem (see Theorem 6.2.2) which holds for crosses in arbitrary complex manifolds. The study of such a theorem has been initiated in [NVA 2005] (see also [NVA 2008], [NVA 2009]). Proofs are based on the former discussions and some new results from pluripotential theory (see Theorem 6.1.3). While in Section 6.1 the necessary tools will be briefly repeated, the proofs are given in Section 6.2.

6.1 Some prerequisites

$\boxed{>}$ § 2.3.

Let us first repeat some notions. An *n-dimensional complex manifold* is a connected Hausdorff space M such that there are families of open subsets $U_\alpha \subset M$, $V_\alpha \subset \mathbb{C}^n$ ($\alpha \in A$), and topological mappings $\varphi_\alpha : U_\alpha \to V_\alpha$ for which:

- $\bigcup_{\alpha \in A} U_\alpha = M$.
- $\varphi_\alpha \circ \varphi_\beta^{-1}|_{\varphi_\beta(U_\alpha \cap U_\beta)} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ is biholomorphic.
  $(U_\alpha, \varphi_\alpha)$ is called a *chart* of M and the system $(U_\alpha, \varphi_\alpha, V_\alpha)_{\alpha \in A}$ is an *atlas of* M. Note that any open connected part G of M is again a complex manifold with the induced charts $U_\alpha \cap G$.
  Using the set of charts one can lift many notions from the classical situation to n-dimensional manifolds:
- A function $f : M \to \mathbb{C}$, M a complex manifold, is said to be *holomorphic*, if $f \circ \varphi_\alpha^{-1} \in \mathcal{O}(V_\alpha)$ for all α's.
- Let $\widetilde{M}$ be another complex manifold of dimension m equipped with an atlas $(\widetilde{U}_\beta, \tilde{\varphi}_\beta, \widetilde{V}_\beta)_{\beta \in B}$. A mapping $F : M \to \widetilde{M}$ is called to be *holomorphic* if it is continuous and if for all α, β the mapping $\tilde{\varphi}_\beta \circ f \circ \varphi_\alpha^{-1}|_{\varphi_\alpha(f^{-1}(\widetilde{U}_\beta) \cap U_\alpha)}$ is holomorphic. We write as usual $f \in \mathcal{O}(M, \widetilde{M})$.
- A function $u : M \to \mathbb{R}_{-\infty}$ is called *plurisubharmonic* (we write $u \in \mathcal{PSH}(M)$), if $u \circ \varphi_\alpha^{-1} \in \mathcal{PSH}(V_\alpha)$, $\alpha \in A$.
- A set $A \subset M$ is called *locally pluripolar* ($A \in \mathcal{PLP}$) if any point $a \in A$ has a connected neighborhood U_a and a function $v_a \in \mathcal{PSH}(U_a)$ with $v_a \not\equiv -\infty$, $M \cap U_a \subset v_a^{-1}(-\infty)$.
- If $f \in \mathcal{O}(M)$, $f \not\equiv 0$, then its zero set $\{z \in M : f(z) = 0\}$ is locally pluripolar.

- Let $D \subset M$ be open and $A \subset D$. Set (see Definition 3.2.1)

$$\boldsymbol{h}_{A,D} := \sup\{u \in \mathcal{PSH}(D) : u \le 1,\ u|_A \le 0\};$$

  $\boldsymbol{h}_{A,D}$ is the *relative extremal function of A with respect to D*. Note that $\boldsymbol{h}^*_{A,D} \in \mathcal{PSH}(D)$ and if A is open, then $\boldsymbol{h}_{A,D} = \boldsymbol{h}^*_{A,D}$.
- We say that a set $A \subset M$ is *pluriregular at a point $a \in \overline{A}$* (see Definition 3.2.8) if $\boldsymbol{h}^*_{A,U}(a) = 0$ for any open neighborhood U of a. Observe that A is pluriregular at a iff there exists a basis $\mathcal{U}(a)$ of neighborhoods of a such that $\boldsymbol{h}^*_{A,U}(a) = 0$ for every $U \in \mathcal{U}(a)$. Define

$$A^* = A^{*,M} := \{a \in \overline{A} : A \text{ is pluriregular at } a\}.$$

  We say that A is *locally pluriregular* if $A \neq \varnothing$ and A is pluriregular at every point $a \in A$, i.e. $\varnothing \neq A \subset A^*$.
- If $A \subset M$ is locally pluriregular, then it is not locally pluripolar at any of its points and any $f \in \mathcal{O}(M)$ with $f|_A = 0$ is automatically the zero function.

For simple properties of the relative extremal functions we refer to Section 3.2. We only mention the following fact.

Lemma 6.1.1. *Let M be an n-dimensional complex manifold, $A \subset M$ locally pluriregular, and $\varepsilon \in (0,1)$ Put*

$$M_\varepsilon := \{z \in M : \boldsymbol{h}^*_{A,M}(z) < 1 - \varepsilon\}.$$

Let D be a connected component of M_ε. Then

(a) $A \cap D \neq \varnothing$;

(b) $\boldsymbol{h}^*_{A\cap D,D}(z) = \frac{\boldsymbol{h}^*_{A,M}(z)}{1-\varepsilon}, \quad z \in D.$

Proof. Exercise (see Proposition 3.2.27). □

The following theorem is due to Royden (see [Roy 1974]) and it will be used together with Theorem 6.1.3 later in this paragraph.

Proposition* 6.1.2. *Let $f : \mathbb{D}(r) \to M$ be a holomorphic embedding (f is proper, injective, and regular) to a complex manifold M of dimension n. If $r' \in (0,r)$, then there exist a neighborhood $U = U(f(\mathbb{D}(r'))$ and a biholomorphic mapping $F : \mathbb{P}_n(r') = \mathbb{D}(r') \times \mathbb{P}_{n-1}(r') \to U$ with $F(\lambda,0) = f(\lambda)$, $\lambda \in \mathbb{D}(r')$.*

The main tool for this general approach to prove the cross theorem is a result on analytic discs due to J. P. Rosay (see [Ros 2003a], [Ros 2003b], and also [Edi 2003]).

Put

$$\boldsymbol{I}(v) := \frac{1}{2\pi}\int_0^{2\pi} v(e^{it})dt.$$

Theorem* 6.1.3. *Let M be a complex manifold and let $u\colon M \to \mathbb{R}_{-\infty}$ be upper semicontinuous. Then the function*

$$\mathfrak{P}_u(z) := \inf\left\{\boldsymbol{I}(u\circ\varphi) : \varphi \in \mathcal{O}(\overline{\mathbb{D}}, M),\ \varphi(0) = z\right\}, \quad z \in M,$$

is plurisubharmonic on M.

$\mathfrak{P}_u$ is sometimes called the *Poisson functional* of u; note that $\mathfrak{P}_u \le u$. Moreover, if $v \in \mathcal{PSH}(M)$ with $v \le u$, then, for a fixed $z \in M$ and a $\varphi \in \mathcal{O}(\overline{\mathbb{D}}, M)$ with $\varphi(0) = z$, we have $v(z) \le \boldsymbol{I}(v\circ\varphi) \le \boldsymbol{I}(u\circ\varphi)$ (use that $v\circ\varphi \in \mathcal{SH}(\overline{\mathbb{D}})$); hence, $v \le \mathfrak{P}_u$, i.e. the Poisson functional of u is the greatest psh minorant of u.

Remark 6.1.4. Let $A \subset M$, $A \ne \varnothing$, be open and $u := \chi_{M\setminus A}$ be the characteristic function of $M \setminus A$. Note that $\mathfrak{P}_u < 1$ (maximum principle for psh functions). Then, for a $z_0 \in M$, there exists a holomorphic disc $\varphi \in \mathcal{O}(\overline{\mathbb{D}}, M)$ with $\varphi(0) = z_0$ such that $\varphi(\mathbb{T}) \cap A \ne \varnothing$. Therefore,

$$\mathfrak{P}_u(z) = \inf\left\{\boldsymbol{I}(u\circ\varphi) : \varphi \in \mathcal{O}(\overline{\mathbb{D}}, M),\ \varphi(0) = z,\ \varphi(\mathbb{T}) \cap A \ne \varnothing\right\}.$$

For the proof of the cross theorem on manifolds the following consequence will be important.

Proposition 6.1.5. *Let M be an n-dimensional complex manifold and $A \subset M$ open, $A \ne \varnothing$. Then $\boldsymbol{h}^*_{A,M} = \mathfrak{P}_{\chi_{M\setminus A}}$.*

Proof. Since A is open, the function $\chi_{M\setminus A}$ is upper semicontinuous on M. Hence, because of Theorem 6.1.3, $v := \mathfrak{P}_{\chi_{M\setminus A}} \in \mathcal{PSH}(M)$. Obviously, $v \le 1$ and $v|_A \le 0$. Therefore, $v \le \boldsymbol{h}_{A,M} = \boldsymbol{h}^*_{A,M}$.

To prove the inverse inequality, let us start with a function $u \in \mathcal{PSH}(M)$, $u \le 1$, and $u|_A \le 0$. Fix $z_0 \in M$ and $\varepsilon \in (0, 1 - v(z_0))$. Then, by the definition of v, there exists a holomorphic disc $\varphi \in \mathcal{O}(\overline{\mathbb{D}}, M)$ with $\varphi(0) = z_0$ and

$$\boldsymbol{I}(\chi_{M\setminus A}\circ\varphi) < v(z_0) + \varepsilon.$$

Note that $\varphi^{-1}(A) \cap \mathbb{T} \ne \varnothing$. Then, using Lemma 3.2.7, we get

$$u(z_0) = u\circ\varphi(0) \le \boldsymbol{h}_{\varphi^{-1}(A)\cap\mathbb{D},\mathbb{D}}(0) \le \boldsymbol{I}(\chi_{\mathbb{T}\setminus\varphi^{-1}(A)}) \le \boldsymbol{I}(\chi_{M\setminus A}\circ\varphi) \le v(z_0) + \varepsilon,$$

or, since z_0, u, and ε were arbitrary, $\boldsymbol{h}^*_{A,M} \le v$. □

Moreover, from Theorem 6.1.3 we get also the following useful information.

Lemma 6.1.6. *Let M be an n-dimensional complex manifold and A a non-empty open subset of M. Then for any $z_0 \in M$ and $\varepsilon > 0$ with $\mathfrak{P}_{\chi_{M\setminus A}}(z_0) + \varepsilon < 1$ there exist an open neighborhood $U = U(z_0)$, an open subset $T \subset \mathbb{C}$, and a family $(\varphi_z)_{z\in U} \subset \mathcal{O}(\overline{\mathbb{D}}, M)$ such that*

- $\varphi \in \mathcal{O}(U \times \mathbb{D})$, *where* $\varphi(z,\lambda) := \varphi_z(\lambda)$*;*
- $\varphi_z(0) = z$, $z \in U$*;*
- $\varphi_z(\lambda) \in A$, $\lambda \in T \cap \mathbb{T}$, $z \in U$*;*
- $\boldsymbol{I}(\chi_{T\cap\mathbb{T}}) < \mathfrak{P}_{\chi_{M\setminus A}}(z_0) + \varepsilon = \boldsymbol{h}^*_{A,M}(z_0) + \varepsilon$.

Note that the last statement implies automatically that $T \cap \mathbb{T} \neq \varnothing$.

Proof. During the proof we will write $u := \chi_{M\setminus A}$. Now fix an arbitrary point $z_0 \in M$ and choose a positive ε such that $\mathfrak{P}_u(z_0) + \varepsilon < 1$. Then according to Theorem 6.1.3, there exist an $r > 1$ and a $\varphi \in \mathcal{O}(\mathbb{D}(r), M)$ satisfying

$$\varphi(0) = z_0 \quad \text{and} \quad \boldsymbol{I}(u \circ \varphi) < \mathfrak{P}_u(z_0) + \varepsilon;$$

in particular, $\varphi^{-1}(A) \cap \mathbb{T} \neq \varnothing$.

Define $f(\lambda) := (\lambda, \varphi(\lambda))$, $|\lambda| < r$, and fix an $r' \in (1, r)$. Then

$$f : \mathbb{D}(r) \to \mathbb{D}(r) \times M$$

is a holomorphic embedding. So Royden's extension theorem (see Proposition 6.1.2) applies. Hence there exist a neighborhood $V = V(f(\mathbb{D}(r')) \subset \mathbb{D}(r') \times M$ and a biholomorphic mapping $F : \mathbb{P}_{n+1}(r') = \mathbb{D}(r') \times \mathbb{P}_n(r') \to V$ with $F(\lambda, 0) = f(\lambda)$, $\lambda \in \mathbb{D}(r')$. Take now a $\rho \in (1, r')$. Note that $(0, z_0) = F(0)$. Thus we find an open neighborhood $U' = U'(z_0) \subset M$ with $\{0\} \times U' \subset V$ and $F^{-1}(\{0\} \times U') \subset \mathbb{D}(r' - \rho) \times \mathbb{P}_n(r')$; in particular, we have

$$\mathbb{D}(\rho) \times \{0\} + F^{-1}(\{0\} \times U') \subset \mathbb{P}_{n+1}(r').$$

Denote by $\mathrm{pr}_M : \mathbb{C} \times M \to M$ the projection map onto the second factor. Then the following mapping

$$\Phi : \mathbb{D}(\rho) \times U' \to M, \quad \Phi(\lambda, z) := \mathrm{pr}_M \circ F\big((\lambda, 0) + F^{-1}(0, z)\big),$$

is holomorphic. For $z \in U'$, put $\varphi_z := \Phi(\cdot, z) : \mathbb{D}(\rho) \to M$. Obviously, $\varphi_z \in \mathcal{O}(\overline{\mathbb{D}}, M)$ and $\varphi_z(0) = z$, $z \in U'$. Moreover, we have $\varphi_{z_0} = \varphi$ on $\mathbb{D}(\rho)$.

It remains to define T and to verify the last two properties in the lemma. Put $T' := \varphi^{-1}(A) = \varphi_{z_0}^{-1}(A)$. Now choose an open subset $T \subset\subset T'$ so large that

$$\boldsymbol{I}(\chi_{\mathbb{T}\setminus T}) < \boldsymbol{I}(u \circ \varphi_{z_0}) + \varepsilon/2.$$

Recall that Φ is continuous on $\overline{\mathbb{D}} \times \overline{U}$, where $U \subset\subset U'$ is a small neighborhood of z_0. Finally applying the continuity of the integral we may shrink U to get the last item in the lemma. □

6.2 Cross theorem for manifolds

$\boxed{>}$ § 5.4.

Now we can describe the situation (see Section 5.1) we will discuss in this section.

- Let D_j be an n_j-dimensional complex manifold and let $\varnothing \neq A_j \subset D_j$, $j = 1, \dots, N$, $N \geq 2$. Let
$$A'_j := A_1 \times \cdots \times A_{j-1}, \quad j = 2, \dots, N,$$
$$A''_j := A_{j+1} \times \cdots \times A_N, \quad j = 1, \dots, N-1.$$
Similarly, for $a = (a_1, \dots, a_N) \in A_1 \times \cdots \times A_N$, we write $a'_j := (a_1, \dots, a_{j-1})$, $a''_j := (a_{j+1}, \dots, a_N)$. Define the *$N$-fold cross*
$$\boldsymbol{X} = \mathbb{X}(A_1, \dots, A_N; D_1, \dots, D_N) = \mathbb{X}((A_j, D_j)_{j=1}^N) := \bigcup_{j=1}^{N} (A'_j \times D_j \times A''_j),$$
where $A'_1 \times D_1 \times A''_1 := D_1 \times A''_1$ and $A'_N \times D_N \times A''_N := A'_N \times D_N$.
- We say that a function $f : \boldsymbol{X} \to \mathbb{C}$ is *separately holomorphic on $\boldsymbol{X}$* ($f \in \mathcal{O}_s(\boldsymbol{X})$) if for any $(a_1, \dots, a_N) \in A_1 \times \cdots \times A_N$ and $j \in \{1, \dots, N\}$, the function
$$D_j \ni z_j \mapsto f(a'_j, z_j, a''_j) \in \mathbb{C}$$
is holomorphic in D_j.
- Assume now in addition that all the sets A_j are locally pluriregular. Then we define
$$\widehat{\boldsymbol{X}} = \widehat{\mathbb{X}}(A_1, \dots, A_N; D_1, \dots, D_N) = \widehat{\mathbb{X}}((A_j, D_j)_{j=1}^N)$$
$$:= \big\{(z_1, \dots, z_N) \in D_1 \times \cdots \times D_N : \textstyle\sum_{j=1}^N \boldsymbol{h}^*_{A_j, D_j}(z_j) < 1\big\}.$$
Note that $\boldsymbol{X} \subset \widehat{\boldsymbol{X}}$, since $\boldsymbol{h}^*_{A_j, D_j}|_{A_j} = 0$.

Lemma 6.2.1. *Let $A_j \subset D_j$ be as before, $j = 1, \dots, N$, and let $U_k \subset D_1$ be a domain with $U_k \cap A_1 \neq \varnothing$, $k = 1, 2$. Put*
$$\boldsymbol{X}_k := \mathbb{X}(U_k \cap A_1, A_2, \dots, A_N; U_k, D_2, \dots, D_N).$$
If $f_k \in \mathcal{O}(\widehat{\boldsymbol{X}_k})$, $k = 1, 2$, such that $f_1 = f_2$ on $(U_1 \cap U_2) \times A_2 \times \cdots \times A_N$, then $f_1 = f_2$ on $\widehat{\boldsymbol{X}_1} \cap \widehat{\boldsymbol{X}_2}$.

Proof. Fix a point $z^0 \in \widehat{\boldsymbol{X}_1} \cap \widehat{\boldsymbol{X}_2}$. Then, for any $(a_3, \dots, a_N) \in A_3 \times \cdots \times A_N$ the functions $f_k(z_1^0, \cdot, a_3, \dots, a_N)$ are holomorphic on the connected component G_2 of
$$\big\{z_2 \in D_2 : \boldsymbol{h}^*_{A_2, D_2}(z_2) < 1 - \max\{\boldsymbol{h}^*_{A_1 \cap U_k, U_k}(z_1^0) : k = 1, 2\}\big\} \ni z_2^0$$

that contains z_2^0. By Lemma 6.1.1 (a), G_2 contains a non-pluripolar subset of A_2. Hence, by the identity theorem, $f_1(z_1^0, \cdot) = f_2(z_1^0, \cdot)$ on $G_2 \times A_3 \times \cdots \times A_N$. In particular, $f_1(z_1^0, z_2^0, \cdot) = f_2(z_1^0, z_2^0, \cdot)$ on $A_3 \times \cdots \times A_N$. Note that if $N = 2$, then $f_1 = f_2$ on their common domain of definition.

Then, for arbitrary $a_j \in A_j$, $j = 4, \dots, N$, we have that

$$f_1(z_1^0, z_2^0, \cdot, a_4, \dots, a_N) = f_2(z_1^0, z_2^0, \cdot, a_4, \dots, a_N)$$

on the connected component G_3 of

$$\big\{z_3 \in D_3 : \boldsymbol{h}^*_{A_3,D_3}(z_3) < 1 - \max\{\boldsymbol{h}^*_{A_1\cap U_k,U_k}(z_1^0) : k = 1,2\} - \boldsymbol{h}^*_{A_2,D_2}(z_2^0)\big\} \ni z_3^0$$

that contains z_3^0. The same argument as above leads to the fact that

$$f_1(z_1^0, z_2^0, z_3^0, \cdot) = f_2(z_1^0, z_2^0, z_3^0, \cdot) \text{ on } A_4 \times \cdots \times A_N.$$

By a similar reasoning one may complete the proof (EXERCISE). □

The main aim of this section is to prove the following result (compare Theorem 5.4.1) due to Nguyên Viêt-Ahn ([NVA 2005], [NVA 2008], and also [NVA 2009]).

Theorem 6.2.2 (Cross theorem for manifolds). *Let D_j be a complex manifold and let $A_j \subset D_j$ be locally pluriregular, $j = 1, \dots, N$. Put $\boldsymbol{X} := \mathbb{X}((A_j, D_j)_{j=1}^N)$. Then*

(∗) *for every $f \in \mathcal{O}_s(\boldsymbol{X})$ there exists a unique $\hat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}})$ such that $\hat{f} = f$ on $\boldsymbol{X}$ and $\hat{f}(\widehat{\boldsymbol{X}}) \subset f(\boldsymbol{X})$; in particular*

- $\|\hat{f}\|_{\widehat{\boldsymbol{X}}} = \|f\|_{\boldsymbol{X}}$,
- $|\hat{f}(z)| \le \|f\|_{\boldsymbol{c}(\boldsymbol{X})}^{1-\sum_{j=1}^N \boldsymbol{h}^*_{A_j,D_j}(z_j)} \|f\|_{\boldsymbol{X}}^{\sum_{j=1}^N \boldsymbol{h}^*_{A_j,D_j}(z_j)}$, $z = (z_1, \dots, z_N) \in \widehat{\boldsymbol{X}}$.

Observe that Lemma 2.1.14 remains true also in the case where D is a connected complex manifold (EXERCISE). Thus the inclusion $\hat{f}(\widehat{\boldsymbol{X}}) \subset f(\boldsymbol{X})$ follows from the generalized version of Lemma 2.1.14.

Note that the only difference from the corresponding result in Section 5.4 is that here the cross lives in a product of arbitrary complex manifolds.

Proof. First, observe that Proposition 3.2.28 may be generalized (EXERCISE) to the case, where D_j is an n_j-dimensional complex manifold, $j = 1, \dots, N$. This permits us to repeat (EXERCISE) the reduction step (P3) from Remark 5.4.4. Thus the situation reduces to the case where $N = 2$. For simplicity, we write $p = n_1$, $q = n_2$, $D = D_1$, $G = D_2$, and $A = A_1$, $B = A_2$. The proof will be divided into the following two steps.

Step 1^0. *Assume that G is a complex manifold and $B \subset G$ is a locally pluriregular set such that (∗) holds for $(\mathbb{D}, G; A, B)$ with an arbitrary non-empty open set $A \subset \mathbb{D}$. Then (∗) holds for $(D, G; A, B)$ with arbitrary complex manifold D and non-empty open set $A \subset D$.*

Consequently:

(1.1) Using Theorem 5.4.1 and Step 1^0 we conclude that condition $(*)$ holds for $(D, G; A, B)$, where D is a complex manifold, $G \subset \mathbb{C}^q$ is a domain, $A \subset D$ is non-empty open, and $B \subset G$ is locally pluriregular.

(1.2) Since the problem $(*)$ is symmetric, property (1.1) implies that $(*)$ holds for $(D, G; A, B)$, where $D \subset \mathbb{C}^p$ is a domain, G is complex manifold, $A \subset D$ is locally pluriregular, and $B \subset G$ is non-empty open.

(1.3) Property (1.2) and Step 1^0 implies that $(*)$ holds for $(D, G; A, B)$, where D, G are complex manifolds and $A \subset D$, $B \subset G$ are non-empty open sets.

(1.4) Observe that in fact we have the following surprising implication: If $(*)$ holds for $(\mathbb{D}, \mathbb{D}; A, B)$, where $A, B \subset \mathbb{D}$ are arbitrary non-empty open sets, then $(*)$ is true for $(D, G; A, B)$, where D, G are arbitrary complex manifolds (of arbitrary dimensions) and $A \subset D$, $B \subset G$ are arbitrary non-empty open sets.

Step 2^0. *Assume that G is a complex manifold and $B \subset G$ is a locally pluriregular set such that*

(2a) $(*)$ *holds for* $(D_0, G; A, B)$, *where* $D_0 \subset \mathbb{C}^p$ *is a fixed bounded domain and* $A \subset D_0$ *is an arbitrary locally pluriregular set,*

(2b) $(*)$ *holds for* $(D, \widetilde{G}_\delta; A, B \cap \widetilde{G}_\delta)$, *where D is an arbitrary p-dimensional complex manifold,* $\widetilde{G}_\delta$ *is a connected component of* $G_\delta := \{w \in G : \boldsymbol{h}^*_{B,G}(w) < 1 - \delta\}$, $0 < \delta < 1/2$, *and $A \subset D$ is an arbitrary non-empty open set.*

Then $(*))$ *holds for* $(D, G; A, B)$ *with arbitrary p-dimensional complex manifold D and locally pluriregular set $A \subset D$.*

Consequently:

(2.1) Using Theorem 5.4.1 (to get (2a)), property (1.1) (to get (2b)), and Step 2^0 we conclude that $(*)$ holds for $(D, G; A, B)$, where D is a complex manifold, $G \subset \mathbb{C}^q$ is a domain, and $A \subset D$, $B \subset G$ are locally pluriregular.

(2.2) By symmetry, $(*)$ holds for $(D, G; A, B)$, where $D \subset \mathbb{C}^p$ is a domain, G is a complex manifold, and $A \subset D$, $B \subset G$ are locally pluriregular.

(2.3) Property (2.2) and Step 1^0 imply that $(*)$ holds for $(D, G; A, B)$, where D, G are complex manifolds, and $A \subset D$ is non-empty open, and $B \subset G$ is locally pluriregular.

(2.4) Finally, properties (2.2) (to get (2a)), (2.3) (to get (2b)) and Step 2^0 imply $(*)$ in its full generality.

Proof of Step 1^0. Put

$$\mathcal{X}_1 := \big\{(z, w) \in D \times G : \exists_{(\lambda,\varphi)\in\mathbb{D}\times\mathcal{O}(\bar{\mathbb{D}},D)} : \varphi^{-1}(A) \cap \mathbb{D} \neq \varnothing, \\ (\lambda, w) \in \widehat{\boldsymbol{X}_\varphi},\ \varphi(\lambda) = z\big\},$$

where $\boldsymbol{X}_\varphi := \mathbb{X}(\varphi^{-1}(A) \cap \mathbb{D}, B; \mathbb{D}, G)$. Put

$$f_\varphi(\lambda, w) := f(\varphi(\lambda), w), \quad (\lambda, w) \in \boldsymbol{X}_\varphi.$$

Then $f_\varphi \in \mathcal{O}_s(\boldsymbol{X}_\varphi)$. By our assumption there exists a function $\widehat{f_\varphi} \in \mathcal{O}(\widehat{\boldsymbol{X}_\varphi})$ such that $\widehat{f_\varphi} = f_\varphi$ on $\boldsymbol{X}_\varphi$. Define $\hat{f} : \mathcal{X}_1 \to \mathbb{C}$:

$$\hat{f}(z,w) := \widehat{f_\varphi}(\lambda, w), \text{ if } (\lambda,\varphi) \text{ is a candidate for } (z,w) \text{ as in the definition of } \mathcal{X}_1.$$

First, we have to prove that this definition is independent of the chosen candidate (λ,φ). For, take $(z_0, w_0) \in \mathcal{X}_1$ and let (λ_j, φ_j) be two candidates for (z_0, w_0). Let V be the connected component of

$$\big\{w \in G : \boldsymbol{h}^*_{B,G}(w) < 1 - \max\{\boldsymbol{h}^*_{\varphi_k^{-1}(A)\cap\mathbb{D},\mathbb{D}}(\lambda_k) : k = 1,2\}\big\}$$

that contains w_0. Applying Lemma 6.1.1, we know that $B \cap V$ is non-pluripolar. Observe that $\widehat{f_{\varphi_j}}(\lambda_j,\cdot) \in \mathcal{O}(V)$ and $\widehat{f_{\varphi_j}}(\lambda_j,\cdot) = f(z_0,\cdot)$ on $B \cap V$, $j = 1,2$. So, by the identity theorem it follows that $\widehat{f_{\varphi_1}}(\lambda_1,\cdot) = \widehat{f_{\varphi_2}}(\lambda_2,\cdot)$ on V. Summarizing, $\hat{f}$ is a well-defined function on $\mathcal{X}_1$.

Next we will prove that $\mathcal{X}_1 = \widehat{\boldsymbol{X}}$. Indeed, let $(z_0, w_0) \in \mathcal{X}_1$. Then, by definition, one may find a $\varphi \in \mathcal{O}(\overline{\mathbb{D}}, D)$ and a $\lambda_0 \in \mathbb{D}$ such that $\varphi(\lambda_0) = z_0$, $\varphi^{-1}(A) \cap \mathbb{D} \neq \varnothing$, and $(\lambda_0, w_0) \in \widehat{\boldsymbol{X}}_\varphi$. Note that $\boldsymbol{h}^*_{A,D} \circ \varphi \le \boldsymbol{h}^*_{\varphi^{-1}(A)\cap\mathbb{D},\mathbb{D}}$ on $\mathbb{D}$. Hence,

$$\boldsymbol{h}^*_{A,D}(z_0) = \boldsymbol{h}^*_{A,D} \circ \varphi(\lambda_0) \le \boldsymbol{h}^*_{\varphi^{-1}(A)\cap\mathbb{D},\mathbb{D}}(\lambda_0) < 1 - \boldsymbol{h}^*_{B,G}(w_0),$$

which implies that $(z_0, w_0) \in \widehat{\boldsymbol{X}}$. To prove the converse inclusion we start with a point $(z_0, w_0) \in \widehat{\boldsymbol{X}}$. Fix an $\varepsilon > 0$ such that $\boldsymbol{h}^*_{A,D}(z_0) + \boldsymbol{h}^*_{B,G}(w_0) + \varepsilon < 1$. Applying Theorem 6.1.3 and Proposition 6.1.5, we find a $\varphi \in \mathcal{O}(\overline{\mathbb{D}}, D)$ with $\varphi(0) = z_0$, $\varphi^{-1}(A) \cap \mathbb{D} \neq \varnothing$, and $\boldsymbol{I}(\chi_{D\setminus A} \circ \varphi) < \boldsymbol{h}^*_{A,D}(z_0) + \varepsilon$. Then Lemma 3.2.7 leads to

$$\boldsymbol{h}^*_{\varphi^{-1}(A)\cap\mathbb{D},\mathbb{D}}(0) + \boldsymbol{h}^*_{B,G}(w_0) \le \boldsymbol{I}(\chi_{D\setminus A} \circ \varphi) + \boldsymbol{h}^*_{B,G}(w_0) < 1;$$

thus $(z_0, w_0) \in \mathcal{X}_1$.

So far we have constructed a function $\hat{f}$ on $\widehat{\boldsymbol{X}} = \mathcal{X}_1$. Let $(z_0, w_0) \in A \times G$. Take an arbitrary $\varphi \in \mathcal{O}(\overline{\mathbb{D}}, D)$ with $\varphi(0) = z_0$. Then $(0, w_0) \in \mathcal{X}_1$ and $\hat{f}(z_0, w_0) = \widehat{f_\varphi}(0, w_0) = f(\varphi(0), w_0) = f(z_0, w_0)$. Hence, $\hat{f} = f$ on $A \times G$.

Now let $(z_0, w_0) \in D \times B$ and choose an $\varepsilon > 0$ such that $\varepsilon < 1 - \boldsymbol{h}^*_{A,D}(z_0)$. Then Theorem 6.1.3 and Proposition 6.1.5 implies that there is a $\varphi \in \mathcal{O}(\overline{\mathbb{D}}, D)$ with $\varphi(0) = z_0$, $\varphi^{-1}(A) \cap \mathbb{D} \neq \varnothing$, and $\boldsymbol{I}(\chi_{D\setminus A} \circ \varphi) < \boldsymbol{h}^*_{A,D}(z_0) + \varepsilon$. Applying Lemma 3.2.7, it follows that

$$\boldsymbol{h}^*_{\varphi^{-1}(A)\cap\mathbb{D},\mathbb{D}}(0) + \boldsymbol{h}^*_{B,G}(w_0) \le \boldsymbol{I}(\chi_{D\setminus A} \circ \varphi) \le \boldsymbol{h}^*_{A,D}(z_0) + \varepsilon < 1;$$

in particular, $(0, w_0) \in \widehat{\boldsymbol{X}_\varphi}$. Therefore, $\hat{f}(z_0, w_0) = \widehat{f_\varphi}(0, w_0) = f(\varphi(0), w_0) = f(z_0, w_0)$ or $\hat{f} = f$ on $D \times B$. So we have, in total, that $\hat{f} = f$ on $\boldsymbol{X}$.

In the remaining step we have to verify that $\hat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}})$. Fix a point $(z_0, w_0) \in \widehat{\boldsymbol{X}}$ and then choose an $\varepsilon > 0$ with $\boldsymbol{h}^*_{A,D}(z_0) + \boldsymbol{h}^*_{B,G}(w_0) < 1 - 2\varepsilon$, $\boldsymbol{h}^*_{B,G}(w_0) < 1 - \varepsilon$.

Let V be the connected component of the set $\{w \in G : \boldsymbol{h}^*_{B,G}(w) < \boldsymbol{h}^*_{B,G}(w_0) + \varepsilon\}$ that contains w_0. Notice that $B \cap V$ is not pluripolar (use Lemma 6.1.1 (a)). By Lemma 6.1.6 we find a neighborhood $U = U(z_0)$, an open set $T \subset \mathbb{C}$, and a family $(\varphi_z)_{z\in U} \subset \mathcal{O}(\overline{\mathbb{D}}, D)$ such that

- $\varphi \in \mathcal{O}(U \times \mathbb{D})$, where $\varphi(z,\lambda) := \varphi_z(\lambda)$;
- $\varphi_z(0) = z$, $z \in U$;
- $\varphi_z(\lambda) \in A$, $\lambda \in T \cap \overline{\mathbb{D}}$, $z \in U$;
- $T \cap \mathbb{T} \neq \varnothing$;
- $\boldsymbol{I}(\chi_{\mathbb{T}\setminus T}) < \mathfrak{P}_{\chi_{D\setminus A}}(z_0) + \varepsilon = \boldsymbol{h}^*_{A,D}(z_0) + \varepsilon$, where the last equality follows from Proposition 6.1.5.

Observe that for each $(z,w) \in U \times V$ the pair $(0, \varphi_z)$ is a competitor for (z,w) in the sense of $\mathcal{X}_1$ (because

$$\begin{aligned}\boldsymbol{h}^*_{\varphi_z^{-1}(A)\cap\mathbb{D},\mathbb{D}}(0) &\le \boldsymbol{I}(\chi_{\mathbb{T}\setminus\varphi_z^{-1}(A)}) \le \boldsymbol{I}(\chi_{\mathbb{T}\setminus T}) \\ &< \boldsymbol{h}^*_{A,D}(z_0) + \varepsilon < 1 - \varepsilon - \boldsymbol{h}^*_{B,G}(w_0) < 1 - \boldsymbol{h}^*_{B,G}(w)).\end{aligned}$$

Thus $\hat{f}(z,w) = \widehat{f_{\varphi_z}}(0,w)$, $(z,w) \in U \times V$. Consequently, $\hat{f}(z,\cdot) \in \mathcal{O}(V)$ for each $z \in U$. If $w \in B \cap V$, then $\hat{f}(z,w) = \widehat{f_{\varphi_z}}(0,w) = f(\varphi_z(0),w) = f(z,w)$, $z \in U$. Thus $\hat{f}(\cdot,w) \in \mathcal{O}(U)$ for each $w \in B \cap V$. Now Terada's theorem (Theorem 4.2.2) implies that $\hat{f} \in \mathcal{O}(U \times V)$ (EXERCISE). □

Proof of Step 2^0. For any $a \in A$ choose an open neighborhood $U_a \subset D$ such that U_a is biholomorphic to D_0. Put $f_a := f|_{\boldsymbol{X}_a}$, where $\boldsymbol{X}_a := \mathbb{X}(A \cap U_a, B; U_a, G)$. In view of (2a) we find a unique holomorphic extension $\hat{f}_a \in \mathcal{O}(\widehat{\boldsymbol{X}_a})$ with $\hat{f}_a = f_a = f$ on $\boldsymbol{X}_a$. For $\delta \in (0, 1/2)$ put

$$U_{a,\delta} := \{z \in U_a : \boldsymbol{h}^*_{A\cap U_a,U_a}(z) < \delta\}, \quad G_\delta := \{w \in G : \boldsymbol{h}^*_{B,G}(w) < 1-\delta\}.$$

Note that $B \subset G_\delta$ and $U_{a,\delta} \times G_\delta \subset \widehat{\boldsymbol{X}_a}$; in particular, $\hat{f}_a \in \mathcal{O}(U_{a,\delta} \times G_\delta)$. Applying Lemma 6.2.1, it follows that $\hat{f}_{a_1} = \hat{f}_{a_2}$ on $\widehat{\boldsymbol{X}_{a_1}} \cap \widehat{\boldsymbol{X}_{a_2}}$ whenever $a_1, a_2 \in A$ and $U_{a_1} \cap U_{a_2} \neq \varnothing$. In particular, $\hat{f}_{a_1} = \hat{f}_{a_2}$ on $(U_{a_1,\delta} \cap U_{a_2,\delta}) \times G_\delta$. Put $A_\delta := \bigcup_{a\in A} U_{a,\delta}$; A_δ is an open neighborhood of A in D. Gluing the functions $(\hat{f}_a|_{U_{a,\delta}\times G_\delta})_{a\in A}$, gives a new function $\tilde{f}_\delta \in \mathcal{O}(A_\delta \times G_\delta)$. Observe that by construction $\tilde{f}_\delta = f$ on $(A \times G_\delta) \cup (A_\delta \times B)$. Hence we may put

$$f_\delta(z,w) := \begin{cases} \tilde{f}_\delta(z,w) & \text{if } (z,w) \in A_\delta \times G_\delta, \\ f(z,w) & \text{if } (z,w) \in D \times B. \end{cases}$$

By definition, $f_\delta \in \mathcal{O}_s(\boldsymbol{Y}_\delta)$, where $\boldsymbol{Y}_\delta := \mathbb{X}(A_\delta, B \cap \widetilde{G}_\delta; D, \widetilde{G}_\delta)$ and $\widetilde{G}_\delta$ an arbitrary connected component of G_δ. By (2b) we get a unique extension $\hat{f}_\delta \in \mathcal{O}(\widehat{\boldsymbol{Y}}_\delta)$ with

$\widehat{f_\delta} = f_\delta$ on $\boldsymbol{Y}_\delta$. Fix a point $(z_0, w_0) \in \widehat{\boldsymbol{X}}$ and put

$$\delta_0 := \frac{1}{3}\big(1 - \boldsymbol{h}^*_{A,D}(z_0) - \boldsymbol{h}^*_{B,G}(w_0)\big).$$

Then choose an open connected neighborhood $U = U(z_0)$, such that

$$\boldsymbol{h}^*_{A,D}(z) < \boldsymbol{h}^*_{A,D}(z_0) + \delta_0, \quad z \in U.$$

Moreover, let V be the connected component of

$$\{w \in G : \boldsymbol{h}^*_{B,G}(w) < \boldsymbol{h}^*_{B,G}(w_0) + \delta_0\}$$

that contains w_0. Note that $B \cap V \neq \varnothing$ (see Lemma 6.1.1 (a)). We have $V \subset G_{\delta_0}$ and $U \times V \in \widehat{\boldsymbol{X}}$. Now let $\delta \in (0, \delta_0]$. Note that $V \subset G_{\delta_0} \subset G_\delta$. Let $\widetilde{G}_\delta$ be the connected component of G_δ that contains V. Thus, for $(z, w) \in U \times V$, we have according to Lemma 6.1.1 (b),

$$\begin{aligned}
\boldsymbol{h}^*_{A_\delta,D}(z) + \boldsymbol{h}^*_{B\cap\widetilde{G}_\delta,\widetilde{G}_\delta}(w) &\leq \boldsymbol{h}^*_{A,D}(z) + \boldsymbol{h}^*_{B\cap\widetilde{G}_{\delta_0},\widetilde{G}_{\delta_0}}(w)\\
&\leq \boldsymbol{h}^*_{A,D}(z) + \frac{\boldsymbol{h}^*_{B,G}(w)}{1-\delta_0}\\
&\leq \boldsymbol{h}^*_{A,D}(z_0) + \delta_0 + \frac{\boldsymbol{h}^*_{B,G}(w_0) + \delta_0}{1-\delta_0} < 1.
\end{aligned}$$

In particular, $U \times V \subset \widehat{\boldsymbol{Y}_\delta}$. Hence, $\widehat{f_\delta}$ is a holomorphic function on $U \times V$. Moreover, $\widehat{f_\delta} = f = \widehat{f_{\delta_0}}$ on $U \times B$. Using Lemma 6.1.1 (a), it follows that $\widehat{f_\delta} = \widehat{f_{\delta_0}}$ on $U \times V$.

To summarize: for each point $x_0 = (z_0, w_0) \in \widehat{\boldsymbol{X}}$ we have found a neighborhood $W(x_0) \subset \widehat{\boldsymbol{X}}$ of x_0 and a positive number $\delta(x_0)$ such that for every $\delta \in (0, \delta(x_0)]$ one has the following equality $\widehat{f_\delta} = \widehat{f_{\delta(x_0)}}$ on $W(x_0)$. Hence, on $\widehat{\boldsymbol{X}}$ we may set $\hat{f} := \lim_{\delta\to 0} \widehat{f_\delta}$ as a holomorphic extension of f with $\hat{f} = f$ on $\boldsymbol{X}$. $\square$

Chapter 7
Non-classical cross theorems

Summary. The aim of this chapter is to present some less standard results on separately holomorphic functions. Section 7.1 presents an extension problem for separately holomorphic functions defined on so-called generalized N-fold crosses. The main result is Theorem 7.1.4, which will be frequently used in the second part of the book. Section 7.2 is devoted to a new class of N-fold crosses (called (N,k)-crosses) and an extension theorem for them (Theorem 7.2.7), which may be a starting point for further research. Finally, Section 7.3 discusses some aspects of continuation problems for separately holomorphic functions defined on Cartesian products of crosses (Theorem 7.3.2).

7.1 Cross theorem for generalized crosses

$\boxed{>}$ §§ 2.3, 3.2, 5.4.

Let $D_j \in \mathfrak{R}_c(\mathbb{C}^{n_j})$, let $\varnothing \neq A_j \subset D_j$, and let $\Sigma_j \subset A'_j \times A''_j$, $j = 1, \dots, N$. Put

$$\mathfrak{X}_j := \{(a'_j, z_j, a''_j) \in A'_j \times D_j \times A''_j : (a'_j, a''_j) \notin \Sigma_j\}, \quad j = 1, \dots, N.$$

Definition 7.1.1. We define a *generalized N-fold cross*

$$\boldsymbol{T} := \mathbb{T}(A_1, \dots, A_N; D_1, \dots, D_N; \Sigma_1, \dots, \Sigma_N) = \mathbb{T}((A_j, D_j, \Sigma_j)_{j=1}^N) := \bigcup_{j=1}^N \mathfrak{X}_j$$

and its *center*

$$\boldsymbol{c}(\boldsymbol{T}) := \boldsymbol{T} \cap (A_1 \times \dots \times A_N).$$

Remark 7.1.2. (a) $\boldsymbol{c}(\boldsymbol{T}) = (A_1 \times \dots \times A_N) \setminus \varDelta_0$, where

$$\varDelta_0 := \bigcap_{j=1}^N \big\{(a'_j, a_j, a''_j) \in A'_j \times A_j \times A''_j : (a'_j, a''_j) \in \Sigma_j\big\}.$$

(b) If one of the sets $\Sigma_1, \dots, \Sigma_N$ is pluripolar, then $\varDelta_0 \in \mathcal{PLP}$.

(c) $\mathbb{T}((A_j, D_j, \varnothing)_{j=1}^N) = \mathbb{X}((A_j, D_j)_{j=1}^N)$.

(d) If $N = 2$, then $\mathbb{T}(A_1, A_2; D_1, D_2; \Sigma_1, \Sigma_2) = \mathbb{X}(A_1 \setminus \Sigma_2, A_2 \setminus \Sigma_1; D_1, D_2)$. Thus, generalized 2-fold crosses are nothing new in comparison with the standard 2-fold crosses. For $N \geq 3$ generalized N-fold crosses are geometrically different than the standard ones – for instance, this makes the theory of extension with singularities for $N \geq 3$ essentially more difficult – cf. Chapter 10.

(e) Observe that in general $\boldsymbol{T}$ need not be connected. For example: Let $D_j = A_j = \mathbb{D}$, $b_j \in \mathbb{D}^n$, $b_j^* := (b_j', b_j'')$, $\Sigma_j := \mathbb{D}^{n-1} \setminus \{b_j^*\}$, $j = 1, \dots, n$, $n \geq 3$. Then $\mathfrak{X}_j = \{b_j'\} \times \mathbb{D} \times \{b_j''\}$. One can easily find points $b_1, \dots, b_n$ such that $\mathfrak{X}_j \cap \mathfrak{X}_k = \varnothing$ for $j \neq k$. Since each branch $\mathfrak{X}_j$ is open in $\boldsymbol{T}$, we conclude that $\boldsymbol{T}$ is disconnected. See Corollary 7.1.6 for the case where $\Sigma_1, \dots, \Sigma_N$ are pluripolar.

Definition 7.1.3. We say that a function $f : \boldsymbol{T} \to \mathbb{C}$ is *separately holomorphic on* $\boldsymbol{T}$ ($f \in \mathcal{O}_s(\boldsymbol{T})$) if for any $j \in \{1, \dots, N\}$ and $(a_j', a_j'') \in (A_j' \times A_j'') \setminus \Sigma_j$, the function

$$D_j \ni z_j \mapsto f(a_j', z_j, a_j'') \in \mathbb{C}$$

is holomorphic in D_j.

Let $\mathcal{O}_s^c(\boldsymbol{T})$ denote the space of all $f \in \mathcal{O}_s(\boldsymbol{T})$ such that for any $j \in \{1, \dots, N\}$ and $b_j \in D_j$, the function

$$(A_j' \times A_j'') \setminus \Sigma_j \ni (z_j', z_j'') \mapsto f(z_j', b_j, z_j'')$$

is continuous.

Theorem 7.1.4 (Extension theorem for generalized crosses). *Assume that D_j is a Riemann domain over $\mathbb{C}^{n_j}$, $A_j \subset D_j$ is locally pluriregular, $\Sigma_j \subset A_j' \times A_j''$ is pluripolar, $j = 1, \dots, N$, $\boldsymbol{X} := \mathbb{X}((A_j, D_j)_{j=1}^N)$, and $\boldsymbol{T} := \mathbb{T}((A_j, D_j, \Sigma_j)_{j=1}^N)$. Then for every $f \in \mathcal{O}_s^c(\boldsymbol{T})$ there exists an $\hat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}})$ such that $\hat{f} = f$ on $\boldsymbol{T}$ and $\hat{f}(\widehat{\boldsymbol{X}}) \subset f(\boldsymbol{T})$; in particular*

- $\|\hat{f}\|_{\widehat{\boldsymbol{X}}} = \|f\|_{\boldsymbol{T}}$,
- $|\hat{f}(z)| \leq \|f\|_{\boldsymbol{c}(\boldsymbol{T})}^{1 - \sum_{j=1}^N \boldsymbol{h}^*_{A_j, D_j}(z_j)} \|f\|_{\boldsymbol{T}}^{\sum_{j=1}^N \boldsymbol{h}^*_{A_j, D_j}(z_j)}$, $z = (z_1, \dots, z_N) \in \widehat{\boldsymbol{X}}$.

Observe that the inclusion $\hat{f}(\widehat{\boldsymbol{X}}) \subset f(\boldsymbol{T})$ follows from Lemma 2.1.14 with

$$(G, D, A_0, A, \mathcal{F}) = (\widehat{\boldsymbol{X}}, \widehat{\boldsymbol{X}}, \boldsymbol{c}(\boldsymbol{T}), \boldsymbol{T}, \mathcal{O}_s^c(\boldsymbol{T})).$$

The inequality is a consequence of Lemma 3.2.5 and Proposition 3.2.28:

$$\boldsymbol{h}^*_{\boldsymbol{c}(\boldsymbol{T}), \widehat{\boldsymbol{X}}}(z) = \boldsymbol{h}^*_{\boldsymbol{c}(\boldsymbol{X}) \setminus \Delta_0, \widehat{\boldsymbol{X}}}(z) = \boldsymbol{h}^*_{\boldsymbol{c}(\boldsymbol{X}), \widehat{\boldsymbol{X}}}(z) = \sum_{j=1}^N \boldsymbol{h}^*_{A_j, D_j}(z_j).$$

Remark 7.1.5. (a) The case where $\Sigma_1 = \dots = \Sigma_N = \varnothing$ follows immediately from Theorem 5.4.1.

(b) If $N = 2$, then Theorem 7.1.4 holds for arbitrary $f \in \mathcal{O}_s(\boldsymbol{T})$ (cf. Remark 7.1.2 (d) and Theorem 5.4.1).

(c) [?] We do not know whether Theorem 7.1.4 remains true for all $f \in \mathcal{O}_s(\boldsymbol{T})$ [?]

(d) [?] It is an open problem to extend Theorem 7.1.4 to the case of arbitrary complex manifolds (cf. Chapter 6) [?]

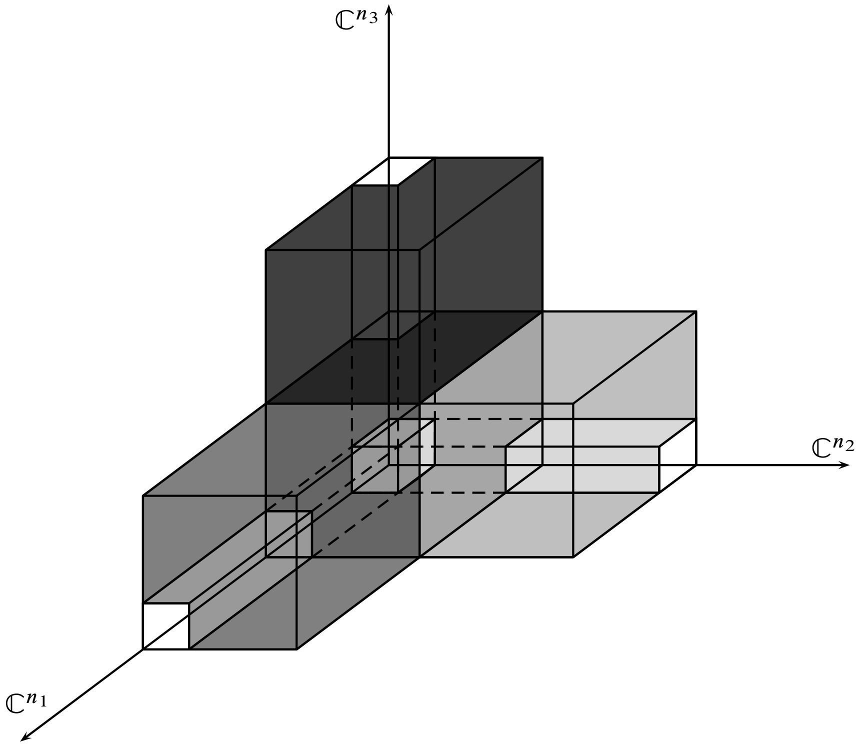


Figure 7.1.1. Generalized 3-fold cross.

Proof of Theorem 7.1.4. We apply induction on N. As we already observed the result is true for $N = 2$. Assume that the result is true for $N - 1 \geq 2$. Take an $f \in \mathcal{O}_s^c(\boldsymbol{T})$. Let

$$Q := \{z_N \in A_N : \exists_{j\in\{1,\dots,N-1\}} : (\Sigma_j)_{(\cdot,z_N)} \notin \mathcal{PLP}\}.$$

Then, by Proposition 2.3.31, $Q \in \mathcal{PLP}$. Take a $z_N \in A_N \setminus Q$ and define

$$\boldsymbol{T}(z_N) := \mathbb{T}((A_j, D_j, (\Sigma_j)_{(\cdot,z_N)})_{j=1}^{N-1}), \quad \boldsymbol{Y} := \mathbb{X}((A_j, D_j)_{j=1}^{N-1}).$$

Put $\widetilde{A}_j'' := A_{j+1} \times \dots \times A_{N-1}$, $\tilde{a}_j'' := (a_{j+1}, \dots, a_{N-1})$, $j = 1, \dots, N-1$. Observe that

$$\boldsymbol{T}_{(\cdot,z_N)} = \boldsymbol{T}(z_N) \cup (A_N' \setminus \Sigma_N).$$

Then $f(\cdot, z_N) \in \mathcal{O}_s^c(\boldsymbol{T}(z_N))$ (Exercise). By the inductive assumption, there exists an $\hat{f}_{z_N} \in \mathcal{O}(\widehat{\boldsymbol{Y}})$ such that $\hat{f}_{z_N} = f(\cdot, z_N)$ on $\boldsymbol{T}(z_N)$. Consider the 2-fold cross

$$\boldsymbol{Z} := \mathbb{X}(A_N' \setminus \Sigma_N, A_N \setminus Q; \widehat{\boldsymbol{Y}}, D_N) = ((A_N' \setminus \Sigma_N) \times D_N) \cup (\widehat{\boldsymbol{Y}} \times (A_N \setminus Q)).$$

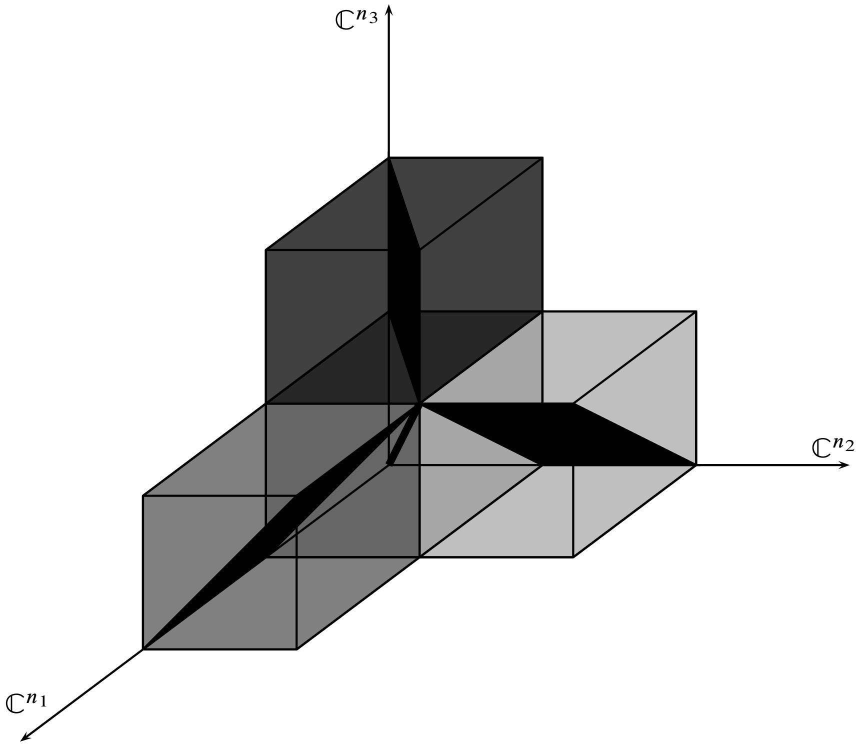


Figure 7.1.2. Generalized 3-fold cross.

Note that $\widehat{\boldsymbol{Z}} = \widehat{\boldsymbol{X}}$ (cf. Proposition 3.2.28). Let $g\colon \boldsymbol{Z} \to \mathbb{C}$ be given by the formula

$$g(z', z_N) := \begin{cases} f(z', z_N) & \text{if } (z', z_N) \in (A'_N \setminus \Sigma_N) \times D_N, \\ \hat{f}_{z_N}(z') & \text{if } (z', z_N) \in \widehat{\boldsymbol{Y}} \times (A_N \setminus Q). \end{cases}$$

Observe that g is well defined.

Indeed, let $(z', z_N) \in ((A'_N \setminus \Sigma_N) \times D_N) \cap (\widehat{\boldsymbol{Y}} \times (A_N \setminus Q))$. If $z' \in \boldsymbol{T}(z_N)$, then obviously $\hat{f}_{z_N}(z') = f(z', z_N)$. Suppose that $z' \notin \boldsymbol{T}(z_N)$. Then $z' \in P(z_N)$, where

$$P(z_N) := \bigcap_{j=1}^{N-1} \big\{(w'_j, w_j, \widetilde{w}''_j) \in A'_j \times A_j \times \widetilde{A}''_j : (w'_j, \widetilde{w}''_j) \in (\Sigma_j)_{(\cdot, z_N)}\big\}.$$

In view of the definition of Q, the set $P(z_N)$ is pluripolar. Take a sequence

$$A'_N \setminus (\Sigma_N \cup P(z_N)) \ni z'^{\nu} \to z'.$$

Then $z'^{\nu} \in \boldsymbol{T}(z_N)$. Thus $\hat{f}_{z_N}(z'^{\nu}) = f(z'^{\nu}, z_N)$ and, using the definition of $\mathcal{O}^c_s(\boldsymbol{T})$ with $j := N$ and $b_N := z_N$, we get $\hat{f}_{z_N}(z') = f(z', z_N)$.

It is clear that $g \in \mathcal{O}_s(\boldsymbol{Z})$. By Theorem 5.4.1, we get a holomorphic extension $\hat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}})$ with $\hat{f} = g$ on $\boldsymbol{Z}$. It remains to show that $\hat{f} = f$ on $\boldsymbol{T}$.

Take a point $a \in \boldsymbol{T}$. If $a \in (A'_N \setminus \Sigma_N) \times D_N \subset \boldsymbol{Z}$, then $\hat{f}(a) = g(a) = f(a)$. In the remaining case we may assume that $a = (a_1, a''_1) \in D_1 \times (A''_1 \setminus \Sigma_1)$. Then

$$T_0 := \bigcup_{z_N \in A_N \setminus Q} \boldsymbol{T}(z_N) \times \{z_N\} \subset \widehat{\boldsymbol{Y}} \times (A_N \setminus Q) \subset \boldsymbol{Z}.$$

On the other hand, $T_0 \subset \bigcup_{z_N \in A_N \setminus Q} \boldsymbol{T}_{(\cdot, z_N)} \times \{z_N\} \subset \boldsymbol{T}$. Observe that if $b = (b', b_N) \in T_0$, then $\hat{f}(b) = g(b) = \hat{f}_{b_N}(b') = f(b)$. Thus, we only need to show that there exists a sequence $(b^\nu)_{\nu=1}^\infty \subset T_0 \cap (\{a_1\} \times (A''_1 \setminus \Sigma_1))$ with $b^\nu \to a$ (and then use the continuity of $f(a_1, \cdot)$ on $A''_1 \setminus \Sigma_1$).

Since Q is pluripolar, we may find a sequence $b_N^\nu \to a_N$ with $b_N^\nu \in A_N \setminus Q$. Let $P := \bigcup_{\nu=1}^\infty (\Sigma_1)_{(\cdot, b_N^\nu)}$. In view of the definition of Q, the set P is pluripolar. In particular, we may find a sequence $(b_2^\nu, \dots, b_{N-1}^\nu) \to (a_2, \dots, a_{N-1})$ with $(b_2^\nu, \dots, b_{N-1}^\nu) \in (A_2 \times \dots \times A_{N-1}) \setminus P$. Put $b^\nu := (a_1, b_2^\nu, \dots, b_N^\nu)$. Then $b^\nu \to a$ and obviously $b^\nu \in \boldsymbol{T}(b_N^\nu) \times \{b_N^\nu\} \subset T_0$. □

Corollary 7.1.6. *Let $\boldsymbol{T}$ be as in Theorem 7.1.4. Then $\boldsymbol{T}$ is connected.*

Proof. Suppose that $\boldsymbol{T} = U_1 \cup U_2$, where U_1, U_2 are non-empty disjoint relatively open subsets of $\boldsymbol{T}$. Define $f(z) := j$, $z \in U_j$, $j = 1, 2$. Then f is continuous and separately holomorphic on $\boldsymbol{T}$. Consequently, by Theorem 7.1.4, f extends to an $\hat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}})$ with $\hat{f}(\widehat{\boldsymbol{X}}) = \{1, 2\}$; a contradiction. □

Remark 7.1.7. In the context of Theorem 7.1.4, one may formulate the following general problem:

Assume that D_j is a Riemann domain of holomorphy over $\mathbb{C}^{n_j}$, $A_j \subset D_j$ is locally pluriregular, $\varnothing \neq B_j \subset A'_j \times A''_j$, $j = 1, \dots, N$,

$$\boldsymbol{W} := \bigcup_{j=1}^{N} \{(a'_j, a_j, a''_j) \in A'_j \times D_j \times A''_j : (a'_j, a''_j) \in B_j\}.$$

We say that a function $f : \boldsymbol{W} \to \mathbb{C}$ is *separately holomorphic* on $\boldsymbol{W}$ ($f \in \mathcal{O}_s(\boldsymbol{W})$) if for any $j \in \{1, \dots, N\}$ and $(a'_j, a''_j) \in B_j$, the function

$$D_j \ni z_j \mapsto f(a'_j, z_j, a''_j) \in \mathbb{C}$$

is holomorphic in D_j.

[?] Given an $f \in \mathcal{O}_s(\boldsymbol{W})$, we look for conditions on $B_1, \dots, B_N$, and f, under which there exists an open neighborhood Ω of $\boldsymbol{W}$ (independent of f) such that f extends holomorphically to Ω [?]

7.2 (N, k)-crosses

$\boxed{>}$ §§ 2.1, 2.3, 2.5, 3.2, 4.2, 5.4.

The problem of continuation of a separately holomorphic function $f : \boldsymbol{X} \to \mathbb{C}$ defined on an N-fold cross $\boldsymbol{X}$ describes the situation when we have $(N-1)$ "fixed" groups of variables and only one group "free", i.e. $f(a_1, \dots, a_{j-1}, \cdot, a_{j+1}, \dots, a_N)$ is holomorphic on D_j for arbitrarily fixed $(a_1, \dots, a_N) \in \boldsymbol{c}(\boldsymbol{X})$ and $j \in \{1, \dots, N\}$. Now we start discussing the situation where we have $(N-k)$ fixed groups and k free groups. More precisely, assume that D_j is a Riemann domain over $\mathbb{C}^{n_j}$ and $A_j \subset D_j$ is locally pluriregular, $j = 1, \dots, N$, $N \geq 2$. For $\alpha = (\alpha_1, \dots, \alpha_N) \in \{0,1\}^N$ put

$$\mathfrak{X}_\alpha := \mathfrak{X}_{1,\alpha_1} \times \dots \times \mathfrak{X}_{N,\alpha_N}, \quad \mathfrak{X}_{j,\alpha_j} := \begin{cases} D_j & \text{if } \alpha_j = 1, \\ A_j & \text{if } \alpha_j = 0. \end{cases}$$

Definition 7.2.1. For $k \in \{1, \dots, N\}$ define an (N,k)-cross

$$\boldsymbol{X}_{N,k} = \mathbb{X}_{N,k}((A_j, D_j)_{j=1}^N) := \bigcup_{\alpha \in \{0,1\}^N,\, |\alpha| = k} \mathfrak{X}_\alpha$$

In analogy to $\widehat{\boldsymbol{X}}$ we define

Definition 7.2.2.

$$\begin{aligned} \widehat{\boldsymbol{X}}_{N,k} &= \widehat{\mathbb{X}}_{N,k}((A_j, D_j)_{j=1}^N) \\ &:= \Big\{(z_1, \dots, z_N) \in D_1 \times \dots \times D_N : \textstyle\sum_{j=1}^N \boldsymbol{h}^*_{A_j, D_j}(z_j) < k\Big\}. \end{aligned}$$

Remark 7.2.3 (Properties of (N,k)-crosses). The reader is asked to complete details (cf. Remark 5.1.8).

(a) $\boldsymbol{X}_{N,1} = \mathbb{X}((A_j, D_j)_{j=1}^N)$, $\widehat{\boldsymbol{X}}_{N,1} = \widehat{\mathbb{X}}((A_j, D_j)_{j=1}^N)$.

(b) $\boldsymbol{X}_{N,N} = D_1 \times \dots \times D_N = \widehat{\boldsymbol{X}}_{N,N}$.

(c) $A_1 \times \dots \times A_N \subset \boldsymbol{X}_{N,k} \subset \widehat{\boldsymbol{X}}_{N,k}$.

(d) $\boldsymbol{X}_{N,k}$ is arcwise connected.

(e) If $(D_{j,k})_{k=1}^\infty$ is a sequence of subdomains of D_j such that $D_{j,k} \nearrow D_j$, $D_{j,k} \supset A_{j,k} \nearrow A_j$, $j = 1, \dots, N$, then

$$\mathbb{X}_{N,k}((A_{j,k}, D_{j,k})_{j=1}^N) \nearrow \boldsymbol{X}_{N,k}, \quad \widehat{\mathbb{X}}_{N,k}((A_{j,k}, D_{j,k})_{j=1}^N) \nearrow \widehat{\boldsymbol{X}}_{N,k}$$

(cf. Proposition 3.2.25).

(f) $\widehat{\boldsymbol{X}}_{N,k}$ is connected.

By (e) we may assume that $D_j \in \mathfrak{R}_b(\mathbb{C}^{n_j})$, $j = 1, \dots, N$. Since $\boldsymbol{X}_{N,k}$ is connected (cf. (d)), it suffices to show that every point $a = (a_1, \dots, a_N) \in \widehat{\boldsymbol{X}}_{N,k}$ may be connected in $\widehat{\boldsymbol{X}}_{N,k}$ with a point from $A_1 \times \dots \times A_N \subset \boldsymbol{X}_{N,k}$. Put $\alpha :=$

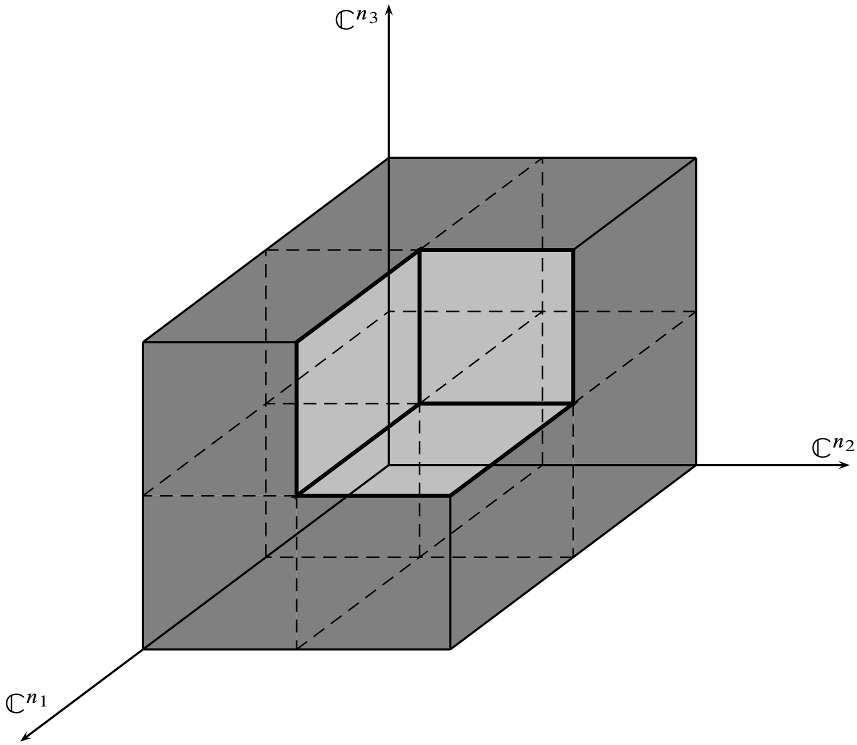


Figure 7.2.1. $\boldsymbol{X}_{3,2} = (A_1 \times D_2 \times D_3) \cup (D_1 \times A_2 \times D_3) \cup (A_1 \times D_2 \times D_3)$.

$\left(\sum_{j=1}^{N-1} \boldsymbol{h}^*_{A_j,D_j}(a_j)\right) - k + 1 < 1$. Observe that the connected component of the open set $\{z_N \in D_N : \boldsymbol{h}^*_{A_N,D_N}(z_N) < 1 - \alpha\}$ that contains a_N, intersects A_N (cf. Proposition 3.2.27). Consequently, a may be connected inside of $\widehat{\boldsymbol{X}}_{N,k}$ with $(a_1, \dots, a_{N-1}, b_N)$, where $b_N \in A_N$. Repeating the above argument, we easily show that a may be connected inside of $\widehat{\boldsymbol{X}}_{N,k}$ with a point $b \in A_1 \times \dots \times A_N$.

(g) If $D_1, \dots, D_N$ are domains of holomorphy, then $\widehat{\boldsymbol{X}}_{N,k}$ is a domain of holomorphy (cf. Theorem 2.5.5 (e)).

(h) $\boldsymbol{X}_{N,k} \subset \boldsymbol{X}_{N,k+1}$, $\widehat{\boldsymbol{X}}_{N,k} \subset \widehat{\boldsymbol{X}}_{N,k+1}$, $k = 1, \dots, N-1$.

(i) $\boldsymbol{X}_{N,k} = (\boldsymbol{X}_{N-1,k-1} \times D_N) \cup (\boldsymbol{X}_{N-1,k} \times A_N)$, $k = 2, \dots, N-1$, $N \geq 3$.

Example 7.2.4. If $\boldsymbol{X}_{N,k} := \mathbb{X}_{N,k}(((-1,1), \mathbb{D})_{j=1}^N)$, then

$$\widehat{\boldsymbol{X}}_{N,k} = \left\{(z_1, \dots, z_N) \in \mathbb{D}^N : \sum_{j=1}^N \left|\operatorname{Arg} \frac{1+z_j}{1-z_j}\right| < \frac{k\pi}{2}\right\}.$$

Definition 7.2.5. We say that a function $f : \boldsymbol{X}_{N,k} \to \mathbb{C}$ is *separately holomorphic* ($f \in \mathcal{O}_s(\boldsymbol{X}_{N,k})$) if for all $(a_1, \dots, a_N) \in A_1 \times \dots \times A_N$ and $\alpha = (\alpha_1, \dots, \alpha_N) \in$

$\{0,1\}^N$ with $|\alpha| = k$, the function $f|_{\mathfrak{X}_\alpha}$ is holomorphic with respect to the k groups of free variables, i.e. the function $D_\alpha \ni z \mapsto f(i_{a,\alpha}(z))$ is holomorphic, where

$$D_\alpha := \prod_{j\in\{1,\dots,N\}:\,\alpha_j=1} D_j, \quad i_{a,\alpha} = (i_{a,\alpha,1},\dots,i_{a,\alpha,N})\colon D_\alpha \to \mathfrak{X}_\alpha,$$

$$i_{a,\alpha,j}(z) := \begin{cases} z_j & \text{if } \alpha_j = 1, \\ a_j, & \text{if } \alpha_j = 0, \end{cases} \quad j = 1,\dots,N.$$

For instance, if $\alpha = (1,\dots,1,0,\dots,0)$, then this means that the function

$$D_1 \times \dots \times D_k \ni (z_1,\dots,z_k) \mapsto f(z_1,\dots,z_k,a_{k+1},\dots,a_N)$$

is holomorphic.

Proposition 7.2.6. *Let $\varphi_j\colon D_j \to \widetilde{D}_j$ be the envelope of holomorphy (cf. Definition 2.1.18). Observe that $\widetilde{A}_j := \varphi_j(A_j) \subset \widetilde{D}_j$ is locally pluriregular (because φ_j is locally biholomorphic; cf. Remark 2.1.11* (b)*), $j = 1,\dots,N$. Put*

$$\varphi\colon D_1 \times \dots \times D_N \to \widetilde{D}_1 \times \dots \times \widetilde{D}_N, \quad \varphi(z_1,\dots,z_N) := (\varphi_1(z_1),\dots,\varphi_N(z_N)).$$

Let

$$\boldsymbol{Y}_{N,k} := \mathbb{X}_{N,k}((\widetilde{A}_j,\widetilde{D}_j)_{j=1}^N), \quad \widehat{\boldsymbol{Y}}_{N,k} := \widehat{\mathbb{X}}_{N,k}((\widetilde{A}_j,\widetilde{D}_j)_{j=1}^N).$$

Then

- $\varphi(\boldsymbol{X}_{N,k}) \subset \boldsymbol{Y}_{N,k}$, $\varphi(\widehat{\boldsymbol{X}}_{N,k}) \subset \widehat{\boldsymbol{Y}}_{N,k}$,
- *for each function $f \in \mathcal{O}_s(\boldsymbol{X}_{N,k})$ there exists a function $\tilde{f} \in \mathcal{O}_s(\boldsymbol{Y}_{N,k})$ such that $\tilde{f} \circ \varphi \equiv f$.*

Proof. The inclusion $\varphi(\boldsymbol{X}_{N,k}) \subset \boldsymbol{Y}_{N,k}$ is trivial. The inclusion $\varphi(\widehat{\boldsymbol{X}}_{N,k}) \subset \widehat{\boldsymbol{Y}}_{N,k}$ follows immediately from the fact that $\boldsymbol{h}^*_{\varphi_j(A_j),\widetilde{D}_j} \circ \varphi_j \le \boldsymbol{h}^*_{A_j,D_j}$, $j = 1,\dots,N$.

Fix an $f \in \mathcal{O}_s(\boldsymbol{X}_{N,k})$. Take $a = (a_1,\dots,a_N), b = (b_1,\dots,b_N) \in A_1 \times \dots \times A_N$ and $\alpha = (\alpha_1,\dots,\alpha_N), \beta = (\beta_1,\dots,\beta_N) \in \{0,1\}^N$ with $|\alpha| = |\beta| = k$. To simplify notation, suppose that $\alpha = (1,\dots,1,0,\dots,0)$.

First observe that if $\varphi_j(a_j) = \varphi_j(b_j)$, $j = k+1,\dots,N$, then

$$f(\cdot,a_{k+1},\dots,a_N) \equiv f(\cdot,b_{k+1},\dots,b_N) \text{ on } D_1 \times \dots \times D_k.$$

Indeed, since $\varphi_j\colon D_j \to \widetilde{D}_j$ is the envelope of holomorphy, for each $g_j \in \mathcal{O}(D_j)$ there exists a $\tilde{g}_j \in \mathcal{O}(\widetilde{D}_j)$ such that $g_j \equiv \tilde{g}_j \circ \varphi_j$. In particular, if $\varphi_j(z_j) = \varphi_j(w_j)$, then $g_j(z_j) = g_j(w_j)$. Take arbitrary $c_j \in A_j$, $j = 1,\dots,k$. Then

$$\begin{aligned} f(c_1,\dots,c_k,a_{k+1},\dots,a_N) &= f(c_1,\dots,c_k,b_{k+1},a_{k+2},\dots,a_N) \\ &= \dots = f(c_1,\dots,c_k,b_{k+1},\dots,b_N). \end{aligned}$$

Thus $f(\cdot, a_{k+1}, \dots, a_N) = f(\cdot, b_{k+1}, \dots, b_N)$ on $A_1 \times \dots \times A_k$. It remains to use the identity principle.

Recall that

$$(\varphi_1 \times \dots \times \varphi_k)\colon D_1 \times \dots \times D_k \to \widetilde{D}_1 \times \dots \times \widetilde{D}_k$$

is the envelope of holomorphy (cf. [Jar-Pfl 2000], Proposition 1.8.15 (b)). Consequently, the formula

$$\tilde{f}_\alpha(\cdot, \varphi_{k+1}(a_{k+1}), \dots, \varphi_N(a_N)) := ((\varphi_1 \times \dots \times \varphi_k)^*)^{-1}(f(\cdot, a_{k+1}, \dots, a_N))$$

defines a holomorphic mapping on

$$\widetilde{\mathfrak{X}}_\alpha := \widetilde{D}_1 \times \dots \times \widetilde{D}_k \times \widetilde{A}_{k+1} \times \dots \times \widetilde{A}_N$$

with $\tilde{f}_\alpha \circ \varphi = f$ on $\mathfrak{X}_\alpha$.

In particular, $\tilde{f}_\alpha \circ \varphi = f = \tilde{f}_\beta \circ \varphi$ on $A_1 \times \dots \times A_N$. Hence, by the identity principle, $\tilde{f}_\alpha = \tilde{f}_\beta$ on $\widetilde{\mathfrak{X}}_\alpha \cap \widetilde{\mathfrak{X}}_\beta$. □

The following result is a generalization of the main extension theorem for crosses (Theorem 5.4.1).

Theorem 7.2.7 (Extension theorem for (N,k)-crosses). *For every $f \in \mathcal{O}_s(\boldsymbol{X}_{N,k})$ there exists an $\hat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}}_{N,k})$ such that $\hat{f} = f$ on $\boldsymbol{X}_{N,k}$ and $\hat{f}(\widehat{\boldsymbol{X}}_{N,k}) \subset f(\boldsymbol{X}_{N,k})$ (in particular, $\|\hat{f}\|_{\widehat{\boldsymbol{X}}_{N,k}} = \|f\|_{\boldsymbol{X}_{N,k}}$).*

See also Theorem 7.2.11.

Proof. The inclusion $\hat{f}(\widehat{\boldsymbol{X}}) \subset f(\boldsymbol{X})$ follows from Lemma 2.1.14 with

$$(G, D, A_0, A, \mathcal{F}) = (\widehat{\boldsymbol{X}}_{N,k}, \widehat{\boldsymbol{X}}_{N,k}, \boldsymbol{X}_{N,k}, \boldsymbol{X}_{N,k}, \mathcal{O}_s(\boldsymbol{X}_{N,k})).$$

Using Proposition 7.2.6, we may assume that D_j is a domain of holomorphy; moreover, by Remark 7.2.3 (e), we may assume that $D_j \in \mathfrak{R}_b(\mathbb{C}^{n_j})$ and $A_j \subset\subset D_j$, $j = 1, \dots, N$.

The case $k = N$ is trivial. The case $k = 1$ is the cross theorem (Theorem 5.4.1). In particular, there is nothing to prove for $N = 2$. We apply induction on N. Suppose that the result is true for $N - 1 \geq 2$.

Now we apply finite induction on k. The case $k = 1$ is known. Suppose that the result is true for $k - 1$ with $2 \leq k \leq N - 1$.

Fix an $f \in \mathcal{O}_s(\boldsymbol{X}_{N,k})$. Recall that

$$\boldsymbol{X}_{N,k} = (\boldsymbol{X}_{N-1,k-1} \times D_N) \cup (\boldsymbol{X}_{N-1,k} \times A_N).$$

For each $z_N \in D_N$ the function $f(\cdot, z_N)$ belongs to $\mathcal{O}_s(\boldsymbol{X}_{N-1,k-1})$. By the inductive assumption there exists a $g_{z_N} \in \mathcal{O}(\widehat{\boldsymbol{X}}_{N-1,k-1})$ such that $g_{z_N} = f(\cdot, z_N)$ on

$\boldsymbol{X}_{N-1,k-1}$. Analogously, for each $z_N \in A_N$ there exists an $h_{z_N} \in \mathcal{O}(\widehat{\boldsymbol{X}}_{N-1,k})$ such that $h_{z_N} = f(\cdot, z_N)$ on $\boldsymbol{X}_{N-1,k}$. Recall that $\widehat{\boldsymbol{X}}_{N-1,k-1} \subset \widehat{\boldsymbol{X}}_{N-1,k}$ and $A_1 \times \cdots \times A_{N-1} \subset \boldsymbol{X}_{N-1,k-1} \cap \boldsymbol{X}_{N-1,k}$. Since the set $A_1 \times \cdots \times A_{N-1}$ is not pluripolar, we get $g_{z_N} = h_{z_N}$ on $\widehat{\boldsymbol{X}}_{N-1,k-1}$ for $z_N \in A_N$.

Consider the cross

$$\boldsymbol{Y} := \mathbb{X}(\widehat{\boldsymbol{X}}_{N-1,k-1}, A_N; \widehat{\boldsymbol{X}}_{N-1,k}, D_N) = (\widehat{\boldsymbol{X}}_{N-1,k-1} \times D_N) \cup (\widehat{\boldsymbol{X}}_{N-1,k} \times A_N)$$

and let $F : \boldsymbol{Y} \to \mathbb{C}$,

$$F(z', z_N) := \begin{cases} g_{z_N}(z') & \text{if } (z', z_N) \in \widehat{\boldsymbol{X}}_{N-1,k-1} \times D_N, \\ h_{z_N}(z') & \text{if } (z', z_N) \in \widehat{\boldsymbol{X}}_{N-1,k} \times A_N. \end{cases}$$

To see that $F \in \mathcal{O}_s(\boldsymbol{Y})$, we have to prove that for each $z' \in \widehat{\boldsymbol{X}}_{N-1,k-1}$, the function $D_N \ni z_N \mapsto F(z', z_N)$ is holomorphic, or equivalently (by the Hartogs theorem), that $F \in \mathcal{O}(\widehat{\boldsymbol{X}}_{N-1,k-1} \times D_N)$. We know that $F(\cdot, z_N)$ is holomorphic for each $z_N \in D_N$. To prove that $F \in \mathcal{O}(\widehat{\boldsymbol{X}}_{N-1,k-1} \times D_N)$ we are going to apply Terada's theorem (Theorem 4.2.2). Let $\boldsymbol{Z}_{N-1,k-1} := \mathbb{X}_{N-1,k-1}((A_j, D_j)_{j=2}^N)$. Analogously as above, for each $z_1 \in D_1$ there exists a $\varphi_{z_1} \in \mathcal{O}(\widehat{\boldsymbol{Z}}_{N-1,k-1})$ such that $\varphi_{z_1} = f(z_1, \cdot)$ on $\boldsymbol{Z}_{N-1,k-1}$. Thus

$$F(z_1, \dots, z_N) = f(z_1, \dots, z_N) = \varphi_{z_1}(z_2, \dots, z_N),$$
$$(z_1, \dots, z_N) \in (\boldsymbol{X}_{N-1,k-1} \times D_N) \cap (D_1 \times \boldsymbol{Z}_{N-1,k-1}) \supset A_1 \times \cdots \times A_{N-1} \times D_N.$$

Consequently, $F(z', \cdot) \in \mathcal{O}(D_N)$ for $z' \in A_1 \times \cdots \times A_{N-1}$.

Now, by the cross theorem (Theorem 5.4.1), there exists an $\hat{f} \in \mathcal{O}(\widehat{\boldsymbol{Y}})$ such that $\hat{f} = F$ on $\boldsymbol{Y}$ (in particular, $\hat{f} = f$ on $\boldsymbol{X}_{N,k}$). Recall that

$$\widehat{\boldsymbol{Y}} = \{(z', z_N) \in \widehat{\boldsymbol{X}}_{N-1,k} \times D_N : \boldsymbol{h}^*_{\widehat{\boldsymbol{X}}_{N-1,k-1}, \widehat{\boldsymbol{X}}_{N-1,k}}(z') + \boldsymbol{h}^*_{A_N, D_N}(z_N) < 1\}.$$

Thus, it remains to apply the following lemma.

Lemma 7.2.8. *For an arbitrary Riemann domain of holomorphy $D_j \in \mathfrak{R}(\mathbb{C}^{n_j})$ and a locally pluriregular set $A_j \subset D_j$, $j = 1, \dots, N$, we have*

$$L_{N,k}(z) := \boldsymbol{h}^*_{\widehat{\boldsymbol{X}}_{N,k-1}, \widehat{\boldsymbol{X}}_{N,k}}(z) = \max\left\{0, \textstyle\sum_{j=1}^N \boldsymbol{h}^*_{A_j, D_j}(z_j) - k + 1\right\} =: R_{N,k}(z),$$
$$z = (z_1, \dots, z_N) \in \widehat{\boldsymbol{X}}_{N,k}.$$

Proof. By Proposition 3.2.25, we may assume that $D_j \in \mathfrak{R}_b(\mathbb{C}^{n_j})$ and $A_j \subset\subset D_j$, $j = 1, \dots, N$. We apply induction on N and the following lemma.

Lemma 7.2.9. *Let $D \in \mathfrak{R}_b(\mathbb{C}^n)$ be a Riemann domain of holomorphy and let $A \subset D$ be non-pluripolar. Put*

$$\Delta(r) := \{z \in D : \boldsymbol{h}^*_{A,D}(z) < r\}, \quad \Delta[r] := \{z \in D : \boldsymbol{h}^*_{A,D}(z) \le r\}, \quad 0 < r \le 1.$$

Then for $0 < r < s \le 1$ *we have*

$$L := \boldsymbol{h}^*_{\Delta(r),\Delta(s)} = \max\left\{0, \frac{\boldsymbol{h}^*_{A,D} - r}{s-r}\right\} =: R \quad \text{on } \Delta(s). \tag{‡}$$

Remark 7.2.10. (a) ? We do not know whether Lemmas 7.2.8 and 7.2.9 are true for an arbitrary Riemann domain $D \in \mathfrak{R}_b(\mathbb{C}^n)$?

(b) Observe that

$$L = \boldsymbol{h}_{\Delta(r),\Delta(s)} \ge \boldsymbol{h}^*_{\Delta[r],\Delta(s)} \ge \boldsymbol{h}_{\Delta[r],\Delta(s)} \ge R,$$
$$L = R = 0 \text{ on } \Delta(r), \quad R = 0 \text{ on } \Delta[r].$$

In particular, if (‡) is true (for given (D, A, r, s)), then

$$\boldsymbol{h}^*_{\Delta(r),\Delta(s)} = \boldsymbol{h}^*_{\Delta[r],\Delta(s)} = \boldsymbol{h}_{\Delta[r],\Delta(s)} = \max\left\{0, \frac{\boldsymbol{h}^*_{A,D} - r}{s-r}\right\} \text{ on } \Delta(s).$$

Proof of Lemma 7.2.9. Step 1^0. Reduction to the case $s = 1$.

Suppose that $0 < r < s < 1$. Observe that $\Delta(s)$ is a Riemann region of holomorphy (provided D is a domain of holomorphy). Moreover, by Proposition 3.2.27, $A \cap S$ is not pluripolar for every connected component S of $\Delta(s)$. By Proposition 3.2.27, we have $\boldsymbol{h}^*_{A,\Delta(s)} = (1/s)\boldsymbol{h}^*_{A,D}$ on $\Delta(s)$. Hence,

$$L = \boldsymbol{h}_{\Delta(r),\Delta(s)} = \boldsymbol{h}_{\{\boldsymbol{h}^*_{A,\Delta(s)} < r/s\},\Delta(s)}, \quad R = \max\left\{0, \frac{\boldsymbol{h}^*_{A,\Delta(s)} - r/s}{1 - r/s}\right\}.$$

Thus the problem for (D, A, r, s) reduces to $(S, A \cap S, r/s, 1)$, where S is a connected component of $\Delta(s)$.

From now on we assume that $s = 1$.

Step 2^0. Approximation. Let $A_\nu \nearrow A$, $D_\nu \nearrow D$, where $A_\nu \subset D_\nu$ is non-pluripolar, $\nu \in \mathbb{N}$. Suppose that (‡) holds for each (D_ν, A_ν, r). Then it holds for (D, A, r).

Indeed, we know (Proposition 3.2.23) that $\boldsymbol{h}^*_{A_\nu,D_\nu} \searrow \boldsymbol{h}^*_{A,D}$. Hence $\{\boldsymbol{h}^*_{A_\nu,D} < r\} \nearrow \Delta(r)$. Thus $\boldsymbol{h}^*_{\{\boldsymbol{h}^*_{A_\nu,D} < r\},D} \searrow \boldsymbol{h}^*_{\Delta(r),D}$.

In particular, we may always assume that D *is strongly pseudoconvex and* $A \subset\subset D$.

Step 3^0. If (‡) holds for all non-pluripolar compact sets A, then it holds for all non-pluripolar sets A.

Indeed, first observe that by Step 2^0, formula (‡) holds for all non-empty open sets A.

Now let A be an arbitrary non-pluripolar set. Using once again Step 2^0 we may assume that $A \subset\subset D$. Since $\Delta(\varepsilon)$ is open, we have

$$\boldsymbol{h}^*_{\{\boldsymbol{h}^*_{\Delta(\varepsilon),D} < r\},D} = \max\left\{0, \frac{\boldsymbol{h}^*_{\Delta(\varepsilon),D} - r}{1-r}\right\}, \quad 0 < \varepsilon < 1.$$

By Proposition 3.2.15, we get

$$\frac{\boldsymbol{h}^*_{A,D} - \varepsilon}{1 - \varepsilon} \le \boldsymbol{h}^*_{\varDelta(\varepsilon),D} \le \boldsymbol{h}^*_{A,D},$$

in particular, $\boldsymbol{h}^*_{\varDelta(\varepsilon),D} \nearrow \boldsymbol{h}^*_{A,D}$ as $\varepsilon \searrow 0$. Moreover,

$$\Big\{\boldsymbol{h}^*_{\varDelta(\varepsilon),D} < \frac{r-\varepsilon}{1-\varepsilon}\Big\} \subset \varDelta(r) \subset \{\boldsymbol{h}^*_{\varDelta(\varepsilon),D} < r\}, \quad 0 < \varepsilon < r.$$

Consequently,

$$\begin{aligned} \max\Big\{0, \frac{\boldsymbol{h}^*_{\varDelta(\varepsilon),D} - \frac{r-\varepsilon}{1-\varepsilon}}{1 - \frac{r-\varepsilon}{1-\varepsilon}}\Big\} &= \boldsymbol{h}^*_{\{\boldsymbol{h}^*_{\varDelta(\varepsilon),D} < \frac{r-\varepsilon}{1-\varepsilon}\},D} \ge \boldsymbol{h}^*_{\varDelta(r),D} \\ &\ge \boldsymbol{h}^*_{\{\boldsymbol{h}^*_{\varDelta(\varepsilon),D} < r\},D} = \max\Big\{0, \frac{\boldsymbol{h}^*_{\varDelta(\varepsilon),D} - r}{1-r}\Big\}, \quad 0 < \varepsilon < r. \end{aligned}$$

Letting $\varepsilon \searrow 0$, we get (‡).

Thus the proof reduces to the case where A is compact.

Step 4^0. The case where D is hyperconvex, A is compact, and $\boldsymbol{h}^*_{A,D}$ is continuous.

Let $u \in \mathcal{PSH}(D)$, $u \le 1$, $u \le 0$ on $\varDelta[r]$. Using continuity of $\boldsymbol{h}^*_{A,D}$ and Theorem 3.2.31 (a), we easily conclude that $\varDelta[r]$ is compact. Let $U := D \setminus \varDelta[r]$. Observe that for $z_0 \in \partial U$ we get

$$\begin{aligned} &\liminf_{U \ni z \to z_0} (\boldsymbol{h}^*_{A,D}(z) - (1-r)u(z) - r) \\ &\qquad \ge \left\{ \begin{matrix} 1 - (1-r)\limsup\limits_{z\to z_0} u(z) - r & \text{if } z_0 \in \partial D \\ r - (1-r)\limsup\limits_{z\to z_0} u(z) - r & \text{if } z_0 \in \varDelta[r] \end{matrix} \right\} \ge 0. \end{aligned}$$

Hence, by Corollary 3.2.33, $(1-r)u + r \le \boldsymbol{h}^*_{A,D}$ in U. This shows that $\boldsymbol{h}_{\varDelta[r],D} \le R$. Thus, we get $\boldsymbol{h}^*_{\varDelta[r],D} \equiv R$ for all $0 < r < 1$. Observe that $\varDelta[r_\nu] \nearrow \varDelta(r)$ for $0 < r_\nu \nearrow r$. Consequently, by Proposition 3.2.25, $L \equiv R$.

Step 5^0. The case where D is hyperconvex and A is compact.

By Theorem 3.2.31 (d) we know that $\boldsymbol{h}_{A^{(\varepsilon)},D} = \boldsymbol{h}^*_{A^{(\varepsilon)},D}$ is continuous. Thus, using Step 3^0, we have

$$\boldsymbol{h}_{\{\boldsymbol{h}_{A^{(\varepsilon)},D} \le r\},D} = \max\Big\{0, \frac{\boldsymbol{h}_{A^{(\varepsilon)},D} - r}{1-r}\Big\}, \quad 0 < \varepsilon \ll 1.$$

By Proposition 3.2.24 we have $\boldsymbol{h}_{A^{(\varepsilon)},D} \nearrow \boldsymbol{h}_{A,D}$ as $\varepsilon \searrow 0$. In particular,

$$\{\boldsymbol{h}_{A^{(\varepsilon)},D} \le r\} \searrow \{\boldsymbol{h}_{A,D} \le r\} \quad \text{as } \varepsilon \searrow 0.$$

Hence, once again by Proposition 3.2.24,

$$\boldsymbol{h}_{\{\boldsymbol{h}_{A^{(\varepsilon)},D}\le r\},D} \nearrow \boldsymbol{h}_{\{\boldsymbol{h}_{A,D}\le r\},D} \quad \text{as } \varepsilon \searrow 0.$$

Consequently,

$$\boldsymbol{h}_{\{\boldsymbol{h}_{A,D}\le r\},D} = \max\left\{0, \frac{\boldsymbol{h}_{A,D} - r}{1-r}\right\} \le R.$$

Thus $\boldsymbol{h}^*_{\{\boldsymbol{h}_{A,D}\le r\},D} \le R$. Observe that the set $\{\boldsymbol{h}_{A,D} \le r\} \setminus \Delta[r]$ is pluripolar. Consequently, $\boldsymbol{h}^*_{\Delta[r],D} \le R$. We finish the proof as in Step 4^0. □

We move to the proof of Lemma 7.2.8. Fix $2 \le k \le N$. Let

$$h_j := \boldsymbol{h}^*_{A_j,D_j}, \ j = 1, \dots, N, \quad h(z_1, \dots, z_N) := h_1(z_1) + \dots + h_N(z_N).$$

It is clear that $L_{N,k} \ge R_{N,k}$ and $L_{N,k} = R_{N,k} = 0$ on $\widehat{\boldsymbol{X}}_{N,k-1}$. Fix an $a = (a_1, \dots, a_N) \in \widehat{\boldsymbol{X}}_{N,k} \setminus \widehat{\boldsymbol{X}}_{N,k-1}$. We may assume that $h_1(a_1) \le \dots \le h_N(a_N)$. Suppose that $h_1(a_1) = \dots = h_s(a_s) = 0$ and $h_{s+1}(a_{s+1}), \dots, h_N(a_N) > 0$ for an $s \in \{0, \dots, N\}$. Since $h(a) \ge k - 1$, we see that in fact $s \le N - k \le N - 2$. In particular, if $N = 2$, then $s = 0$.

Let $\widehat{\boldsymbol{Y}}_{N-s,p} = \widehat{\mathbb{X}}_{N-s,p}((A_j, D_j)_{j=s+1}^N)$, $p \in \{k-1, k\}$. Observe that

$$\{a_1, \dots, a_s\} \times \widehat{\boldsymbol{Y}}_{N-s,p} \subset \widehat{\boldsymbol{X}}_{N,p}, \quad p \in \{k-1, k\}.$$

Consequently,

$$\boldsymbol{h}^*_{\widehat{\boldsymbol{X}}_{N,k-1},\widehat{\boldsymbol{X}}_{N,k}}(a) \le \boldsymbol{h}^*_{\widehat{\boldsymbol{Y}}_{N-s,k-1},\widehat{\boldsymbol{Y}}_{N-s,k}}(a_{s+1}, \dots, a_N).$$

Thus, if we know that $L_{N-s,k}(a_{s+1}, \dots, a_N) \le R_{N-s,k}(a_{s+1}, \dots, a_N)$, then

$$L_{N,k}(a) \le R_{N-s,k}(a_{s+1}, \dots, a_N) = R_{N,k}(a).$$

This reduces the proof to the case $s = 0$, i.e. $h_j(a_j) > 0$, $j = 1, \dots, N$.

Put

$$\Delta_{j,t} := \{z_j \in D_j : h_j(z_j) < t\}, \quad j = 1, \dots, N.$$

Take $0 < r_j < s_j \le 1$, $j = 1, \dots, N$, such that $r_1 + \dots + r_N = k - 1$ and $s_1 + \dots + s_N = k$. Observe that

$$\Delta_{1,r_1} \times \dots \times \Delta_{N,r_N} \subset \widehat{\boldsymbol{X}}_{N,k-1}, \quad \Delta_{1,s_1} \times \dots \times \Delta_{N,s_N} \subset \widehat{\boldsymbol{X}}_{N,k}.$$

Hence, using the product property for the relative extremal function (cf. Theorem 3.2.17) and Lemma 7.2.9, we get

$$\begin{aligned} L_{N,k}(z) &\le \boldsymbol{h}^*_{\Delta_{1,r_1}\times\dots\times\Delta_{N,r_N},\Delta_{1,s_1}\times\dots\times\Delta_{N,s_N}}(z) \\ &= \max\{\boldsymbol{h}^*_{\Delta_{1,r_1},\Delta_{1,s_1}}(z_1), \dots, \boldsymbol{h}^*_{\Delta_{N,r_N},\Delta_{N,s_N}}(z_N)\} \\ &= \max\left\{0, \frac{h_1(z_1) - r_1}{s_1 - r_1}, \dots, \frac{h_N(z_N) - r_N}{s_N - r_N}\right\}, \\ &\qquad\qquad z = (z_1, \dots, z_N) \in \Delta_{1,s_1} \times \dots \times \Delta_{N,s_N}. \end{aligned}$$

Observe that there exist numbers $s_1, \dots, s_N \in (0, 1]$ such that $s_1 + \dots + s_N = k$ and

$$h_j(a_j) < s_j < \frac{h_j(a_j)}{h(a) - k + 1}, \quad j = 1, \dots, N.$$

Indeed, since the case where $h(a) = k - 1$ is trivial, we may assume that $h(a) > k - 1$. Note that $h_j(a_j) < \frac{h_j(a_j)}{h(a)-k+1}$, $j = 1, \dots, N$. Suppose that

$$\frac{h_j(a_j)}{h(a) - k + 1} \le 1,\ j = 1, \dots, \sigma, \qquad \frac{h_j(a_j)}{h(a) - k + 1} > 1,\ j = \sigma + 1, \dots, N,$$

for a $\sigma \in \{0, \dots, N\}$. Observe that

$$\sum_{j=1}^{N} \frac{h_j(a_j)}{h(a) - k + 1} = \frac{h(a)}{h(a) - k + 1} > k,$$

so the case $\sigma = N$ is simple. Thus, assume that $\sigma \le N - 1$. We only need do show that

$$\Big(\sum_{j=1}^{\sigma} \frac{h_j(a_j)}{h(a) - k + 1}\Big) + N - \sigma > k.$$

The case where $\sigma \le N - k$ is obvious. Thus assume that $\sigma \ge N - k + 1$. We have to show that

$$\begin{aligned}\sum_{j=1}^{\sigma} h_j(a_j) &> (h(a) - k + 1)(k - N + \sigma)\\ &= (k - 1 - N + \sigma)h(a) + \Big(\sum_{j=1}^{\sigma} h_j(a_j)\Big)\\ &\quad + \Big(\sum_{j=\sigma+1}^{N} h_j(a_j)\Big) + (-k + 1)(k - N + \sigma),\end{aligned}$$

or equivalently,

$$(k - 1 - N + \sigma)h(a) + \Big(\sum_{j=\sigma+1}^{N} h_j(a_j)\Big) < (k - 1)(k - N + \sigma).$$

We have

$$\begin{aligned}(k - 1 - N + \sigma)h(a) &+ \Big(\sum_{j=\sigma+1}^{N} h_j(a_j)\Big)\\ &< (k - 1 - N + \sigma)k + N - \sigma \le (k - 1)(k - N + \sigma),\end{aligned}$$

which gives the required inequality.

Now define

$$r_j := \frac{h_j(a_j) - s_j(h(a) - k + 1)}{k - h(a)}, \quad j = 1, \dots, N.$$

Then

- $r_j > 0$ because $s_j < \frac{h_j(a_j)}{h(a)-k+1}$,
- $r_j < s_j$ because $h_j(a) < s_j$,
- $r_1 + \cdots + r_N = k - 1$,
- $\frac{h_j(a_j)-r_j}{s_j-r_j} = h(a) - k + 1$, $j = 1, \dots, N$.

Thus

$$\begin{aligned} L_{N,k}(a) &\le \max\left\{0, \frac{h_1(a_1) - r_1}{s_1 - r_1}, \dots, \frac{h_N(a_N) - r_N}{s_N - r_N}\right\} \\ &= \max\{0, h(a) - k + 1\} = R_{N,k}(a). \end{aligned}$$

□ □

Proposition 7.2.6 permits us to strengthen Theorem 7.2.7.

Theorem 7.2.11. *Let $\varphi_j : D_j \to \widetilde{D}_j$ be the envelope of holomorphy. Put*

$$\varphi: D_1 \times \cdots \times D_N \to \widetilde{D}_1 \times \cdots \times \widetilde{D}_N, \quad \varphi(z_1, \dots, z_N) := (\varphi_1(z_1), \dots, \varphi_N(z_N)).$$

Let

$$\boldsymbol{Y}_{N,k} := \mathbb{X}_{N,k}((\widetilde{A}_j, \widetilde{D}_j)_{j=1}^N), \quad \widehat{\boldsymbol{Y}}_{N,k} := \widehat{\mathbb{X}}_{N,k}((\widetilde{A}_j, \widetilde{D}_j)_{j=1}^N).$$

Then for every $f \in \mathcal{O}_s(\boldsymbol{X}_{N,k})$ there exists an $\hat{f} \in \mathcal{O}(\widehat{\boldsymbol{Y}}_{N,k})$ such that $\hat{f} \circ \varphi = f$ on $\boldsymbol{X}_{N,k}$ and $\hat{f}(\widehat{\boldsymbol{X}}_{N,k}) \subset f(\boldsymbol{X}_{N,k})$ (in particular, $\|\hat{f}\|_{\widehat{\boldsymbol{X}}_{N,k}} = \|f\|_{\boldsymbol{X}_{N,k}}$).

Corollary 7.2.12. *Under the notation of Proposition 7.2.6, $\varphi: \widehat{\boldsymbol{X}}_{N,k} \to \widehat{\boldsymbol{Y}}_{N,k}$ is the envelope of holomorphy.*

Proof. We know (Remark 7.2.3 (g) that $\widehat{\boldsymbol{Y}}_{N,k}$ is a domain of holomorphy. Take an $f \in \mathcal{O}(\widehat{\boldsymbol{X}}_{N,k})$. Then $f|_{\boldsymbol{X}_{N,k}} \in \mathcal{O}_s(\boldsymbol{X}_{N,k})$. Consequently, by Theorem 7.2.11, there exists an $\hat{f} \in \mathcal{O}(\widehat{\boldsymbol{Y}}_{N,k})$ such that $\hat{f} \circ \varphi = f$ on $\boldsymbol{X}_{N,k}$. Thus, by the identity principle (recall that $\widehat{\boldsymbol{X}}_{N,k}$ is connected – Remark 7.2.3 (f)), $\hat{f} \circ \varphi = f$ on $\widehat{\boldsymbol{X}}_{N,k}$. □

? It would be interesting to know whether Theorem 7.2.7 is true in the case of arbitrary complex manifolds (cf. Chapter 6) ?

7.3 Hartogs type theorem for 2-separately holomorphic functions

Let

$$X^q = \mathbb{X}_{N(q),k(q)}((A_{q,j}, D_{q,j})_{j=1}^{N(q)}), \quad q = 1, \dots, Q \ \text{(cf. § 7.2)}.$$

Definition 7.3.1. We say that a function $f: \boldsymbol{X}^1 \times \dots \times \boldsymbol{X}^Q \to \mathbb{C}$ is 2-*separately holomorphic* ($f \in \mathcal{O}_{ss}(\boldsymbol{X}^1 \times \dots \times \boldsymbol{X}^Q)$) if for all $q \in \{1, \dots, Q\}$ and $(x^1, \dots, x^Q) \in \boldsymbol{X}^1 \times \dots \times \boldsymbol{X}^Q$, the function $f(x^1, \dots, x^{q-1}, \cdot, x^{q+1}, \dots, x^Q)$ is separately holomorphic on $\boldsymbol{X}^q$.

Theorem 7.3.2 (Hartogs type theorem for 2-separately holomorphic functions). *For every $f \in \mathcal{O}_{ss}(\boldsymbol{X}^1 \times \dots \times \boldsymbol{X}^Q)$ there exists an $\hat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}^1} \times \dots \times \widehat{\boldsymbol{X}^Q})$ such that $\hat{f} = f$ on $\boldsymbol{X}^1 \times \dots \times \boldsymbol{X}^Q$ and $\hat{f}(\widehat{\boldsymbol{X}^1} \times \dots \times \widehat{\boldsymbol{X}^Q}) \subset f(\boldsymbol{X}^1 \times \dots \times \boldsymbol{X}^Q)$ (in particular, $\|\hat{f}\|_{\widehat{\boldsymbol{X}^1} \times \dots \times \widehat{\boldsymbol{X}^Q}} = \|f\|_{\boldsymbol{X}^1 \times \dots \times \boldsymbol{X}^Q}$).*

Proof. As always, the inclusion $\hat{f}(\widehat{\boldsymbol{X}^1} \times \dots \times \widehat{\boldsymbol{X}^Q}) \subset f(\boldsymbol{X}^1 \times \dots \times \boldsymbol{X}^Q)$ follows from Lemma 2.1.14.

We apply induction on Q. For $Q = 1$ the result follows from Theorem 7.2.7. Suppose that the result is true for $Q - 1 \geq 1$. Fix an $f \in \mathcal{O}_{ss}(\boldsymbol{X}^1 \times \dots \times \boldsymbol{X}^Q)$. For each $x^Q \in \boldsymbol{X}^q$, the function $f(\cdot, x^Q)$ belongs to $\mathcal{O}_{ss}(\boldsymbol{X}^1 \times \dots \times \boldsymbol{X}^{Q-1})$. Consequently, by the inductive assumption, there exists an $\hat{f}_{x^Q} \in \mathcal{O}(\widehat{\boldsymbol{X}^1} \times \dots \times \widehat{\boldsymbol{X}^{Q-1}})$ such that $\hat{f}_{x^Q} = f(\cdot, x^Q)$ on $\boldsymbol{X}^1 \times \dots \times \boldsymbol{X}^{Q-1}$. Define

$$g: \widehat{\boldsymbol{X}^1} \times \dots \times \widehat{\boldsymbol{X}^{Q-1}} \times \boldsymbol{X}^Q \to \mathbb{C}, \quad g(x^1, \dots, x^Q) := \hat{f}_{x^Q}(x^1, \dots, x^{Q-1}).$$

Then $g(x^1, \dots, x^{Q-1}, \cdot) \in \mathcal{O}_s(\boldsymbol{X}^Q)$ for arbitrary $(x^1, \dots, x^{Q-1}) \in \widehat{\boldsymbol{X}^1} \times \dots \times \widehat{\boldsymbol{X}^{Q-1}}$.

Indeed, take a branch $\mathfrak{X}_{Q,\alpha}$ of $\boldsymbol{X}^Q$. To simplify notation, suppose that

$$\mathfrak{X}_{Q,\alpha} = D_{Q,1} \times \dots \times D_{Q,k(Q)} \times A_{Q,k(Q)+1} \times \dots \times A_{Q,N(Q)}.$$

We have to prove that for arbitrary

$$(a_{Q,k(Q)+1}, \dots, a_{Q,N(Q)}) \in A_{Q,k(Q)+1} \times \dots \times A_{Q,N(Q)},$$

the function $g(x^1, \dots, x^{Q-1}, \cdot, a_{Q,k(Q)+1}, \dots, a_{Q,N(Q)})$ is holomorphic on $D_{Q,1} \times \dots \times D_{Q,k(Q)}$. Define $h: (\widehat{\boldsymbol{X}^1} \times \dots \times \widehat{\boldsymbol{X}^{Q-1}}) \times (D_{Q,1} \times \dots \times D_{Q,k(Q)}) \to \mathbb{C}$ by

$$\begin{aligned} &h(x^1, \dots, x^{Q-1}, x_{Q,1}, \dots, x_{Q,k(Q)}) \\ &\quad = g(x^1, \dots, x^{Q-1}, x_{Q,1}, \dots, x_{Q,k(Q)}, a_{Q,k(Q)+1}, \dots, a_{Q,N(Q)}). \end{aligned}$$

Observe that $h(\cdot, x_{Q,1}, \dots, x_{X,k(Q)})$ is holomorphic on $\widehat{\boldsymbol{X}^1} \times \dots \times \widehat{\boldsymbol{X}^{Q-1}}$ for arbitrary $(x_{Q,1}, \dots, x_{X,k(Q)}) \in D_{Q,1} \times \dots \times D_{Q,k(Q)}$. Moreover, $h(x^1, \dots, x^{Q-1}, \cdot)$ is holomorphic on $D_{Q,1} \times \dots \times D_{Q,k(Q)}$ for arbitrary $(x^1, \dots, x^{Q-1}) \in \boldsymbol{X}^1 \times \dots \times \boldsymbol{X}^{Q-1}$.

Now we use Terada's theorem ((Theorem 4.2.2) and we conclude that $h \in \mathcal{O}(\widehat{\boldsymbol{X}^1} \times \dots \times \widehat{\boldsymbol{X}^{Q-1}} \times D_{Q,1} \times \dots \times D_{Q,k(Q)})$.

Thus, for arbitrary $(x^1, \dots, x^{Q-1}) \in \widehat{\boldsymbol{X}^1} \times \dots \times \widehat{\boldsymbol{X}^{Q-1}}$, the function

$$g(x^1, \dots, x^{Q-1}, \cdot)$$

extends holomorphically to an $\hat{f}_{(x^1,\dots,x^{Q-1})} \in \mathcal{O}(\widehat{\boldsymbol{X}^Q})$. Finally, Proposition 1.1.10 implies that the function

$$\hat{f} : \widehat{\boldsymbol{X}^1} \times \dots \times \widehat{\boldsymbol{X}^Q} \to \mathbb{C}, \quad \hat{f}(x^1, \dots, x^Q) := \hat{f}_{(x^1,\dots,x^{Q-1})}(x^Q),$$

is a holomorphic extension of f to $\widehat{\boldsymbol{X}^1} \times \dots \times \widehat{\boldsymbol{X}^Q}$. □

Exercise 7.3.3. Formulate and prove a Hartogs type theorem for k-separately holomorphic functions with $k \geq 3$.

Chapter 8
Boundary cross theorems

Summary. We discuss the so-called boundary crosses and the extension of a separately holomorphic function for such crosses. We mainly study here the classical results due to [Zern 1961], [Dru 1980], [Gon 1985], [Gon 2000].

8.1 Boundary crosses

$\boxed{>}$ § 3.7.

Recall that so far N-fold crosses were built from domains D_j and subsets $A_j \subset D_j$, $j = 1, \dots, N$. Now we deal with sets $A_j \subset \partial D_j$. We define:

Definition 8.1.1. Let $D_j \subset \mathbb{C}^{n_j}$ be a domain and $\varnothing \neq A_j \subset \partial D_j$, $j = 1, \dots, N$, $N \geq 2$. The associated *N-fold boundary cross* is given by

$$\begin{aligned} \boldsymbol{X}_b &= \mathbb{X}_b(A_1, \dots, A_N; D_1, \dots, D_N) = \mathbb{X}_b((A_j, D_j)_{j=1}^N) \\ &= \bigcup_{j=1}^N (A'_j \times (D_j \cup A_j) \times A''_j), \end{aligned}$$

where we have used the abbreviations from Section 5.1. Moreover, we set

$$\boldsymbol{X}_b^o = \bigcup_{j=1}^N (A'_j \times D_j \times A''_j).$$

$\boldsymbol{X}_b^o$ is called the associated *inner boundary cross*.

Let $\mathfrak{A}_j = \big((\mathcal{A}_{j,\alpha_j}(a_j))_{\alpha_j \in I_{j,a_j}}\big)_{a_j \in A_j}$ be a system of approach regions for (A_j, D_j) (see Definition 3.7.1), $j = 1, \dots, N$. Put

$$\begin{aligned} \widehat{\boldsymbol{X}}^*_{\mathfrak{A},b} &= \big\{z \in (D_1 \cup A_1) \times \dots \times (D_N \cup A_N) : \textstyle\sum_{j=1}^N \boldsymbol{h}^*_{\mathfrak{A}_j, A_j, D_j}(z_j) < 1\big\}, \\ \widehat{\boldsymbol{X}}_{\mathfrak{A},b} &= \big\{z \in D_1 \times \dots \times D_N : \textstyle\sum_{j=1}^N \boldsymbol{h}^*_{\mathfrak{A}_j, A_j, D_j}(z_j) < 1\big\}, \end{aligned}$$

where $\mathfrak{A} := (\mathfrak{A}_1, \dots, \mathfrak{A}_N)$. In case we are dealing with the canonical system of approach regions we simply write $\widehat{\boldsymbol{X}}^*_b := \widehat{\boldsymbol{X}}^*_{(\mathfrak{K},\dots,\mathfrak{K}),b}$ and $\widehat{\boldsymbol{X}}_b := \widehat{\boldsymbol{X}}_{(\mathfrak{K},\dots,\mathfrak{K}),b}$.

Note that if $N = 2$ and $\boldsymbol{h}^*_{\mathfrak{A}_j, A_j, D_j}|_{A_j} = 0$, $j = 1, 2$, then $\widehat{\boldsymbol{X}}^*_{\mathfrak{A},b} = \widehat{\boldsymbol{X}}_{\mathfrak{A},b} \cup \boldsymbol{X}_b$.

Lemma 8.1.2. *Let $D_j, A_j, \mathfrak{A}_j$ be as above and assume that A_j is locally $\mathfrak{A}_j$-pluriregular, $j = 1, \dots, N$. Then $\boldsymbol{X}_b \subset \widehat{\boldsymbol{X}}^*_{\mathfrak{A},b} \subset \overline{\widehat{\boldsymbol{X}}_{\mathfrak{A},b}}$ and $\widehat{\boldsymbol{X}}_{\mathfrak{A},b}$ is connected.*

Proof. For the first claim use Remark 3.7.8 (a). Now fix two points $z', z'' \in \widehat{\boldsymbol{X}}_{\mathfrak{A},b}$. Choose a positive δ such that $\sum_{j=1}^{N} \boldsymbol{h}^*_{\mathfrak{A}_j,A_j,D_j}(z_j) + N\delta < 1$ for $z \in \{z', z''\}$. Put

$$\widetilde{A}_j := \{z_j \in D_j : \boldsymbol{h}^*_{\mathfrak{A}_j,A_j,D_j}(z_j) < \delta\}.$$

$\widetilde{A}_j$ is a non-empty open subset of D_j (see Remark 3.7.8 (b)) and so it is locally pluriregular. Moreover, we have $\boldsymbol{h}^*_{\widetilde{A}_j,D_j} \le \boldsymbol{h}^*_{\mathfrak{A}_j,A_j,D_j} \le \boldsymbol{h}^*_{\widetilde{A}_j,D_j} + \delta$ on D_j. Therefore,

$$\sum_{j=1}^{N} \boldsymbol{h}^*_{\widetilde{A}_j,D_j}(z_j) + N\delta < 1, \quad z \in \{z', z''\}.$$

Applying the proof of Remark 5.1.8 (d), we conclude that the points z', z'' can be connected inside of $\{z \in D_1 \times \dots \times D_N : \sum_{j=1}^{N} \boldsymbol{h}^*_{\widetilde{A}_j,D_j}(z_j) < 1 - N\delta\}$ and hence in $\widehat{\boldsymbol{X}}_{\mathfrak{A},b}$ (using the above inequality). □

We say that a function $f : \boldsymbol{X}_b \to \mathbb{C}$ is *separately holomorphic on $\boldsymbol{X}_b$ with respect to* $\mathfrak{A} = (\mathfrak{A}_1, \dots, \mathfrak{A}_N)$ ($f \in \mathcal{O}_{\mathfrak{A},s}(\boldsymbol{X}_b)$) if for any point $a = (a_1, \dots, a_N) \in A_1 \times \dots \times A_N$ and any $j \in \{1, \dots, N\}$ the following conditions hold:

- $f(a'_j, \cdot, a''_j) \in \mathcal{O}(D_j)$,
- $\lim_{\mathcal{A}_{j,\alpha_j}(a_j) \ni z_j \to a_j} f(a'_j, z_j, a''_j) = f(a), \alpha_j \in I_{j,a_j}$.

For the first condition we write in brief $f \in \mathcal{O}_s(\boldsymbol{X}^o_b)$.

(S-$\mathcal{O}_{\mathrm{B}}$) Similar to the classical cross theorem one may ask

- whether $\boldsymbol{X}_b \subset \partial\widehat{\boldsymbol{X}}_{\mathfrak{A},b}$ (see Lemma 8.1.2),
- whether for every function $f \in \mathcal{O}_{\mathfrak{A},s}(\boldsymbol{X}_b)$ there exists a unique $\hat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}}_{\mathfrak{A},b})$ such that f is in a certain sense the limit of $\hat{f}$.

We do not give a precise formulation since in the future we will restrict our discussion to the case where $D_j \subset \mathbb{C}$, $\mathcal{A}_j = \mathcal{K}$, and f is "continuous" up to the boundary sets A_j, $j = 1, \dots, N$. In particular, we are interested in the case when $f \in \mathcal{C}(\boldsymbol{X}_b) \cap \mathcal{O}_s(\boldsymbol{X}^o_b)$ asking whether then the function given by $\hat{f}$ on $\widehat{\boldsymbol{X}}_{\mathfrak{K},b}$ and by f on $\boldsymbol{X}_b$ is continuous on $\widehat{\boldsymbol{X}}_{\mathfrak{K},b} \cup \boldsymbol{X}_b$.

8.2 Classical results

8.2.1 The Laplace-Fourier transform method

The first results for boundary crosses, proved by different methods, seem to be due to B. Malgrange (1961) (unpublished, see [Eps 1966]) and by M. Zerner [Zern 1961]. They discuss the following geometric situation:

Let $S := \{z \in \mathbb{C} : 0 < \operatorname{Im} z < 1\}$, $\boldsymbol{S}_b = \mathbb{X}_b(\mathbb{R}, \dots, \mathbb{R}; S, \dots, S)$. Using the canonical system $\mathfrak{K}$ of approach regions for $(\mathbb{R}, S)$ it is clear that the relative boundary extremal function is given by $\boldsymbol{h}^*_{\mathfrak{K},\mathbb{R},S}(z) = \operatorname{Im} z$ and that $\mathbb{R}$ is locally $\mathfrak{K}$-pluriregular. Therefore, $\widehat{\boldsymbol{S}}_b = \{z \in S^N : \operatorname{Im} z_1 + \dots + \operatorname{Im} z_N < 1\}$ and $\boldsymbol{S}_b \subset \overline{\widehat{\boldsymbol{S}}_b}$. Note that $\widehat{\boldsymbol{S}}^*_b = \operatorname{conv} \boldsymbol{S}_b$.

Malgrange and Zerner proved similar results like the following one under various additional assumptions on f (see also [Eps 1966], [Kom 1972], [Akh-Ron 1973], Theorem 3, or the book [Ron 1977], Theorem 7.2).

Theorem 8.2.1 (See [Rud 1970], [Dru 1980]). *Let $\boldsymbol{S}_b$ be as above and $f : \boldsymbol{S}_b \to \mathbb{C}$ such that*

- $f \in \mathcal{O}_s(\boldsymbol{S}^o_b)$;
- $f(a'_j, \cdot, a''_j) \in \mathcal{C}(S \cup \mathbb{R})$ *for any* $a = (a'_j, a_j, a''_j) \in \mathbb{R}^N$, $j = 1, \dots, N$;
- $f|_{\mathbb{R}^N} \in \mathcal{C}(\mathbb{R}^N)$;
- f *is locally bounded on* $\boldsymbol{S}_b$.

Then there exists exactly one $\hat{f} \in \mathcal{C}(\widehat{\boldsymbol{S}}^*_b)$ *with* $\hat{f}|_{\widehat{\boldsymbol{S}}_b} \in \mathcal{O}(\widehat{\boldsymbol{S}}_{\mathfrak{K},b})$ *and* $\hat{f} = f$ *on* $\boldsymbol{S}_b$.

Note that all the assumptions in the theorem are necessary for the existence of an $\hat{f}$ with the desired properties.

In other words, any function f as in Theorem 8.2.1 is the boundary value of a holomorphic function $\hat{f}|_{\widehat{\boldsymbol{S}}_b}$. This kind of result may be understood as a one sided "Edge-of-the-Wedge" Theorem which has been studied before by theoretical physicists (see, for example, [Vla 1966]).

Applying biholomorphic mappings Theorem 8.2.1 immediately implies the following result.

Corollary 8.2.2. *Let $D_j \subset \mathbb{C}$ be a Jordan domain, A_j a connected relatively open subset of ∂D_j, $j = 1, \dots, N$. Put $\boldsymbol{X}_b := \mathbb{X}_b(A_1, \dots, A_N; D_1, \dots, D_N)$. Let $f : \boldsymbol{X}_b \to \mathbb{C}$ be such that:*

- $f \in \mathcal{O}_s(\boldsymbol{X}^o_b)$;
- $f(a'_j, \cdot, a''_j) \in \mathcal{C}(D_j \cup A_j)$ *for every* $a \in A := A_1 \times \dots \times A_N$ *and* $j = 1, \dots, N$;
- $f|_A \in \mathcal{C}(A)$;
- f *is locally bounded on* $\boldsymbol{X}_b$.

Then there exists a unique $\hat{f} \in \mathcal{C}(\widehat{\boldsymbol{X}}^*_b) \cap \mathcal{O}(\widehat{\boldsymbol{X}}_b)$ *with* $\hat{f} = f$ *on* $\boldsymbol{X}_b$.

The proof of Theorem 8.2.1 will be mainly based on Fourier transforms and the existence of certain conformal mappings.

Proof of Theorem 8.2.1. Step 1^0. *It is enough to prove the theorem in the case where, in addition to the above assumptions, $f \in \mathcal{C}(\overline{\boldsymbol{S}}_b)$ and f is bounded on $\overline{\boldsymbol{S}}_b$.*

Let f be as in the theorem. Put

$$D(\tau,\alpha) := \Big\{z \in \mathbb{C} : 0 < \operatorname{Arg}\frac{\tau+z}{\tau-z} < \alpha\Big\}, \quad \tau > 0,\ \alpha \in (0,\pi).$$

Note that $D(\tau,\alpha)$ is given by the domain in the upper halfplane which is bounded by the real segment $[-\tau,\tau]$ and the circle passing through $\pm\tau$ and $i\tau\tan(\alpha/2)$. Recall that the mapping $\Phi_{\tau,\alpha}$,

$$D(\tau,\alpha) \ni z \mapsto \frac{1}{\alpha}\operatorname{Log}\frac{\tau+z}{\tau-z} \in S,$$

is biholomorphic and extends to a homeomorphism between $\overline{D}(\tau,\alpha)\setminus\{\pm\tau\}$ and $\overline{S}$. Obviously, $\Phi^{-1}_{\tau,\alpha}(\zeta) = \tau\tanh\frac{\alpha\zeta}{2}$.

Fix an $h \in (0,1)$. Put

$$\alpha_k = \alpha_{h,k} := \frac{1}{h}\arctan\frac{h}{k}, \quad D_k = D_{h,k} := D(k,\alpha_{h,k}), \quad \varphi_k = \varphi_{h,k} := \Phi^{-1}_{k,\alpha_{h,k}}.$$

Then φ_k is a biholomorphic mapping from S onto D_k which is a homeomorphism from $\overline{S}$ onto $D_k\setminus\{\pm k\}$. Moreover, we define

$$g_k(z) := f(\varphi_1(z_1),\dots,\varphi_N(z_N)), \quad z \in \overline{\boldsymbol{S}}_b.$$

It is clear that g_k satisfies all the conditions of f in the theorem on $\boldsymbol{S}_b$ and, in addition, it is bounded there.

Let $|g_k| \le M_k$ on $\boldsymbol{S}_b$. We claim that $g_k \in \mathcal{C}(\boldsymbol{S}_b)$. Indeed, instead of discussing g_k it suffices to verify this condition for the function $\tilde g_k$,

$$\tilde g_k(z) := g_k(z)\exp\Big(-\sum_{j=1}^N z_j^2\Big).$$

Since $\tilde g_k$ is continuous and bounded on $\mathbb{R}^N$, it follows that $\tilde g_k$ is uniformly continuous on $\mathbb{R}^N$.

Fix a point $a = (a'_N, a_N) \in \mathbb{R}^{N-1}\times S$ and let $\varepsilon > 0$. Then, for $z = (z'_N, z_N) \in \boldsymbol{S}_b$ we have

$$\begin{aligned}&|\tilde g_k(z'_N,z_N) - \tilde g_k(a'_N,a_N)|\\ &\qquad \le |\tilde g_k(a'_N,z_N) - \tilde g_k(a'_N,a_N)| + |\tilde g(z'_N,z_N) - \tilde g_k(a'_N,z_N)|.\end{aligned}$$

Applying the continuity of $\tilde{g}_k(a'_N,\cdot)$ we see that $|\tilde{g}_k(a'_N,z_N)-\tilde{g}_k(a'_N,a_N)|<\varepsilon/2$ whenever $|z_N-a_N|<\delta$ for a sufficiently small positive δ with

$$0<\operatorname{Im}(a_N)-\delta<\operatorname{Im}a_N+\delta=:1-p<1.$$

Observe that $u_{z'_N}:=\log|g(z'_N,\cdot)-\tilde{g}_k(a'_N,\cdot)|$ is continuous and bounded by $\widetilde{M}_k=\log(2M_k)$ on $\overline{S}$ and subharmonic in S. Using the uniform continuity of $\tilde{g}_k|_{\mathbb{R}^N}$ we get $u_{z'_N}|_{\mathbb{R}}\le(\log(\varepsilon/2)-\widetilde{M}_k(1-p))/(1-p)=:m$ when z'_N is near to a'_N. Then, by the two-constant lemma for subharmonic function, it follows that

$$u_{z'_N}(z_N)\le m+(M_k-m)(1-p),\quad z'_N \text{ near } a'_N,\ |z_N-a_n|<\delta.$$

Thus $|g(z'_N,z_N)-\tilde{g}_k(a'_N,z_N)|\le\varepsilon/2$ for all the z's as before.

Continuity at the points $a\in\mathbb{R}^N$ can be shown in a similar way (EXERCISE).

Let $t\in(0,1)$. Then our hypothesis applies to $g_k|_{\overline{\boldsymbol{S}}_b(t)}$, where

$$\boldsymbol{S}_b(t):=\mathbb{X}_b(\mathbb{R},\dots,\mathbb{R};S_t,\dots,S_t)$$

with $S_t:=tS$. Put

$$\widehat{\boldsymbol{S}}_b(t)=\big\{z\in S_t\times\dots\times S_t:\textstyle\sum_{j=1}^N\operatorname{Im}z_j<t\big\}.$$

Then there exists a function $\widehat{g}_{k,t}\in\mathcal{C}(\widehat{\boldsymbol{S}}_b^*(t))\cap\mathcal{O}(\widehat{\boldsymbol{S}}_b(t))$ with $\widehat{g}_{k,t}=g_k$ on $\boldsymbol{S}_b(t)$.

Let $0<t<t'<1$. Note that $\widehat{g}_{k,t}$ and $\widehat{g}_{k,t'}$ are both defined on $\widehat{\boldsymbol{S}}_b^*(t)$. For an arbitrary $x\in\mathbb{R}^N$ we know that $\widehat{g}_{k,t}(x'_N,\cdot)-\widehat{g}_{k,t'}(x'_N,\cdot)\in\mathcal{C}(S_t\cup\mathbb{R})\cap\mathcal{O}(S_t)$ and that it vanishes on the real axis. Hence, the refection principle implies that it even vanishes identically on S_t. Now let $(x'_{N-1},z_N)\in\mathbb{R}^{N-2}\times S_{t/N}$. Then for $\nu\ge N$ the function

$$\widehat{g}_{k,t}(x_1+i/\nu,\dots,x_{N-2}+i/\nu,\cdot,z_N)-\widehat{g}_{k,t'}(x_1+i/\nu,\dots,x_{N-2}+i/\nu,\cdot,a_N)$$

is bounded and holomorphic on $S_{t/N}$. If $\nu\to\infty$, then, by Montel, we have that

$$\widehat{g}_{k,t}(x_1,\dots,x_{N-2},\cdot,z_N)-\widehat{g}_{k,t'}(x_1,\dots,x_{N-2},\cdot,a_N)$$

is holomorphic on $S_{t/N}$, continuous on $S_{t/N}\cup\mathbb{R}$, and vanishes on $\mathbb{R}$. As above we conclude that this function vanishes identically on $S_{t/N}$. Repeating the argument we get $\widehat{g}_{k,t}=\widehat{g}_{k,t'}$ on $S_{t/N}\times\dots\times S_{t/N}$. Then the identity theorem gives that $\widehat{g}_{k,t}=\widehat{g}_{k,t'}$ on $\widehat{\boldsymbol{S}}_b(t)$ and hence on $\widehat{\boldsymbol{S}}_b^*(t)$. Finally, gluing these functions we attain a function $\widehat{g}_k\in\mathcal{C}(\widehat{\boldsymbol{S}}_b^*)\cap\mathcal{O}(\widehat{\boldsymbol{S}}_b)$ and $\widehat{g}_k=g_k$ on $\boldsymbol{S}_b$.

Transforming back we define the function $\hat{f}_{k,h}:=\widehat{g}_k(\varphi_k^{-1},\dots,\varphi_k^{-1})$. Then

$$\hat{f}_{k,h}\in\mathcal{C}(\widehat{\boldsymbol{Y}}_b^*(h,k))\cap\mathcal{O}(\widehat{\boldsymbol{Y}}_b(h,k)),$$

where $\boldsymbol{Y}_b(h,k):=\mathbb{X}_b((-k,k),\dots,(-k,k);D_{h,k},\dots,D_{h,k})$, such that $\hat{f}_{h,k}=f$ on $\boldsymbol{Y}_b(h,k)$. Note that

$$\widehat{\boldsymbol{Y}}_b(h,k)=\big\{z\in D_{h,k}\times\dots\times D_{h,k}:\textstyle\sum_{j=1}^N\operatorname{Im}\psi_k(z_j)<1\big\},$$

where $\psi_k := \varphi_k^{-1}$.

Observe that

- $D_{h,k} \subset D_{h,k+1}$, $\bigcup_k D_{h,k} = S_h$;
- $\boldsymbol{Y}_b(h,k) \subset \boldsymbol{Y}_b(h,k+1)$, $\lim_{k\to\infty} \operatorname{Im} \psi_k(z) = \operatorname{Im} z/h$;
- $\bigcup_{k=1}^{\infty} \boldsymbol{Y}_b(h,k) = \boldsymbol{S}_b(h)$, $\bigcup_{k=1}^{\infty} \widehat{\boldsymbol{Y}}_b(h,k) = \widehat{\boldsymbol{S}}_b(h)$.

As above one can prove that $\hat{f}_{h,k} = \hat{f}_{h,k+1}$ on $\widehat{\boldsymbol{Y}}_b^*(h,k)$. Gluing now all these functions leads to an $\hat{f}_h \in \mathcal{C}(\widehat{\boldsymbol{S}}_b^*(h)) \cap \mathcal{O}(\widehat{\boldsymbol{S}}_b(h))$ with $\hat{f}_h = f$ on $\boldsymbol{S}_b(h)$.

Finally we have to repeat the gluing process from above for the functions $\hat{f}_h$ to get the desired function $\hat{f}$ on $\widehat{\boldsymbol{S}}_b^*$. Hence, the first step is completely verified.

Step 2^0. *A preparation to Step* 3^0.

Let $u \in L^1(\mathbb{R}^N)$, $K \subset \mathbb{R}^N$ compact and assume that $K \ni y \mapsto u \cdot e_y \in L^1(\mathbb{R}^N)$ is continuous, where $e_y(x) := \exp(-\langle x, y\rangle)$, $x, y \in \mathbb{R}^n$. Then

(a) the function $\operatorname{conv}(K) \ni y \mapsto ue_y \in L^1(\mathbb{R}^N)$ is continuous;

(b) if $|\int_{\mathbb{R}^N} u(t)e^{i\langle z,t\rangle} d\mathcal{L}^N(t)| \le b$, $y \in K$, then the same inequality holds for every $y \in \operatorname{conv}(K)$. (The reader who is familiar with these results from real analysis or who is not interested in their proofs may jump directly to Step 3^0.)

Indeed, to prove (a) fix an arbitrary positive ε. Then, by virtue of the compactness of K, we find points $y_1, \dots, y_m \in K$ such that for every $y \in K$ there is an index $k \in \{1, \dots, m\}$ such that

$$\int_{\mathbb{R}^N} |e^{-\langle x,y\rangle} - e^{-\langle x,y_k\rangle}||u(x)| d\mathcal{L}^N(x) < \varepsilon.$$

Looking at the $L^1(\mathbb{R}^N)$-functions $u \cdot e_{y_1}, \dots, u \cdot e_{y_m}$, there exists a closed ball $B' := \overline{\mathbb{B}}_N^{\mathbb{R}}(r)$ such that

$$\int_{\mathbb{R}^N \setminus B'} e_{y_k}(x)|u(x)| d\mathcal{L}^N(x) < \varepsilon, \quad k = 1, \dots, m.$$

Hence for an arbitrary $y \in K$ we get

$$\begin{aligned}\int_{\mathbb{R}^N\setminus B'} & e_y(x)|u(x)| d\mathcal{L}^N(x) \\ &\le \int_{\mathbb{R}^N\setminus B'} |e_y(x) - e_{y_k}(x)||u(x)| d\mathcal{L}^N(x) + \int_{\mathbb{R}^N\setminus B'} e_{y_k}(x)|u(x)| d\mathcal{L}^N(x) < 2\varepsilon,\end{aligned}$$

when k is correctly chosen.

Denote by $K \subset Y$ the set of all y's in $\mathbb{R}^N$ such that the former estimate holds for y. Then Y is convex (and so $\operatorname{conv}(K) \subset Y$). For, take two points $y', y'' \in Y$ and let

$\tau \in (0,1)$. Then

$$\begin{aligned}\int_{\mathbb{R}^N\setminus B'} & e_{\tau y'+(1-\tau)y''}(x)|u(x)|d\mathcal{L}^N(x)\\ &= \int_{\mathbb{R}^N\setminus B'} (e_{y'}(x))^{\tau}(e_{y''}(x))^{1-\tau}|u(x)|d\mathcal{L}^N(x)\\ &\le \Big(\int_{\mathbb{R}^N\setminus B'} e_{y'}(x)|u(x)|d\mathcal{L}^N(x)\Big)^{\tau}\Big(\int_{\mathbb{R}^N\setminus B'} e_{y''}(x)|u(x)|d\mathcal{L}^N(x)\Big)^{1-\tau} < 2\varepsilon,\end{aligned}$$

where the last inequality follows because of the Hölder inequality with respect to the measure $|u|d\mathcal{L}^N(x)$.

Finally, fix a $y_0 \in \operatorname{conv}(K)$. Then, for $y \in \operatorname{conv}(K)$ we have

$$\int_{\mathbb{R}^N} |e_y(x) - e_{y_0}(x)||u(x)|d\mathcal{L}^N(x) \le 4\varepsilon + \mathcal{L}^N(B')\|e_y - e_{y_0}\|_{B'} + 4\varepsilon < 5\varepsilon$$

if y is sufficiently near to y_0.

To verify (b) define $U(z) := \int_{\mathbb{R}^N} u(t)e^{i\langle z,t\rangle}d\mathcal{L}^N(t)$, $z \in \mathbb{R}^N + i\operatorname{conv}(K)$. By virtue of (a) we see that U is continuous and bounded on $\mathbb{R}^N + i\operatorname{conv}(K)$. Let Y be the set of all $y \in \operatorname{conv}(K)$ with $\sup\{|U(x+iy)| : x \in \mathbb{R}\} \le b$. By assumption, $K \subset Y$. It suffices to show that Y itself is convex.

So take $y', y'' \in Y$ and $\sigma_0 \in (0,1)$. Put $y := (1-\sigma_0)y' + \sigma_0 y''$. Fix an $x \in \mathbb{R}^N$ and set $z' := x + iy'$ and $z'' := x + iy''$. Finally define the function

$$\gamma(s) := U((1-s)z' + sz''), \quad s = \sigma + i\tau,\ 0 \le \sigma \le 1.$$

Note that $\operatorname{Im}\big((1-s)z' + sz''\big) = (1-\sigma)y' + \sigma y'' \in \operatorname{conv}(K)$. Hence, γ is well defined on the strip $S = [0,1] + i\mathbb{R}$ and bounded and continuous there. Moreover, γ is bounded in the interior of this strip. Observe that $|\gamma(i\tau)| \le b$ and $|\gamma(1+i\tau)| \le b$, $\tau \in \mathbb{R}$ (since $y', y'' \in Y$). Then, using the maximum principle, one concludes that $|U(x+iy)| = |\gamma(\sigma_0)| \le b$. Since x was arbitrarily chosen it follows that $y \in Y$ meaning that Y is convex.

Step 3^0. *The case where f belongs, in addition, to $\mathcal{C}(\overline{S}_b)$ and is bounded on $\overline{S}_b$.*

Put $g(z) := f(z)\exp\big(-(z_1^2 + \cdots + z_N^2)\big)$, $z \in \overline{S}_b$. Obviously, it is enough to prove the extension for the function g. Note that

$$|g(z)| \le \|f\|_{\overline{S}_b} e^{-(x_1^2+\cdots+x_N^2)+1}, \quad z \in \overline{S}_b.$$

Therefore, g is uniformly continuous on $\overline{S}_b$.

Let $V := \{y \in [0,1]^N : \exists_k :\ y_j = 0, j = 1,\dots,N, j \ne k\}$. Then for a $y \in V$ put $g_y := g(\cdot + iy)$. By virtue of the above estimate, we get $g_y \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$. Then the Fourier transform of g_y is given by

$$\mathcal{F}g_y(t) := \Big(\frac{1}{2\pi}\Big)^N \int_{\mathbb{R}^N} g_y(x)e^{-i\langle x,t\rangle}d\mathcal{L}^N(x), \quad t \in \mathbb{R}^N.$$

Specifying $y = (0, \dots, 0, y_j, 0, \dots, 0)$ (the $y_j \in (0,1)$ is at the j-th place) we have

$$\begin{aligned}\mathcal{F} g_y(t) &= \left(\frac{1}{2\pi}\right)^N \int_{\mathbb{R}^{N-1}} \left(\int_{\mathbb{R}} g(x_j', x_j + iy_j, x_j'') e^{-iz_j t_j} d\mathcal{L}^1 x_j \right) \\ &\qquad \times e^{-i \sum\limits_{k=1, k\neq j}^{N} x_k t_k} d\mathcal{L}^{N-1}(x_j', x_j'') \\ &= \left(\frac{1}{2\pi}\right)^N \int_{\mathbb{R}^{N-1}} \left(\int_{\mathbb{R}} g(x_j', x_j + i\varepsilon, x_j'') e^{-i(x_j + i\varepsilon)t_j} d\mathcal{L}^1 (x_j) \right) \\ &\qquad \times e^{-i \sum\limits_{k=1, k\neq j}^{N} x_k t_k} d\mathcal{L}^{N-1}(x_j', x_j'') \underset{\varepsilon \to 0}{\to} \mathcal{F} g_0(t),\end{aligned}$$

where the second equation is true for any $\varepsilon \in (0,1)$ because of the Cauchy integral theorem and the limit follows because of the uniform continuity of g. Hence, $\mathcal{F} g_y(t) = e^{-\langle y,t\rangle} \mathcal{F} g_0(t)$, $t \in \mathbb{R}^N$, $y \in V$.

By virtue of Plancherel's theorem we have $\mathcal{F} g_y \in L^2(\mathbb{R}^N)$, $y \in V$. But we cannot conclude that $\mathcal{F} g_y \in L^1(\mathbb{R}^N)$. Choose a non-negative cut-off function $\psi_r \in \mathcal{C}^\infty(\mathbb{R}^N)$, $\psi_r = 0$ if $\|x\| \geq r$, such that $\int_{\mathbb{R}^N} \psi_r(t) d\mathcal{L}^N(t) = 1$. Define

$$\boldsymbol{h}_{r,y}(x) = h_r(z) := \int_{\mathbb{R}^N} \psi_r(\xi) g(z - \xi) d\mathcal{L}^N(\xi), \quad z = x + iy \in \overline{\boldsymbol{S}}_b.$$

Then

$$\begin{aligned}\mathcal{F} h_y(t) &= \mathcal{F} \psi_r(t) \mathcal{F} g_y(t) \\ &= \mathcal{F} \psi_r(t) e^{-\langle y,t\rangle} \mathcal{F} g_0(t) = e^{-\langle y,t\rangle} \mathcal{F} h_0(t), \quad t \in \mathbb{R}^n, \ y \in V.\end{aligned}$$

Moreover note that $V \ni y \to g_y \in L^2(\mathbb{R}^n)$ is continuous. Therefore, $V \ni y \mapsto \mathcal{F} g_y \in L^2(\mathbb{R}^N)$ is also continuous. Then the Schwarz inequality implies that $V \ni y \mapsto \mathcal{F} h_y \in L^1(\mathbb{R}^N)$ is continuous. In particular, the Fourier inversion formula applies to $\mathcal{F} h_y$. Hence,

$$h_r(z) = \boldsymbol{h}_{r,y}(x) = \int_{\mathbb{R}^N} \mathcal{F} h_0(t) e^{i\langle z,t\rangle} d\mathcal{L}^N(t), \quad z \in \overline{\boldsymbol{S}}_b. \tag{$*$}$$

Using Step 2^0 it follows that $\operatorname{conv}(V) \ni y \mapsto \mathcal{F} h_y \in L^1(\mathbb{R}^N)$ is continuous meaning that the function h_r in $(*)$ is even continuous on $\overline{\widehat{\boldsymbol{S}}_b}$ and holomorphic in $\widehat{\boldsymbol{S}}_b$. Moreover, (b) in Step 2^0 leads to $\|h_r\|_{\overline{\widehat{\boldsymbol{S}}_b}} = \|h_r\|_{\boldsymbol{S}_b}$.

Recall that $h_r \underset{r \to 0}{\to} g$ uniformly on $\boldsymbol{S}_b$. Then $(h_{1/m})_m$ is a uniform Cauchy sequence on $\overline{\widehat{\boldsymbol{S}}_b}$; so it converges uniformly to a function which we call $\widehat{g}$. Then $\widehat{g} \in \mathcal{C}(\overline{\widehat{\boldsymbol{S}}_b}) \cap \mathcal{O}(\widehat{\boldsymbol{S}}_b)$ with $\widehat{g} = g$ on $\boldsymbol{S}_b$. □

Proof of Corollary 8.2.2. The case where $\overline{A_j} \subsetneq \partial D_j$, $j = 1, \dots, N$, is an easy application of Theorem 8.2.1 by a suitable biholomorphic mapping (EXERCISE). It remains the case where $A_j = \partial D_j$, $j = 1, \dots, s_1$, $A_j = \partial D_j \setminus \{a_j\}$ with $a_j \in \partial D_j$, $j = s_1 + 1, \dots, s_2$, for $s_1 \le s_2 \in \{1, \dots, N\}$, and $A_j \subsetneq \partial D_j$, $j = s_2 + 1, \dots, N$. Moreover, we may assume that $D_j = \mathbb{D}$ and $-1 \in A_j$, $j = 1, \dots, N$, and $a_{s_1+1} = \dots = a_{s_2} = 1$.

Define

$$A_{j,k} := \{z \in \mathbb{T} : |\operatorname{Arg} z| > \pi/(2k)\}, \quad j = 1, \dots, s_2.$$

Then the former case implies that there exists a function $\hat{f}_k \in \mathcal{C}(\widehat{\boldsymbol{X}}_b^*(k)) \cap \mathcal{O}(\widehat{\boldsymbol{X}}_b(k))$ such that $\hat{f}_k = f$ on $\boldsymbol{X}_b(k)$, where

$$\boldsymbol{X}_b(k) := \mathbb{X}_b(A_{1,k}, \dots, A_{s_2,k}, A_{s_2+1}, \dots, A_N; \mathbb{D}, \dots, \mathbb{D}), \quad k \in \mathbb{N}.$$

Note that $\boldsymbol{h}^*_{A_{j,k},\mathbb{D}} \ge \boldsymbol{h}^*_{A_{j,k+1},\mathbb{D}}$ $(1 \le j \le s_2)$ and therefore, $\widehat{\boldsymbol{X}}_b^*(k) \subset \widehat{\boldsymbol{X}}_b^*(k+1)$. Let $u_j := \lim_{k\to\infty} \boldsymbol{h}^*_{A_{j,k},\mathbb{D}}$. Then $0 \le u_j \in \mathcal{SH}(\mathbb{D})$ with $u_j^{*,\mathfrak{K}} \le 0$ on $\mathbb{T}^* := \mathbb{T} \setminus \{1\}$. So by the general maximum principle for subharmonic functions (see [Ran 1995], Theorem 3.6.9) we have that $u_j = 0$ on $\mathbb{D}$. Hence,

$$\bigcup_{k=1}^{\infty} \widehat{\boldsymbol{X}}_b^*(k) = \widehat{\boldsymbol{Y}}_b^* \quad \text{and} \quad \bigcup_{k=1}^{\infty} \widehat{\boldsymbol{X}}_b(k) = \widehat{\boldsymbol{X}}_b,$$

where $\boldsymbol{Y}_b := \mathbb{X}_b(\mathbb{T}^*, \dots, \mathbb{T}^*, A_{s_2+1}, \dots, A_N; \mathbb{D}, \dots, \mathbb{D})$.

Fix a k. Then we have two holomorphic functions $\hat{f}_k$, $\hat{f}_{k+1}$ on $\widehat{\boldsymbol{X}}_b(k)$ which are continuous on $\widehat{\boldsymbol{X}}_b^*(k)$ such that $f = \hat{f}_k = \hat{f}_{k+1}$ on $\boldsymbol{X}_b(k)$. We want to show that $\hat{f}_k = \hat{f}_{k+1}$ on $\widehat{\boldsymbol{X}}_b^*(k)$. By assumption, one finds a positive ε such that

- $\mathbb{D}(-1,\varepsilon) \cap \mathbb{T} \subset \bigcap_{j=1}^{s_2} A_{j,k} \cap \bigcap_{j=s_2+1}^{N} A_j$, $k \in \mathbb{N}$;
- $\mathbb{P}_N(-\mathbb{1},\varepsilon) \cap \overline{\mathbb{D}}^N \subset \widehat{\boldsymbol{X}}_b^*(k)$;
- $G := \mathbb{P}_N(-\mathbb{1},\varepsilon) \cap \mathbb{D}^N \subset \widehat{\boldsymbol{X}}_b(k)$.

Fix an arbitrary point $\tilde{z} \in (\mathbb{T} \cap \mathbb{D}(-1,\varepsilon))^{N-1}$. Then the function

$$\hat{f}(\tilde{z},\cdot) - \hat{f}_{k+1}(\tilde{z},\cdot)$$

is holomorphic on $\mathbb{D}$, continuous on $\overline{\mathbb{D}} \cap \mathbb{D}(-1,\varepsilon)$, and vanishes on $\mathbb{T} \cap \mathbb{D}(-1,\varepsilon)$. Hence it vanishes on $\mathbb{D}$. In a next step take a point $(\tilde{z}, z_N) \in (\mathbb{T} \cap \mathbb{D}(-1,\varepsilon))^{N-2} \times (\mathbb{D} \cap \mathbb{D}(-1,\varepsilon))$. Let $t_m := 1 - 1/m$, $m \in \mathbb{N}$. Then for sufficiently large m the functions

$$\hat{f}_k(t_m\tilde{z},\cdot,z_N) - \tilde{f}_{k+1}(t_m\tilde{z},\cdot,z_N)$$

are bounded holomorphic on $\mathbb{D} \cap \mathbb{D}(-1,\varepsilon)$. Taking $m \to \infty$ it follows that

$$\hat{f}_k(\tilde{z},\cdot,z_N) - \tilde{f}_{k+1}(\tilde{z},\cdot,z_N)$$

is holomorphic on $\mathbb{D} \cap \mathbb{D}(-1, \varepsilon)$ with vanishing boundary values on $\mathbb{T} \cap \mathbb{D}(-1, \varepsilon)$. Hence it vanishes identically on $\mathbb{D} \cap \mathbb{D}(-1, \varepsilon)$. Continuing with the same argument we get that $\hat{f}_k - \hat{f}_{k+1}$ coincide on G. Then the identity theorem gives that $\hat{f}_k = \hat{f}_{k+1}$ on $\widehat{\boldsymbol{X}}_b(k)$. Finally, continuity implies equality on $\widehat{\boldsymbol{X}}_b^*(k)$.

Gluing all these $\hat{f}_k$'s we end up with a function $\widehat{g}_1 \in \mathcal{C}(\widehat{\boldsymbol{Y}}_b^*) \cap \mathcal{O}(\widehat{\boldsymbol{X}}_b)$ such that $\widehat{g}_1 = f$ on $\boldsymbol{Y}_b$.

If $s_1 = 0$, then we are done. Otherwise take $s_1 + 1$ pairwise different points $c_1 = 1, c_2, \dots, c_{s_1+1} \in \mathbb{T}$ near 1. Following the same procedure as above leads to functions

$$\widehat{g}_j \in \mathcal{C}(\widehat{\boldsymbol{Y}}_b^*(j)) \cap \mathcal{O}(\widehat{\boldsymbol{X}}_b) \quad \text{with } \widehat{g}_j = f \text{ on } \boldsymbol{X}_b,$$

where $\widehat{\boldsymbol{Y}}_b(j) := \mathbb{X}(\mathbb{T} \setminus \{c_j\}, \dots, \mathbb{T} \setminus \{c_j\}, A_{s_2+1}, \dots, A_N; \mathbb{D}, \dots, \mathbb{D})$, $j = 1, \dots, s_1 + 1$. Then, gluing the $\widehat{g}_j$'s gives the final extension $\hat{f} \in \mathcal{C}(\widehat{\boldsymbol{X}}_b^*) \cap \mathcal{O}(\widehat{\boldsymbol{X}}_b)$, $\hat{f} = f$ on $\boldsymbol{X}_b$. □

As a consequence we get a boundary cross theorem due to N. I. Akhiezer and L. I. Ronkin (see [Akh-Ron 1973] and [Akh-Ron 1976], where this result is proved under additional assumptions).

Corollary 8.2.3. *Let $D_j := \mathbb{A}(1, b_j)$ and $A_j := \mathbb{T} \subset \partial D_j$, $j = 1, \dots, N$. Put $\boldsymbol{X}_b := \mathbb{X}_b((A_j, D_j)_{j=1}^N)$. Let $f : \boldsymbol{X}_b \to \mathbb{C}$ be such that*

- *$f(a_j', \cdot, a_j'') \in \mathcal{C}(D_j \cup \mathbb{T}) \cap \mathcal{O}(\mathbb{D}_j)$ for any $a \in \mathbb{T}^N$ and $j = 1, \dots, N$;*
- *$f|_{\mathbb{T}^N} \in \mathcal{C}(\mathbb{T}^N)$;*
- *f is locally bounded.*

Then there exists a unique $\hat{f} \in \mathcal{C}(\widehat{\boldsymbol{X}}_b^*) \cap \mathcal{O}(\widehat{\boldsymbol{X}}_b)$ with $f = \hat{f}$ on $\boldsymbol{X}_b$ and $\widehat{\boldsymbol{X}}_b = \operatorname{int} \widehat{\boldsymbol{X}}_b^*$. Notice that

$$\widehat{\boldsymbol{X}}_b^* := \Big\{z \in \mathbb{A}[1, b_1) \times \dots \times \mathbb{A}[1, b_N) : \textstyle\sum_{j=1}^N \frac{\log |z_j|}{\log b_j} < 1\Big\}.$$

Proof. We only present the proof for $N = 2$; the general case is left as an Exercise. First introduce the following domains:

$$D_j^\pm := \{z \in \mathbb{A}(1, b_j) : |\operatorname{Arg}(\pm z)| > \pi/8\},$$
$$\widetilde{D}_j^\pm := \{z \in \mathbb{A}(1, b_j) : |\operatorname{Arg}(\pm z)| > \pi/4\}.$$

Note that $\widetilde{D}_j^\pm \subset D_j^\pm$ are Jordan domains. Put

$$A_j^\pm := \{z \in \mathbb{T} : |\operatorname{Arg}(\pm z)| > \pi/8\}$$

and let $u_j := \boldsymbol{h}^*_{\mathfrak{K}, A_j^+, D_j^+}$ be the relative boundary extremal function with respect to the canonical system of approach regions. Note that $D_j^+ = -D_j^-$ and therefore, $\boldsymbol{h}^*_{\mathfrak{K}, A_j^-, D_j^-}(z) = u_j(-z)$ on D_j^-.

Now let δ be a positive number with $1+2\delta < \min\{b_1, b_2\}$. Then there are positive numbers $\delta'' < \delta' < \delta$ such that

$$\{z_j \in \widetilde{D}_j^{\pm} : 1 < |z_j| < b_j - \delta\} \subset \{z_j \in D_j^{\pm} : u_j(z_j) < 1 - \delta'\},$$
$$\{z_j \in \widetilde{D}_j^{\pm} : 1 < |z_j| < 1 + \delta''\} \subset \{z_j \in D_j^{\pm} : u_j(z_j) < \delta'\}.$$

Put $\boldsymbol{X}_b(\pm) := \mathbb{X}(A_1^{\pm}, A_2^{\pm}; D_1^{\pm}, D_2^{\pm})$. Then, applying Corollary 8.2.2, we get two continuous functions $\hat{f}^{\pm} \in \mathcal{C}(\widehat{\boldsymbol{X}}_b^*(\pm)) \cap \mathcal{O}(\widehat{\boldsymbol{X}}_b(\pm))$ with $\hat{f}^{\pm} = f$ on $\boldsymbol{X}_b(\pm)$. In particular, $\hat{f}^{\pm}$ is continuous on

$$\begin{aligned}\boldsymbol{Z}^{\pm}(\delta) := \{z \in \overline{\widetilde{D}_1^{\pm}} \times \overline{\widetilde{D}_2^{\pm}} : 1 \le |z_1| < \delta'', 1 \le |z_2| < b_2 - \delta\} \\ \cup \{z \in \overline{\widetilde{D}_1^{\pm}} \times \overline{\widetilde{D}_2^{\pm}} : 1 \le |z_1| < b_1 - \delta, 1 \le |z_2| < \delta''\}\end{aligned}$$

and holomorphic on $\operatorname{int} \boldsymbol{Z}^{\pm}(\delta)$.

Fix a $z_2^0 \in \widetilde{D}_2^{+} \cap \widetilde{D}_2^{-}$ with $1 < |z_2| < b_2 - \delta$. Then $\hat{f}^{\pm}$ are holomorphic functions on the annulus

$$G := \{z \in \mathbb{A}(1, \delta'') : |\operatorname{Arg}(\pm z_1)| > \pi/4\}$$

having the same boundary values, namely $f(\cdot, z_2^0)$, at the inner boundary of G. Arguing as above, the reflection principle shows that $\hat{f}^{+}(\cdot, z_2^0) = \hat{f}^{-}(\cdot, z_2^0)$ on the two connected components of G. The same argument leads to the conclusion that $\hat{f}^{+} = \hat{f}^{-}$ on their common region of definition. Hence we end up with a function $g_\delta \in \mathcal{O}_s(\boldsymbol{Y}(\delta))$, where

$$\boldsymbol{Y}(\delta) := \mathbb{X}(\mathbb{A}(1, 1+\delta''), \mathbb{A}(1, 1+\delta''); \mathbb{A}(1, b_1 - \delta), \mathbb{A}(1, b_2 - \delta)).$$

By virtue of the cross theorem there exists a $\hat{g}_\delta \in \mathcal{O}(\widehat{\boldsymbol{Y}}(\delta))$ with $\hat{g}_\delta = g_\delta$ on $\boldsymbol{Y}(\delta)$.

Note that

$$\widehat{\boldsymbol{Y}}(\delta) = \{z \in \mathbb{A}(1, b_1 - \delta) \times \mathbb{A}(1, b_2 - \delta) : \boldsymbol{h}_{1,\delta}(z_1) + \boldsymbol{h}_{2,\delta}(z_2) < 1\},$$

where

$$\boldsymbol{h}_{j,\delta}(z_j) := \frac{\log \frac{|z_j|}{1+\delta''}}{\log(b_j - \delta)}, \quad z_j \in \mathbb{A}(1, b_j - \delta).$$

Now take a sequence $\delta_k \searrow 0$ ($1 + 2\delta_1 < \min\{b_1, b_2\}$) and the corresponding sequence $\delta_k'' \searrow 0$. Put $h_{j,k} := \boldsymbol{h}_{j,\delta_k}$, $\boldsymbol{Y}_k := \boldsymbol{Y}(\delta_k)$, $\hat{g}_k := g_{\delta_k}$, and $\hat{g}_k := \hat{g}_{\delta_k}$. Moreover, put $h_j := \frac{\log|\cdot|}{\log b_j}$ on $\mathbb{A}(1, b_j)$, $j = 1, 2$.

Fix a point $z^0 \in \widehat{\boldsymbol{X}}_b$ with $h_1(z_1^0) + h_2(z_2^0) + 4\eta < 1$ and $z_j^0 \in \mathbb{A}(1, b_j - 2\eta)$ ($\eta > 0$ sufficiently small). Then $\overline{\mathbb{D}}(z_j^0, r) \subset \mathbb{A}(1, b_j - \eta)$ for a sufficiently small r and $h_j < h_j(z_j^0) + \eta$ on $\overline{\mathbb{D}}(z_j^0, r)$. There is a k_0 such that for all $k \ge k_0$ we have $\boldsymbol{h}_{j,k} \le h_j(z_j^0) + \eta$ on $\overline{\mathbb{D}}(z_j^0, r) \subset \mathbb{A}(1, b_j - \delta_k)$, $j = 1, 2$. Thus, $\mathbb{P}_2(z^0, r) \subset\subset \boldsymbol{Y}_k$.

Moreover, for a fixed $z_2' \in \mathbb{D}(z_2^0, r)$, the function $\widehat{g}_k(\cdot, z_2')$ is holomorphic on the annulus $\{z \in \mathbb{A}(1, b_1 - \delta_k) : \boldsymbol{h}_{1,k}(z_1) < \eta\}$, since for these z_1 we have $\boldsymbol{h}_{1,k}(z_1) + \boldsymbol{h}_{2,k}(z_2') < h_2(z_2^0) + 2\eta < 1$.

Observe that $\widehat{g}_{k+1}(\cdot, z_2') - \widehat{g}_k(\cdot, z_2') = g_{k+1}(\cdot, z_2') - g_k(\cdot, z_2')$, $k \geq k_0$, on the annulus $\mathbb{A}(1, 1+\varepsilon_k)$ for a sufficiently small positive ε_k. But on this annulus the above difference has zero boundary values on $\{z \in \mathbb{T} : |\operatorname{Arg}(\pm z)| > \pi/4\}$. Hence, the function $\widehat{g}_{k+1}(\cdot, z_2') - \widehat{g}_k(\cdot, z_2')$ is identically zero on $\mathbb{A}(1, 1+\varepsilon_k)$. Finally, using the identity theorem, we get that this function vanishes identically on the annulus

$$\{z_1 \in \mathbb{A}(1, b_1 - \delta_k) : \boldsymbol{h}_{1,j}(z_1) < 1 - \boldsymbol{h}_{2,j}(z_2'),\ j = k, k+1\}$$

which contains $\mathbb{D}(z_1^0, r)$. Now we define a new function

$$\hat{f} := \lim_{k \to \infty} \widehat{g}_k \quad \text{on } \mathbb{P}_2(z^0, r).$$

Obviously, $\hat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}}_b)$. It remains to check that $\hat{f}$ satisfies the desired properties. □

The following two examples show that the properties in the third and fourth bullet in Corollary 8.2.2 are essential.

Example 8.2.4. [Dru 1980] The reader is asked to complete all details. Let $A := \{z \in \mathbb{T} : \operatorname{Re} z > 0\}$ and put $\boldsymbol{X}_b := \mathbb{X}_b(A, A; \mathbb{D}, \mathbb{D})$.

(a) Define for $z = (z_1, z_2) \in \boldsymbol{X}_b$ the following function f:

$$f(z) := \begin{cases} \exp(-(\operatorname{Log}(1-z_1) + \operatorname{Log}(1-z_2)) \operatorname{Log} \frac{2+z_1z_2}{3}) & \text{if } z_1 \neq 1 \neq z_2, \\ 0 & \text{if } z_1 = 1 \text{ or } z_2 = 1. \end{cases}$$

Then f satisfies all the assumptions in Corollary 8.2.2 except the third one. Note that $1 = f(e^{it}, e^{-it}) \underset{t \to 0}{\not\to} f(1,1)$. Moreover, f is holomorphic on $(\overline{\mathbb{D}} \setminus \{1\}) \times (\overline{\mathbb{D}} \setminus \{1\})$. Note that $(t,t) \in \widehat{\boldsymbol{X}}_b$ for $t \in (0,1)$ near to 1, but nevertheless the radial limit $f(t,t) \underset{t \nearrow 1}{\to} 1 \neq f(1,1)$.

(b) Now let $f_a : \boldsymbol{X}_b \to \mathbb{C}$, $0 < a < 1/2$, be given by the formula

$$f_a(z) := \begin{cases} \exp(-(z_1 - a)(\operatorname{Log} \frac{3+z_2}{1-z_2})^2) & \text{if } z_2 \neq 1, \\ 0 & \text{if } z_2 = 1, \end{cases} \quad z = (z_1, z_2) \in \boldsymbol{X}_b.$$

Then f_a satisfies all the properties in Corollary 8.2.2 except the last one. The function f_a is not locally bounded at the point $(-a + ia, 1)$ since $|f_a(\gamma(t))| \underset{t \searrow 0}{\to} \infty$, where

$$\gamma(t) := \Big(-a + \Big(\operatorname{Re} \Big(\operatorname{Log} \frac{3 + e^{it}}{1 - e^{it}} \Big)^2 \Big)^{-1} + ia, e^{it} \Big), \quad 0 < t < 1.$$

Moreover, f_a is holomorphic on $\overline{\mathbb{D}} \times (\overline{\mathbb{D}} \setminus \{1\})$, but

$$f_a\Big(-a + \Big(\log\frac{4-t}{t}\Big)^{-2} + ia, 1-t\Big) = \exp\Big(-1 - ia\log^2\frac{4-t}{t}\Big)$$

has no limit when $t \to 0+$. Observe that

$$\Big(-a + \Big(\log\frac{4-t}{t}\Big)^{-2} + ia, 1-t\Big) \in \widehat{\boldsymbol{X}}_b$$

for t sufficiently near 0, i.e. $f_a|_{\widehat{\boldsymbol{X}}_b}$ has no limit when approaching $(-a+ia, 1) \in \boldsymbol{X}_b$.

8.2.2 The Carleman operator method

A more general result was proved by Gonchar (see [Gon 1985], [Gon 2000]).

Theorem 8.2.5. *Let $D_j \subset \mathbb{C}$ be a Jordan domain, $\varnothing \neq A_j \subsetneq \partial D_j$ a relatively open subset, $j = 1, 2$. Put*

$$\boldsymbol{X}_b = \mathbb{X}(A_1, A_2; D_1, D_2).$$

Then for any $f \in \mathcal{C}(\boldsymbol{X}_b) \cap \mathcal{O}_s(\boldsymbol{X}_b^o)$, bounded, there exists a unique function $\hat{f} \in \mathcal{C}(\boldsymbol{X}_b \cup \widehat{\boldsymbol{X}}_b) \cap \mathcal{O}(\widehat{\boldsymbol{X}}_b)$ with $f = \hat{f}$ on $\boldsymbol{X}_b$. Moreover,

$$|\hat{f}(z)| \le \|f\|_{A_1\times A_2}^{1-u(z)}\|f\|_{\boldsymbol{X}_b}^{u(z)}, \quad z \in \widehat{\boldsymbol{X}}_b,$$

where $u(z) := \boldsymbol{h}^*_{\mathfrak{R},A_1,D_1}(z_1) + \boldsymbol{h}^*_{\mathfrak{R},A_2,D_2}(z_2)$, $z = (z_1, z_2) \in D_1 \times D_2$.

The proof will be based on:

- a Carleman formula ([Gol-Kry 1933], see also [Aiz 1990], Theorem 1.1) (see (a) below),
- a theorem of Korànyi-Vagi (see [Rud 1980], Theorem 6.3.1) (see (b) below),
- classical results on boundary values in the theory of one complex variable; the reader is asked to consult [Gol 1983], Chapter X, for more details.

(a) *Carleman formula*:

Lemma 8.2.6 ([Gol-Kry 1933]). *Let $B \subset \mathbb{T}$ be relatively open and let $f \in L_h^\infty(\mathbb{D})$. Then*

$$\Big|f(z) - \frac{1}{2\pi i}\int_B \frac{f^*(\zeta)}{\zeta - z}\Big(\frac{\varphi(z)}{\varphi^*(\zeta)}\Big)^m d\zeta\Big| \le \frac{\|f\|_{\mathbb{D}}}{\operatorname{dist}(z, \mathbb{T}\setminus B)}\Big(\frac{|\varphi(z)|}{e}\Big)^m, \quad z \in \mathbb{D}, m \in \mathbb{N},$$

where $\varphi \in L_h^\infty(\mathbb{D})$ with $|\varphi^*| = e$ almost everywhere on $\mathbb{T} \setminus A$ and $1 \le |\varphi| < e$ on $\mathbb{D}$ (φ^* denotes the radial boundary value of φ which exists almost everywhere because of Fatou's lemma). In particular,

$$f(z) = \lim_{m\to\infty}\frac{1}{2\pi i}\int_B \frac{f^*(\zeta)}{\zeta - z}\Big(\frac{\varphi(z)}{\varphi^*(\zeta)}\Big)^m d\zeta, \quad z \in \mathbb{D}.$$

Proof. Note that $f\varphi^{-m} \in L_h^\infty(\mathbb{D})$. Therefore, using the Cauchy integral formula (see [Rud 1974], Corollary to Theorem 17.12) for $f\varphi^{-m}$, we have

$$f(z) = \frac{1}{2\pi i}\int_B \frac{f^*(\zeta)}{\zeta - z}\left(\frac{\varphi(z)}{\varphi^*(\zeta)}\right)^m d\zeta + \frac{1}{2\pi i}\int_{\mathbb{T}\setminus B} \frac{f^*(\zeta)}{\zeta - z}\left(\frac{\varphi(z)}{\varphi^*(\zeta)}\right)^m d\zeta, \quad z \in \mathbb{D},$$

which immediately gives the estimate in the lemma. □

(b) *Korànyi-Vagi inequality*:

In order to formulate the result of Korànyi-Vagi the following definitions are needed:

$$\mathcal{C}f(z) := \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(\zeta)}{\zeta - z} d\zeta,\ z \in \mathbb{D}, \quad \text{when } f \in L^1(\mathbb{T}),$$
$$M_{\text{rad}}[g](e^{i\theta}) := \sup_{0\le r<1} |g(re^{i\theta})|, \quad \text{when } g\colon \mathbb{D} \to \mathbb{C};$$

$\mathcal{C}f$ is the *Cauchy integral* of f and $M_{\text{rad}}[g]$ the so-called *radial maximal function* for g.

Now we can give the precise statement (for a proof see [Rud 1980], Theorem 6.3.1).

Lemma 8.2.7. *There exists a positive number C such that for every $f \in L^2(\mathbb{T})$ the following inequality holds:*

$$\int_0^{2\pi} \big(M_{\text{rad}}[\mathcal{C}f](e^{i\theta})\big)^2 d\theta \le C\int_0^{2\pi} |f(e^{i\theta})|^2 d\theta.$$

After all these preparations we start the proof of Theorem 8.2.5.

Proof of Theorem 8.2.5. Note that we may assume that $D_1 = D_2 = \mathbb{D}$ (use biholomorphic mappings).

Recall that the function $h_j := \boldsymbol{h}^*_{\mathfrak{R},A_j,\mathbb{D}}$ is harmonic on $\mathbb{D}$ with boundary value $\chi_{\mathbb{T}\setminus A_j}$. Let $g_j := h_j + i\tilde{h}_j$, where $\tilde{h}_j$ denotes a conjugate function to h_j. Then $g_j \in \mathcal{O}(\mathbb{D})$. Put $\varphi_j := e^{g_j}$. Then $\varphi_j \in L_h^\infty(\mathbb{D})$, $|\varphi_j| < e$ on $\mathbb{D}$, and $|\varphi_j^*| = e$ on $\mathbb{T}\setminus \overline{A}_j$, j=1,2.

Step 1^0. *Construction of the holomorphic extension $K(f) \in \mathcal{O}(\widehat{\boldsymbol{X}}_b)$.*

Set $A = A_1 \times A_2$. For $m \in \mathbb{N}_0$ put

$$K_m(z) = K_m(f)(z) := \frac{1}{(2\pi i)^2}\int_A \frac{f(\zeta)}{\zeta - z}\left(\frac{\varphi_1(z_1)\varphi_2(z_2)}{\varphi_1^*(\zeta_1)\varphi_2^*(\zeta_2)}\right)^m d\zeta, \quad z = (z_1, z_2) \in \mathbb{D}^2,$$

where $\zeta - z := (\zeta_1 - z_1)(\zeta_2 - z_2)$. Note that $K_m(f) \in \mathcal{O}(\mathbb{D}^2)$.

We claim that the sequence $(K_m)_{m=0}^\infty$ converges locally uniformly on $\widehat{\boldsymbol{X}}_b$.

Indeed, it suffices to prove that the series $\sum_{m=0}^\infty (K_{m+1} - K_m)$ converges locally uniformly on $\widehat{\boldsymbol{X}}_b$. We write

$$K_{m+1} - K_m = (K_{m+1,m+1} - K_{m+1,m}) + (K_{m+1,m} - K_{m,m}) =: K'_m + K''_m,$$

where

$$K_{n,m}(z) := \frac{1}{(2\pi i)^2}\int_A \frac{f(\zeta)}{\zeta - z}\Big(\frac{\varphi_1(z_1)}{\varphi_1^*(\zeta_1)}\Big)^n\Big(\frac{\varphi_2(z_2)}{\varphi_2^*(\zeta_2)}\Big)^m d\zeta.$$

Then

$$\begin{aligned}K_m'(z) &= \frac{1}{(2\pi i)^2}\int_A \frac{f(\zeta)}{\zeta - z}\Big(\frac{\varphi_1(z_1)}{\varphi_1^*(\zeta_1)}\Big)^{m+1}\Big(\frac{\varphi_2(z_2)}{\varphi_2^*(\zeta_2)}\Big)^m\Big(\frac{\varphi_2(z_2)}{\varphi_2^*(\zeta_2)} - 1\Big)d\zeta \\ &= \frac{1}{(2\pi i)^2}\int_{A_1} \frac{1}{\zeta_1 - z_1}\Big(\frac{\varphi_1(z_1)}{\varphi_1^*(\zeta_1)}\Big)^{m+1} \\ &\qquad \Big(\int_{A_2}\frac{f(\zeta)}{\zeta_2 - z_2}\Big(\frac{\varphi_2(z_2)}{\varphi_2^*(\zeta_2)}\Big)^m\Big(\frac{\varphi_2(z_2)}{\varphi_2^*(\zeta_2)} - 1\Big)d\zeta_2\Big)d\zeta_1.\end{aligned}$$

Recall that for any $\zeta_1 \in A_1$, $z_2 \in \mathbb{D}$ the integrand under the second integral

$$\mathbb{D} \ni \zeta_2 \mapsto f(\zeta_1,\zeta_2)\Big(\frac{\varphi_2(z_2)}{\varphi_2(\zeta_2)}\Big)^m\Big(\frac{\varphi_2(z_2)}{\varphi_2(\zeta_2)} - 1\Big)\frac{1}{\zeta_2 - z_2}$$

is holomorphic on $\mathbb{D}$ (note that the singularity at z_2 is a removable one). Hence, using the Cauchy integral theorem, we get

$$\begin{aligned}K_m'(z) = \frac{1}{(2\pi i)^2}&\int_{A_1} \frac{1}{\zeta_1 - z_1}\Big(\frac{\varphi_1(z_1)}{\varphi_1^*(\zeta_1)}\Big)^{m+1} \\ &\times\Big(-\int_{\mathbb{T}\setminus A_2}\frac{f^*(\zeta)}{\zeta_2 - z_2}\Big(\frac{\varphi_2(z_2)}{\varphi_2^*(\zeta_2)}\Big)^m\Big(\frac{\varphi_2(z_2)}{\varphi_2^*(\zeta_2)} - 1\Big)d\zeta_2\Big)d\zeta_1,\end{aligned}$$

where $f^*(\zeta_1,\zeta_2)$ denotes the radial boundary value of $f(\zeta_1,\cdot) \in L_h^\infty(\mathbb{D})$, $\zeta_1 \in A_1$, which exists almost everywhere on $\mathbb{T}$. Hence, for $z \in \widehat{\boldsymbol{X}}_b$ we arrive at the estimate

$$|K_m'(z)| \le C_1\frac{\|f\|_{\boldsymbol{X}_b}}{(1-|z_1|)(1-|z_2|)}e^{m(u(z)-1)}, \quad u(z) = h_1(z_1) + h_2(z_2),$$

where $C_1 = 2e$ is independent of m, z, and f. Hence, $\sum_{m=0}^\infty K_m'$ converges locally uniformly on $\widehat{\boldsymbol{X}}_b$ to K' with

$$|K'(z)| \le C_1\frac{\|f\|_{\boldsymbol{X}_b}}{(1-|z_1|)(1-|z_2|)(1-\theta(z))}, \quad z \in \widehat{\boldsymbol{X}}_b,$$

where $\theta(z) = e^{u(z)-1}$. In an analogous way one gets the convergence of $K'' = \sum_{m=0}^\infty K_m''$ together with an analogous estimate for $K''(z)$. Hence,

$$K(f) := \lim_{m\to\infty} K_m(f) = K_0(f) + K' + K'' \in \mathcal{O}(\widehat{\boldsymbol{X}}_b)$$

with

$$|K(f)(z)| \le C\frac{\|f\|_{\boldsymbol{X}_b}}{(1-|z_1|)(1-|z_2|)(1-\theta(z))}, \quad z \in \widehat{\boldsymbol{X}}_b,$$

where C is a constant which is independent of z and f. Note that the operator K is a linear one.

Put

$$\hat{f}(z) := \begin{cases} K(f)(z) & \text{if } z \in \widehat{\boldsymbol{X}}_b, \\ f(z) & \text{if } z \in \boldsymbol{X}_b. \end{cases}$$

We will prove that $\hat{f}$ is continuous on $\boldsymbol{X}_b \cup \widehat{\boldsymbol{X}}_b$.

Step 2^0: *Boundary behavior of* $K(f)$.

(a) Local Jordan domains.

Let $a \in \mathbb{T}$. Fix positive numbers $s_1 < s_2 \leq 1$ and $\alpha < \pi/4$. Then we define the following simple Jordan domain with piecewise $\mathcal{C}^1$-boundary:

$$\begin{aligned}\Omega(a,\alpha;s_1,s_2) &:= \{z \in \mathbb{A}(s_1,s_2) : |\operatorname{Arg} z - \operatorname{Arg} a| < \alpha\} \\ &\quad \setminus \left\{z \in \mathbb{A}(s_1,s_2) : \alpha/2 \leq |\operatorname{Arg} z - \operatorname{Arg} a|,\ |z| \leq s_1 + \tfrac{2|\operatorname{Arg} z - \operatorname{Arg} a| - \alpha}{\alpha}(s_2 - s_1)\right\}\end{aligned}$$

and $\Omega(a,\alpha;s) = \Omega(a,\alpha;s,1)$ when $s \in (0,1)$. Observe that if $s,t \in (0,1)$, $s(1+t) < 1-t$, $t < t_0$, and t_0 sufficiently small, then $\Omega(a,(1-t)\alpha;(1+t)s,(1-t)) \subset\subset \Omega(a,\alpha;s)$ and $\bigcup_{0<t<t_0} \Omega(a,(1-t)\alpha;(1+t)s,1-t) = \Omega(a,\alpha;s)$.

(b) Discussion on the boundary values of $K(f)$.

Fix an arbitrary point $b = (b_1,b_2) \in A_1 \times \mathbb{D}$ and choose a positive δ such that $h_2(b_2) + 3\delta < 1$. Take an $r > 0$ such that $h_2(z_2) < h_2(b_2) + \delta$, $z_2 \in V := \mathbb{D}(b_2,r)$, and $\mathbb{D}(b_2,2r) \subset \mathbb{D}$. Finally, choose an angle $\alpha \in (0,\pi/4)$ and an $s \in (0,1)$ such that

$$\Omega := \Omega(a,\alpha;s) \subset \{z_1 \in \mathbb{D} : h_1(z_1) < \delta\}, \quad \partial\Omega \cap \mathbb{T} \subset A.$$

Then $K(f)$ is holomorphic on $\Omega \times V$.

We will prove that the function $K(f)(\cdot,z_2)$, $z_2 \in V$, has non-tangential boundary values almost everywhere on $\partial\Omega$. To do so it suffices to verify that the function $K(f)(\cdot,w_2) \in E_2(\Omega)$, where $E_2(\Omega)$ denotes the Smirnow class of order 2 on Ω (see [Gol 1983]).

First we choose a sequence $t_j \searrow 0$ such that

$$\Omega_j = \Omega(a,(1-t_j)\alpha;(1+t_j)s,1-t_j) \nearrow \Omega.$$

It suffices to find a uniform estimate for

$$\sup_{j\in\mathbb{N}} \int_{\Gamma_j} |K(f)(\zeta_1,z_2)|^2 d\sigma_j(\zeta_1),$$

where $d\sigma_j$ is given by $|\gamma_j'|/L_j$, γ_j is a parametrization of $\Gamma_j := \partial\Omega_j$, and L_j is the length of Γ_j. Recall that $|K_{m+1} - K_m|^2 \le 2(|K'_m|^2 + |K''_m|^2)$. Then (see above)

$$\begin{aligned}
&K'_m(z)\\
&= \Big(\frac{1}{2\pi i}\Big)^2 \int_{A_1} \Big(\int_{\mathbb{T}\setminus A_2} \frac{f^*(\zeta_1,\zeta_2)}{\zeta - z} \Big(\frac{\varphi_1(z_1)}{\varphi_1^*(\zeta_1)}\Big)^{m+1} \Big(\frac{\varphi_2(z_2)}{\varphi_2^*(\zeta_2)}\Big)^m \Big(1 - \frac{\varphi_2(z_2)}{\varphi_2^*(\zeta_2)}\Big) d\zeta_2 \Big) d\zeta_1\\
&= \Big(\frac{\varphi_1(z_1)}{e^\delta}\Big)^{m+1} \frac{1}{2\pi i} \int_{A_1} \frac{\widehat{h}_m(\zeta_1)}{\zeta_1 - z_1} d\zeta_1,
\end{aligned}$$

where

$$\widehat{h}_m(\zeta_1) := \frac{1}{2\pi i} \int_{\mathbb{T}\setminus A_2} \frac{f^*(\zeta_1,\zeta_2)}{\zeta_2 - z_2} \Big(\frac{e^\delta}{\varphi_1(\zeta_1)}\Big)^{m+1} \Big(\frac{\varphi_2(z_2)}{\varphi_2(\zeta_2)}\Big)^m \Big(1 - \frac{\varphi_2(z_2)}{\varphi_2(\zeta_2)}\Big) d\zeta_2, \quad \zeta_1 \in A_1.$$

Note that

- $|\big(\frac{\varphi_1(z_1)}{e^\delta}\big)^{m+1}| \le 1$ whenever $z_1 \in \Omega$,
- $|\widehat{h}_m(\zeta_1)| \le \frac{2e\|f\|_{X_b}}{r} e^{-m\delta}$, $\zeta_1 \in A_1$.

We have to estimate the Cauchy integral

$$\frac{1}{2\pi i} \int_{A_1} \frac{\widehat{h}_m(\zeta_1)}{\zeta_1 - z_1} d\zeta_1 = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{H_m(\zeta_1)}{\zeta_1 - z_1} d\zeta_1 = \mathcal{C}H_m(z)$$

along Γ_j, where

$$H_m(\zeta_1) := \begin{cases} 0 & \text{if } \zeta_1 \in \mathbb{T} \setminus \partial\Omega, \\ \widehat{h}_m(\zeta_1) & \text{if } \zeta_1 \in \mathbb{T} \cap \partial\Omega. \end{cases}$$

Put $\Gamma_{j,1} = \Gamma_j \cap (1 - t_j)\mathbb{T}$ and $\Gamma_{j,2} := \Gamma_j \setminus \Gamma_{j,1}$. Then we get

$$\int_{\Gamma_j} |K'_m(z)|^2 d\sigma_j(z_1) \le \sum_{k=1}^{2} \int_{\Gamma_{j,k}} (M_{\text{rad}}[\mathcal{C}H_m](z_1/|z_1|))^2 d\sigma_j(z_1) =: I_j.$$

Put $\theta_0 := \operatorname{Arg} a$, $F := M_{\text{rad}}[\mathcal{C}H_m]$, $s_{j,1} := (1 + t_j)s$, $s_{j,2} := 1 - t_j$, and $\alpha_j := (1 - t_j)\alpha$. Moreover, let

$$\begin{aligned}
r_{j,1}(\theta) &:= s_{j,1} + \frac{2(\theta_0 - \theta) - \alpha_j}{\alpha_j}(s_{j,2} - s_{j,1}),\\
r_{j,2}(\theta) &:= s_{j,1} + \frac{2(\theta - \theta_0) - \alpha_j}{\alpha_j}(s_{j,2} - s_{j,1}).
\end{aligned}$$

Then the second term in I_j can be written as a sum of the following three integrals:

$$\int_{\theta_0-\alpha_j}^{\theta_0-\alpha_j/2} F^2(e^{i\theta})\sqrt{(r_{j,1}(\theta))^2+(r'_{j,1}(\theta))^2}\,\frac{d\theta}{L_j},$$

$$\int_{\theta_0-\alpha_j/2}^{\theta_0+\alpha_j/2} F^2(e^{i\theta})s_{j,1}\,\frac{d\theta}{L_j},$$

$$\int_{\theta_0+\alpha_j/2}^{\theta_0+\alpha_j} F^2(e^{i\theta})\sqrt{(r_{j,2}(\theta))^2+(r'_{j,2}(\theta))^2}\,\frac{d\theta}{L_j}.$$

Therefore, using Lemma 8.2.7,

$$\begin{aligned} I_j &\le C_2\int_{\Gamma_{j,1}} (M_{\text{rad}}[\mathcal{C}H_m](z_1/|z_1|))^2 d\sigma_j(z_1) \\ &\le C_2'\int_{\partial\Omega\cap\mathbb{T}} (M_{\text{rad}}[\mathcal{C}H_m](z_1))^2 d\sigma(z_1) \le C_3\int_{\mathbb{T}} |H_m(z_1)|^2 d\sigma(z_1), \quad d\sigma = \frac{d\theta}{2\pi}. \end{aligned}$$

Hence,

$$\int_{\Gamma_j} |K'_m(z)|^2 d\sigma_j(z_1) \le C_4\cdot e^{-2m\delta}, \quad z_2 \in V.$$

A similar estimate holds for the term $K_0(f)$ (EXERCISE).

On the other hand, the estimate for K''_m will be much simpler. Namely,

$$\begin{aligned} &K''_m(z) \\ &= \left(\frac{1}{2\pi i}\right)^2\int_A \frac{f(\zeta)}{\zeta-z}\left(\frac{\varphi_1(z_1)\varphi_2(z_2)}{\varphi_1^*(\zeta_1)\varphi_2^*(\zeta_2)}\right)^m\left(\frac{\varphi_1(z_1)}{\varphi_1^*(\zeta_1)-1}\right)d\zeta_1 d\zeta_2 \\ &= \left(\frac{1}{2\pi i}\right)^2\int_{\mathbb{T}\setminus A_1}\left(\int_{A_2}\frac{f^*(\zeta_1,\zeta_2)}{(\zeta_1-z_1)(\zeta_2-z_2)}\left(\frac{\varphi_1(z_1)\varphi_2(z_2)}{\varphi_1^*(\zeta_1)\varphi_2^*(\zeta_2)}\right)^m\left(\frac{\varphi_1(z_1)}{\varphi_1^*(\zeta_1)}-1\right)d\zeta_2\right)d\zeta_1, \end{aligned}$$

where the last equality is, similarly as above, a consequence of the Cauchy integral theorem. Hence, we get

$$|K''_m(z)| \le \frac{2\|f\|_{X_b}}{r\,\text{dist}(\Omega,\mathbb{T}\setminus A)}e^{m(u(z)-1)} \le C_5 e^{-m\delta}, \quad z_1\in\Omega,\ z_2\in V.$$

Therefore, $\sup_j \int_{\Gamma_j} |K''_m(z_1,z_2)|^2 d\sigma_j(z_1) \le C_6 e^{-2m\delta}$.

Summarizing, we end with

$$\int_{\Gamma_j} |K(z_1,z_2)|^2 d\sigma_j(z_1) \le C_7,$$

which shows that $K(f)(\cdot,z_2) \in E_2(\Omega)$, $z_2 \in V$. Finally, applying [Gol 1983], Chapter X, §5, shows that $K(f)(\cdot,z_2)$ has almost everywhere non-tangential boundary values $K(f)^*(\cdot, z_2)$ on $\partial\Omega$, $z_2 \in V$, and

$$\int_{\partial\Omega} |K(f)^*(z_1, z_2)|^2 d\sigma(z_1) = \sup_j \int_{\Gamma_j} |K(z_1, z_2)|^2 d\sigma_j(z_1).$$

Step 3^0. *Evaluation of $K(f)^*$ along $\partial\Omega \cap \mathbb{T}$.*

We want to verify that $K(f)^*(\cdot, z_2) = f(\cdot, z_2)$ almost everywhere on $\partial\Omega \cap \mathbb{T}$, $z_2 \in V$. Indeed:

Write $f = f_1 + i f_2$. Then f_j are bounded harmonic functions on $\mathbb{D}$. Define $\tilde{f}_j(\zeta) := \limsup_{k\to\infty} f_j((1-1/k)\zeta_1, \zeta_2)$, $\zeta \in \mathbb{T} \times A_2$. Note that $\tilde{f}_j$ are integrable along $\mathbb{T} \times A_2$, $j = 1, 2$. Then $\tilde{f} := \tilde{f}_1 + i\tilde{f}_2$ is integrable along $\mathbb{T} \times A_2$ and for $\zeta_2 \in A_2$ we have $\tilde{f}(\cdot, \zeta_2) = f^*(\cdot, \zeta_2)$ almost everywhere on $\mathbb{T}$.

Put

$$K^*_{\infty,m}(z) := \frac{1}{2\pi i} \int_{A_2} \frac{\tilde{f}(z_1, \zeta_2)}{\zeta_2 - z_2} \left(\frac{\varphi_2(z_2)}{\varphi_2(\zeta_2)}\right)^m d\zeta_2, \quad z_1 \in \mathbb{T}.$$

To decide whether this function is the boundary value of its Cauchy integral we will apply Theorem 1 in [Gol 1983], Chapter X. So we have to calculate

$$\int_{\mathbb{T}} K^*_{\infty,m}(\zeta_1, z_2)\zeta_1^k d\zeta_1 = \frac{1}{2\pi i} \int_{A_2} \left(\int_{\mathbb{T}} f^*(\zeta_1, z_2)\zeta_1^k d\zeta_1\right) \left(\frac{\varphi_2(z_2)}{\varphi_2(\zeta_2)}\right)^m d\zeta_2 = 0;$$

use that $f^*(\cdot, z_2)$ is a boundary value of $f(\cdot, z_2)$, $z_2 \in V$. Hence we conclude that $K^*_{\infty,m}(\cdot, z_2)$ is along $\mathbb{T}$ the boundary value of its Cauchy integral

$$K_{\infty,m}(z_1, z_2) := \mathcal{C}[K^*_{\infty,m}(\cdot, z_2)](z_1), \quad z_1 \in \mathbb{D},\ z_2 \in V.$$

To evaluate $K^*(f)(\cdot, z_2)$, $z_2 \in V$, we discuss the integral

$$\begin{aligned}
&\int_{\partial\Omega\cap\mathbb{T}} |K^*(f)(z_1, z_2) - f(z_1, z_2)|^2 d\sigma(z_1) \\
&\quad \le 2\int_{\partial\Omega\cap\mathbb{T}} |K^*(f)(z_1, z_2) - K^*_{\infty,m}(z_1, z_2)|^2 d\sigma(z_1) \\
&\qquad + 2\int_{\partial\Omega\cap\mathbb{T}} |K^*_{\infty,m}(z_1, z_2) - f(z_1, z_2)|^2 d\sigma(z_1) \\
&\quad \le 2\sup_j \int_{\Gamma_j} |K(f)(z_1, z_2) - K_{\infty,m}(z_1, z_2)|^2 d\sigma_j(z_1) \\
&\qquad + 2\int_{\partial\Omega\cap\mathbb{T}} |K^*_{\infty,m}(z_1, z_2) - f(z_1, z_2)|^2 d\sigma(z_1) =: \mathrm{I}_m + \mathrm{II}_m.
\end{aligned}$$

The second inequality is a consequence that the integrand belongs to the Smirnow class $E_2(\Omega)$ (see [Gol 1983], Chapter X). First, let us discuss the second integral (use

Lemma 8.2.6):

$$\mathrm{II}_m = 2\int_{\partial\Omega\cap\mathbb{T}} \left|\frac{1}{2\pi i}\int_{A_2} \frac{f(z_1,\zeta_2)}{\zeta_2 - z_2}\left(\frac{\varphi_2(z_2)}{\varphi_2(\zeta_2)}\right)^m d\zeta_2 - f(z_1,z_2)\right|^2 d\sigma(z_1)$$
$$\le 2\,\mathrm{mes}(A_1)\left(C\frac{\|f\|_{\boldsymbol{X}_b}}{r}\left(\frac{|\varphi(z_2)|}{e}\right)^m\right)^2 \le C_8 e^{-2m\delta},$$

where $\mathrm{mes}(A_1)$ means the arc length of A_1.

Finally,

$$\mathrm{I}_m \le 4\sup_j \int_{\Gamma_j} \Big|\sum_{k=m}^{\infty}(K_{m+1}(z_1,z_2) - K_m(z_1,z_2))\Big|^2 d\sigma_j(z_1)$$
$$+ 4\sup_j \int_{\Gamma_j} \big|K_m(z_1,z_2) - K_{\infty,m}(z_1,z_2)\big|^2 d\sigma_j(z_1) \le C_9 e^{-m\delta} + \mathrm{III}_m.$$

To study III_m we write the integrand in full details for $z = (z_1, z_2) \in \Omega \times V$:

$$|K_{\infty,m}(z) - K_m(z)|$$
$$= \left|\left(\frac{1}{2\pi i}\right)^2 \int_{\mathbb{T}\setminus A_1}\left(\int_{A_2} \frac{f^*(\zeta_1,\zeta_2)}{(\zeta_1 - z_1)(\zeta_2 - z_2)}\left(\frac{\varphi_1(z_1)\varphi_2(z_2)}{\varphi_1^*(\zeta_1)\varphi_2^*(\zeta_2)}\right)^m d\zeta_2\right) d\zeta_1\right|$$
$$\le \frac{\|f\|_{\boldsymbol{X}_b}}{\mathrm{dist}(\Omega, \mathbb{T}\setminus A)} e^{-m\delta}.$$

Therefore, $\mathrm{III}_m \le 4\big(\frac{\|f\|_{\boldsymbol{X}_b}}{\mathrm{dist}(\Omega,\mathbb{T}\setminus A)} e^{-m\delta}\big)^2$. Taking all the estimates into account we finally get

$$\int_{\partial\Omega\cap\mathbb{T}} |K^*(f)(z) - f(z)|^2 d\sigma(z_1) \le C_{10} e^{-m\delta} \underset{m\to\infty}{\to} 0;$$

i.e. $K^*(f)(\cdot, z_2) = f(\cdot, z_2)$ almost everywhere on $\mathbb{T} \cap \partial\Omega$, $z_2 \in V$.

Step 4^0. *Proof of* $\|K(f)\|_{\widehat{\boldsymbol{X}}_b} \le \|f\|_{\boldsymbol{X}_b}$.

We will use the geometric objects from Steps 2^0, 3^0. Then $K(f)(\cdot, z_2)$ and $K(f^N)(\cdot, z_2)$ have almost everywhere on $\mathbb{T} \cap \partial\Omega$ non-tangential boundary values $f(\cdot, z_2)$ and $f^N(\cdot, z_2)$, respectively, $z_2 \in V$. Using the Privalov uniqueness result (see [Gol 1983], Chapter X, §2, Theorem 1) we see that $(K(f)(\cdot, z_2))^N = K(f^N)(\cdot, z_2)$ on Ω, $z_2 \in V$. Applying the identity theorem, it follows that $(K(f))^N = K(f^N)$ on $\widehat{\boldsymbol{X}}_b$. Moreover, taking into account the estimate from Step 1^0 we have

$$|K(f)(z)|^N = |K(f^N)(z)| \le C\frac{\|f\|_{\boldsymbol{X}_b}^N}{(1 - |z_1|)(1 - |z_2|)(1 - \theta(z))}$$

and, since $N \in \mathbb{N}$ is arbitrarily chosen, $|K(f)(z)| \le \|f\|_{\boldsymbol{X}_b}$, $z \in \widehat{\boldsymbol{X}}_b$.

Step 5^0. *Continuity at points $a \in A$.*

Fix a point $a = (a_1, a_2) \in A$. We may assume that $f(a) = 0$ (substituting f by $f - f(a)$). Now choose a positive ε. Then there exists an $r > 0$ such that $|f| < \varepsilon$ on the new boundary cross

$$\boldsymbol{X}_b' := \big((\mathbb{D}(a_1, r) \cap \overline{\mathbb{D}}) \times (\mathbb{D}(a_2, r) \cap \mathbb{T})\big) \cup \big((\mathbb{D}(a_1, r) \cap \mathbb{T}) \times (\mathbb{D}(a_2, r) \cap \overline{\mathbb{D}})\big).$$

Using biholomorphic mappings we find with the previous methods a Carleman extension $K(f|_{\boldsymbol{X}_b'}) \in \mathcal{O}(\widehat{\boldsymbol{X}}_b')$ with $|K(f|_{\boldsymbol{X}_b'})| \le \varepsilon$ on $\widehat{\boldsymbol{X}}_b'$. Fix a point $a_2' \in \mathbb{D}(a_2, r) \cap \mathbb{D}$. Following Steps 2^0, 3^0, we conclude that near a_1 the functions $K(f)(\cdot, z_2)$ and $K(f|_{\boldsymbol{X}_b'})(\cdot, z_2)$ have almost everywhere the same non-tangential boundary values, namely, $f(\cdot, z_2)$, where $z_2 \in \mathbb{D}(a_2', s)$, s sufficiently small. Therefore, by virtue of Privalov's uniqueness theorem and the identity theorem, it follows that $K(f) = K(f|_{\boldsymbol{X}_b'})$ on $\widehat{\boldsymbol{X}}_b'$. Then, taking a $\delta_\varepsilon > 0$ such that

$$U := (\mathbb{D}(a_1, \delta_\varepsilon) \cap \mathbb{D}) \times (\mathbb{D}(a_2, \delta_\varepsilon) \cap \mathbb{D}) \subset \widehat{\boldsymbol{X}}_b' \subset \widehat{\boldsymbol{X}}_b,$$

it follows that $|K(f)| < \varepsilon$ on U; i.e. $\hat{f}$ is continuous at the point a.

Step 6^0. *Continuity at points $a \in A_1 \times \mathbb{D}$.*

Fix a point $a = (a_1, a_2) \in A_1 \times \mathbb{D}$. We may assume that $f(a) = 0$ and $\|f\|_{\boldsymbol{X}_b} \le 1$. Now let a positive $\varepsilon < 1$ be given. Choose δ, r, V, and Ω as in Step 3^0. Then we can find a positive s, $s < r$, such that $|f| < \varepsilon$ on $(\mathbb{T} \cap \mathbb{D}(a_1, s)) \times \mathbb{D}(a_2, s)$ and $\mathbb{D} \cap \mathbb{D}(a_1, s) \subset \Omega$. As in Step 3^0, the function $K(f)(\cdot, z_2)$ has almost everywhere along $\mathbb{T} \cap \mathbb{D}(a_1, s)$ the non-tangential boundary value $f(\cdot, z_2)$, $z_2 \in V$.

Put, on $D' := \mathbb{D} \cap \mathbb{D}(a_2 s)$,

$$u_w := \frac{\log|K(f)(\cdot, w)| - \log\varepsilon}{-\log\varepsilon}, \quad w \in \mathbb{D}(a_2, s).$$

Then $u \in \mathcal{SH}(D')$, $\limsup_{D' \ni \zeta \to z} u_w(\zeta) \le 1$, $z \in \partial D' \cap \mathbb{D}$. Moreover, u has the non-tangential limit 0 almost everywhere along $A' := \partial D' \cap \mathbb{T}$. Take a biholomorphic mapping $\Phi\colon \mathbb{D} \to D'$. Then, in fact, Φ is a homeomorphism from $\overline{\mathbb{D}}$ onto $\overline{D'}$. Observe that $v_w := u_w \circ \Phi \in \mathcal{SH}(\mathbb{D})$ with $\limsup_{\mathbb{D} \ni \zeta \to z} v_w(\zeta) \le 1$ for all $z \in \mathbb{T} \setminus \overline{A''}$, where $A'' := \Phi^{-1}(A')$. Moreover, v_w has the non-tangential limit 0 almost everywhere along B. Applying Lemma 3.7.6, it follows that $v_w = u_w \circ \Phi \le \boldsymbol{h}^*_{\mathfrak{K}, A'', \mathbb{D}} = \boldsymbol{h}^*_{\mathfrak{K}, A', D'} \circ \Phi$ on $\mathbb{D}$. Hence, $u_w \le \boldsymbol{h}^*_{\mathfrak{K}, A', D'}$ on D'. In particular, $|K(f)(z)| \le \varepsilon^{1 - \boldsymbol{h}^*_{\mathfrak{K}, A', D'}(z_1)} \le \sqrt{\varepsilon}$, if z_1 is sufficiently near to a_1 and $z_2 \in \mathbb{D}(a_2, s)$, which proves the continuity.

Step 7^0. *Estimate for $\hat{f}$.*

Fix an $a_2 \in A_2$. Then $f(\cdot, a_2)$ is holomorphic on $\mathbb{D}$ and continuous on $\mathbb{D} \cup A_1$. Using the two-constant theorem one concludes that

$$|f(z_1, a_2)| \le \|f\|_{A_1}^{1 - \boldsymbol{h}^*_{\mathfrak{K}, A_1, \mathbb{D}}(z_1)} \|f\|_{\boldsymbol{X}_b}^{\boldsymbol{h}^*_{\mathfrak{K}, A_1, \mathbb{D}}(z_1)}, \quad z_1 \in \mathbb{D}.$$

Now fix a point $z^0 \in \widehat{\boldsymbol{X}}_b$. Put

$$D(\delta) := \{z_2 \in \mathbb{D} : \boldsymbol{h}^*_{\mathfrak{R},A_2,\mathbb{D}}(z_2) < 1 - \delta\},$$

where $\delta := \boldsymbol{h}^*_{A,\mathbb{D}}(z_1^0)$. Note that $A_2 \subset \partial D(\delta)$. As above we get

$$|\hat{f}(z_1^0, z_2)| \le \|\hat{f}(z_1^0, \cdot)\|_{A_2}^{1-\boldsymbol{h}^*_{\mathfrak{R},A_2,D(\delta)}(z_2)} \|\hat{f}(z_1^0, \cdot)\|_{D(\delta)}^{\boldsymbol{h}^*_{\mathfrak{R},A_2,D(\delta)}(z_2)}, \quad z_2 \in D(\delta).$$

Finally, we only have to recall that $\boldsymbol{h}^*_{\mathfrak{R},A_2,D(\delta)} = \frac{\boldsymbol{h}^*_{\mathfrak{R},A_2,\mathbb{D}}}{1-\boldsymbol{h}^*_{\mathfrak{R},A_1,\mathbb{D}}(z_1^0)}$ on $D(\delta)$ (EXERCISE; for a similar result see Proposition 3.2.27) to get the claimed estimate for the point z^0. □

Remark 8.2.8. The extension problem for boundary crosses may be formulated in the following general context.

(S-$\mathcal{O}_{\mathrm{B}}$) We ask whether $\boldsymbol{X}_b \subset \partial\widehat{\boldsymbol{X}}_{\mathfrak{A},b}$ and whether there are subsets $\widetilde{A}_j \subset A_j$, $\widetilde{A}_j \neq \varnothing$, such that:

- for any $a \in \mathbb{X}_b((\widetilde{A}_j, D_j)_{j=1}^N) =: \boldsymbol{Y}_b$ with $a \in \widetilde{A}_1 \times \cdots \times \widetilde{A}_N =: \widetilde{A}$ and any $\alpha = (\alpha_1, \dots \alpha_N) \in I_{1,a_1} \times \cdots \times I_{N,a_N}$ there exist open neighborhoods $U_j = U_j(a, \alpha)$ of a_j with

$$\mathcal{B}_\alpha(a) := (U_1 \cap \mathcal{A}_{1,\alpha_1}(a_1)) \times \cdots \times (U_N \cap \mathcal{A}_{N,\alpha_n}(a_N)) \subset \widehat{\boldsymbol{X}}_{\mathfrak{A},b};$$

- for any $a \in \boldsymbol{Y}_b$ with $a_j \in D_j$ and any

$$(\alpha'_j, \alpha''_j) \in \Big(\prod_{s=1}^{j-1} I_{s,a_s}\Big) \times \Big(\prod_{s=j+1}^{N} I_{s,a_s}\Big) =: I_{a'_j} \times I_{a''_j}$$

there exist open neighborhoods $V_s = V_s(a, \alpha'_j, \alpha''_j)$ of a_s, $s = 1, \dots, N$, with $V_j \subset D_j$ such that

$$\mathcal{B}_{\alpha'_j,\alpha''_j}(a) := \prod_{s=1}^{j-1} (V_s \cap \mathcal{A}_{s,\alpha_s}(a_s)) \times V_j \times \prod_{s=j+1}^{N} (V_s \cap \mathcal{A}_{s,\alpha_s}(a_s)) \subset \widehat{\boldsymbol{X}}_{\mathfrak{A},b};$$

note that the family $(\mathcal{B}_\alpha(a), \mathcal{B}_{\alpha'_j,\alpha''_j}(a))$ defines a system of approach regions for $(\boldsymbol{Y}_b, \widehat{\boldsymbol{X}}_{\mathfrak{A},b})$.

- for every function $f \in \mathcal{O}_{\mathfrak{A},s}(\boldsymbol{X}_b)$ there exists a unique $\hat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}}_{\mathfrak{A},b})$ with

$$f(a) = \begin{cases} \lim\limits_{\mathcal{B}_\alpha(a) \ni z \to a} \hat{f}(z) & \text{if } a \in \widetilde{A},\ \alpha \in I_{1,a_1} \times \cdots \times I_{N,a_N}, \\ \lim\limits_{\mathcal{B}_{\alpha'_j,\alpha''_j}(a) \ni z \to a} \hat{f}(z) & \text{if } a \in \boldsymbol{Y}_b,\ a_j \in D_j, \\ & (\alpha'_j, \alpha''_j) \in I_{a'_j} \times I_{a''_j},\ 1 \le j \le N. \end{cases}$$

Results discussing various aspects of the above general problem may be found in [Pfl-NVA 2003], [Pfl-NVA 2004], [Pfl-NVA 2007], [NVA 2008], [NVA 2009] and [NVA 2010].

Part II

Cross theorems with singularities

Chapter 9
Extension with singularities

Summary. Section 9.1 shows that almost all slices (parallel to coordinate hyperplanes) of an $\mathcal{S}$-region of holomorphy are regions of holomorphy for certain families (induced by $\mathcal{S}$) of holomorphic functions. The next section starts with a discussion of the Gonchar class $\boldsymbol{R}^0$ of functions which allow a strong approximation by rational functions. Then the theorem of Oka–Nishino (Theorem 9.2.19) is presented. It clarifies the analytic structure of a pseudoconcave set in $D \times \mathbb{C}$ under certain assumptions on the nature of its fibers over the points of D. The section finishes with a result of Chirka–Sadullaev (Theorem 9.2.24) explaining that fiberwise holomorphic extension outside of "thin" sets leads to a global holomorphic extension outside of a global pluripolar, respectively analytic set. This result will play the main role in the following discussion of cross theorems with singularities. The chapter concludes with theorems due to Grauert–Remmert, Dloussky, and Chirka (Theorem 9.4.1, Theorem 9.4.2) describing the envelope of holomorphy of $D \setminus M$ in terms of the envelope of D itself and some exceptional set.

9.1 Sections of regions of holomorphy

$\boxed{>}$ §§ 2.1, 2.3.

This section is based on [Jar-Pfl 2005].

Let $(X, p) \in \mathfrak{R}(\mathbb{C}^n)$, $\mathbb{C}^n = \mathbb{C}^k \times \mathbb{C}^\ell$, $p = (u, v)\colon X \to \mathbb{C}^k \times \mathbb{C}^\ell$. Put $\Omega := p(X)$, $\Omega_k := u(X) = \operatorname{pr}_{\mathbb{C}^k}(\Omega)$, $\Omega^\ell := v(X) = \operatorname{pr}_{\mathbb{C}^\ell}(\Omega)$. For $a \in \Omega_k$ define

$$X_a := u^{-1}(a) = p^{-1}(\{a\} \times \mathbb{C}^\ell), \quad p_a := v|_{X_a}.$$

Similarly, for $b \in \Omega^\ell$, put $X^b := v^{-1}(b) = p^{-1}(\mathbb{C}^k \times \{b\})$, $p^b := u|_{X^b}$.

Exercise 9.1.1. (a) (X_a, p_a) is a Riemann region over $\mathbb{C}^\ell$ for every $a \in \Omega_k$. If (X, p) is countable at infinity, then so is (X_a, p_a).

(b) By Theorem 2.5.5 (h), if X is a region of holomorphy, then each (X_a, p_a) is a region of holomorphy.

Let $\varnothing \neq \mathcal{S} \subset \mathcal{O}(X)$. For $a \in \Omega_k$ define $f_a := f|_{X_a}$, $\mathcal{S}_a := \{f_a : f \in \mathcal{S}\} \subset \mathcal{O}(X_a)$.

In view of Exercise 9.1.1 (b), it is natural to ask whether each (X_a, p_a) is an $\mathcal{S}_a$-region of holomorphy provided that (X, p) is an $\mathcal{S}$-region of holomorphy. In general, the answer is negative as the following example shows.

Let $X \subset \mathbb{C}^2$ $(k = \ell = 1)$ be a fat bounded domain of holomorphy with $0 \in X$ (recall that X is *fat* if $X = \operatorname{int} \overline{X}$). Take a non-continuable function $g \in \mathcal{O}(X)$, put $f(z) := z_1 g(z)$, $z = (z_1, z_2) \in X$, $\mathcal{S} := \{f\}$. One can easily check that f is also non-continuable beyond X (EXERCISE). Observe that $f_0 = f(0,\cdot) \equiv 0$. Thus $X_0 = X_{(0,\cdot)} \subsetneq \mathbb{C}$ is not an $\mathcal{S}_0$-region of holomorphy.

Theorem 9.1.2. *Let $(X, p) \in \mathfrak{R}_\infty(\mathbb{C}^n)$ be an $\mathcal{S}$-region of holomorphy. Then there exists a pluripolar set $P_k \subset \Omega_k$ such that (X_a, p_a) is an $\mathcal{S}_a$-region of holomorphy for every $a \in \Omega_k \setminus P_k$.*

Proof. By Proposition 2.1.26, we may assume that $\mathcal{S}$ is finite or countable.

Step 1^0. *There exists a pluripolar set $P \subset \Omega_k$ such that for any $a \in \Omega_k \setminus P$, (X_a, p_a) is an $\mathcal{S}_a$-region of existence.*

Define $R_{f,b}(x) := d(T_x f_{u(x)})$, $f \in \mathcal{S}$, $b \in \Omega^\ell$, $x \in X^b$. Recall (§ 2.1.2) that

$$1/R_{f,b}(x) = \limsup_{\nu\to+\infty} \Big(\max_{\beta\in\mathbb{Z}_+^\ell:\,|\beta|=\nu} \frac{1}{\beta!}|D^{(0,\beta)} f(x)|\Big)^{1/\nu}, \quad x \in X^b.$$

Obviously, $R_{f,b}(x) \ge d_X(x)$, $x \in X^b$. By the Cauchy inequalities, we get

$$\frac{1}{\beta!}|D^{(0,\beta)} f(x)| \le \frac{\sup_{\widehat{\mathbb{P}}_X(x_0,r)} |f|}{r^{|\beta|}}, \quad 0 < r < d_X(x_0),\ x \in \widehat{\mathbb{P}}_X(x_0, r/2),\ \beta \in \mathbb{Z}_+^\ell.$$

Consequently, the function $-\log(R_{f,b})_*$ (where $_*$ denotes the lower semicontinuous regularization on X^b) is plurisubharmonic on X^b. Put

$$P_{f,b} := u(\{x \in X^b : (R_{f,b})_*(x) < R_{f,b}(x)\}) \subset \Omega_k.$$

Then $P_{f,b}$ is pluripolar (cf. Theorem 2.3.33 (b)). Put

$$R_b := \inf_{f\in\mathcal{S}} R_{f,b}, \quad \widehat{R}_b := \inf_{f\in\mathcal{S}} (R_{f,b})_*.$$

Observe that $-\log(\widehat{R}_b)_*$ is plurisubharmonic on X^b. Put

$$P_b := u(\{x \in X^b : (\widehat{R}_b)_*(x) < \widehat{R}_b(x)\}) \subset \Omega_k.$$

The set P_b is also pluripolar (cf. Theorem 2.3.33 (a)). Now let $B \subset \Omega^\ell$ be a dense countable set. Define

$$P := \Big(\bigcup_{f\in\mathcal{S},\, b\in B} P_{f,b}\Big) \cup \Big(\bigcup_{b\in B} P_b\Big) \subset \Omega_k.$$

Then P is pluripolar.

Take an $a \in \Omega_k \setminus P$ and suppose that X_a is not an $\mathcal{S}_a$-region of existence. Then there exist a point $x_0 \in X_a$ and a number $r > d_{X_a}(x_0)$ such that $b := v(x_0) \in B$ and $R_b(x_0) > r$. Since $a \notin P$, we have

$$(\widehat{R}_b)_*(x_0) = \widehat{R}_b(x_0) = \inf_{f\in\mathcal{S}} (R_{f,b})_*(x_0) = \inf_{f\in\mathcal{S}} R_{f,b}(x_0) = R_b(x_0) > r.$$

In particular, there exists $0 < \varepsilon < d_X(x_0)$ such that $(\widehat{R}_b)_*(x) > r$, $x \in \widehat{\mathbb{P}}_{X^b}(x_0, \varepsilon)$. Since

$$R_b(x) = \inf_{f \in \mathcal{S}} R_{f,b}(x) \geq \inf_{f \in \mathcal{S}} (R_{f,b})_*(x) = \widehat{R}_b(x) \geq (\widehat{R}_b)_*(x),$$

we conclude that $R_b(x) > r$, $x \in \widehat{\mathbb{P}}_{X^b}(x_0, \varepsilon)$. Put $U := \widehat{\mathbb{P}}_X(x_0, \varepsilon)$. Hence, by Proposition 1.1.10, for every $f \in \mathcal{S}$, the function $f \circ (p|_U)^{-1}$ extends holomorphically to $V := \mathbb{P}(a, \varepsilon) \times \mathbb{P}(b, r)$. Since (X, p) is an $\mathcal{S}$-domain of holomorphy, by Remark 2.1.24, there exists a univalent domain $W \subset X$ with $U \subset W$ such that $p(W) = V$. In particular, $d_{X_a}(x_0) \geq r$; a contradiction.

Step 2^0. *There exists a pluripolar set $P \subset \Omega_k$ such that for any $a \in \Omega_k \setminus P$ the family $\mathcal{S}_a$ weakly separates points in X_a.*

Take $a \in \Omega_k$, $x', x'' \in X_a$ with $x' \neq x''$, $p_a(x') = p_a(x'') =: b$, and $f \in \mathcal{S}$ such that $T_{x'} f \neq T_{x''} f$. Put $r := \min\{d(T_{x'} f), d(T_{x''} f)\}$ and let

$$\begin{aligned} P_{a,x',x'',f} &:= \{z \in \mathbb{P}(b, r) : (T_{x'} f)(z, \cdot) \equiv (T_{x''} f)(z, \cdot)\} \\ &= \bigcap_{w \in \mathbb{P}(b,r)} \{z \in \mathbb{P}(a, r) : T_{x'} f(z, w) = T_{x''} f(z, w)\}. \end{aligned}$$

Then $P_{a,x',x'',f} \subsetneq \mathbb{P}(a, r)$ is an analytic subset. For any $z \in \mathbb{P}(a, r) \setminus P_{a,x',x'',f}$ we have $(T_{x'} f)(z, \cdot) \not\equiv (T_{x''} f)(z, \cdot)$ on $\mathbb{P}(b, r)$.

Take a countable dense set $A \subset \Omega_k$. For any $a \in A$ let $B_a \subset X_a$ be a countable dense subset such that $p_a^{-1}(p_a(B_a)) = B_a$. Then

$$P := \bigcup_{\substack{a \in A,\ x', x'' \in B_a,\ f \in \mathcal{S} \\ x' \neq x'',\ p_a(x') = p_a(x''),\ T_{x'} f \neq T_{x''} f}} P_{a,x',x'',f}$$

is a pluripolar set.

Fix $a_0 \in \Omega_k \setminus P$, $x_0', x_0'' \in X_{a_0}$, with $x_0' \neq x_0''$ and $p_{a_0}(x_0') = p_{a_0}(x_0'')$. Put $r_0 := \min\{d_X(x_0'), d_X(x_0'')\}$. Since (X, p) is an $\mathcal{S}$-region of holomorphy, there exists an $f_0 \in \mathcal{S}$ be such that $T_{x_0'} f_0 \neq T_{x_0''} f_0$. Let $\varepsilon \in (0, r_0/2)$ be such that $T_{x'} f_0 \neq T_{x''} f_0$ for arbitrary $x' \in \widehat{\mathbb{P}}_X(x_0', \varepsilon)$, $x'' \in \widehat{\mathbb{P}}_X(x_0'', \varepsilon)$. Take $a \in A \cap \mathbb{P}(a_0, \varepsilon)$ and $x', x'' \in B_a$ such that $x' \in \widehat{\mathbb{P}}_X(x_0', \varepsilon)$, $x'' \in \mathbb{P}_X(x_0'', \varepsilon)$, $p_a(x') = p_a(x'') =: b$. Observe that $r := \min\{d(T_{x'} f_0), d(T_{x''} f_0)\} \geq r_0/2$. Since $a_0 \notin \mathbb{P}(a, r) \setminus P_{a,x',x'',f_0}$, we conclude that $(T_{x'} f_0)(a_0, \cdot) \not\equiv (T_{x''} f_0)(a_0, \cdot)$ on $\mathbb{P}(b, r)$. Consequently, $(T_{x_0'} f)(a_0, \cdot) \not\equiv (T_{x_0''} f)(a_0, \cdot)$, which implies that $T_{x_0'} f_{a_0} \neq T_{x_0''} f_{a_0}$. □

Example 9.1.3. The following example shows that in general the set P_k from Theorem 9.1.2 is not contained in any analytic subset of Ω_k of dimension $\leq k - 1$.

Let $A \subset \mathbb{C}$ be a dense $\mathcal{F}_\sigma$ polar set (e.g. $A := \mathbb{Q}^2$). By Theorem 4.2.5 there exists a function $f : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ such that

- $f(\cdot, b) \in \mathcal{O}(\mathbb{C})$ for each $b \in \mathbb{C}$,

- $f(a,\cdot) \in \mathcal{O}(\mathbb{C})$ for each $a \in A$,
- f is unbounded near some point $(z_0, 0) \in \mathbb{C} \times \mathbb{C}$.

By Theorem 1.4.7 there exists a non-empty domain $V \subset \mathbb{C}$ such that $f \in \mathcal{O}(\mathbb{C} \times V)$. Let $\varphi\colon (\mathbb{C} \times V, \mathrm{id}) \to (X, p)$ be the $\{f|_{\mathbb{C}\times V}\}$-maximal extension. Put $\hat{f} := (f|_{\mathbb{C}\times V})^{\varphi} \in \mathcal{O}(X)$, $\mathcal{S} := \{\hat{f}\}$. Then (X, p) is an $\mathcal{S}$-domain of holomorphy (cf. Remark 2.1.19). Observe that $\mathbb{C} \times V \subset p(X)$. We will prove that for every $a \in A$, the region (X_a, p_a) is not an $\mathcal{S}_a$-region of holomorphy, which shows that $A \subset P_k$ for any set P_k as in Theorem 9.1.2.

We may assume that the maximal extension (X, p) is a domain in the sheaf of germs of holomorphic functions in $\mathbb{C}^2$ (as in Remark 2.1.13).

First, observe that X must be univalent and $\hat{f} = f \circ p$.

Indeed, take a sequence of bidiscs $Q_j = Q_j' \times Q_j'' \subset \mathbb{C}^2$, $j = 1, \dots, N$, and a sequence of functions $f_j \in \mathcal{O}(Q_j)$, $j = 1, \dots, N$, such that

- $Q_1'' \subset V$, $f_1 = f|_{Q_1}$,
- $Q_j \cap Q_{j+1} \neq \varnothing$, $f_j = f_{j+1}$ on $Q_j \cap Q_{j+1}$, $j = 1, \dots, N-1$.

We want to show that $f_N = f|_{Q_N}$.

We apply induction on N. Suppose that we already know that $f_{N-1} = f|_{Q_{N-1}}$. Take an $a \in A \cap Q_{N-1}' \cap Q_N' \neq \varnothing$. Then $f(a,\cdot), f_N(a,\cdot) \in \mathcal{O}(Q_N'')$ and they coincide on $Q_{N-1}'' \cap Q_N''$. Consequently, $f = f_N$ on $(A \cap Q_{N-1}' \cap Q_N') \times Q_N''$. Now take a $b \in Q_N''$. Then $f(\cdot, b), f_N(\cdot, b) \in \mathcal{O}(Q_N')$ and they coincide on $A \cap Q_{N-1}' \cap Q_N'$. Since A is dense, we conclude that they coincide on Q_N'.

Thus, we may assume that X is a domain in $\mathbb{C}^2$, $\mathbb{C} \times V \subset X$, $\varphi = \mathrm{id}$, and $\hat{f} = f$ on X. Take an $a \in A$ and suppose that X_a is the $\hat{f}(a,\cdot)$-region of existence. Since $f(a,\cdot) \in \mathcal{O}(\mathbb{C})$, we conclude that $X_a = \mathbb{C}$. Consequently, for every $R > 0$ there exists an $r > 0$ such that $\mathbb{D}(a, r) \times \mathbb{D}(R) \subset X$. Hence $f \in \mathcal{O}(\mathbb{D}(a, r) \times \mathbb{D}(R))$, which implies (Proposition 1.1.10) that $f \in \mathcal{O}(\mathbb{C} \times \mathbb{D}(R))$. Finally, $f \in \mathcal{O}(\mathbb{C} \times \mathbb{C})$; a contradiction.

The following consequences of Theorem 9.1.2 will be used in Chapter 10.

Proposition 9.1.4. *Let $(D, \pi_D) \in \mathfrak{R}_c(\mathbb{C}^k)$, $(G, \pi_G) \in \mathfrak{R}_c(\mathbb{C}^\ell)$. Let $\Omega \subset D \times G$ be a Riemann region of holomorphy, and let $M \subset \Omega$ be a relatively closed pluripolar set that is singular with respect to a family $\mathcal{S} \subset \mathcal{O}(\Omega \setminus M)$. Then there exists a pluripolar set $P \subset D$ such that for any $a \in \mathrm{pr}_D(\Omega) \setminus P$, the fiber $M_{(a,\cdot)}$ is singular with respect to the family $\mathcal{S}_a := \{f(a,\cdot) : f \in \mathcal{S}\} \subset \mathcal{O}(\Omega_{(a,\cdot)} \setminus M_{(a,\cdot)})$.*

Proof. First assume that $(D, \pi_D) = (\mathbb{C}^n, \mathrm{id})$. Observe that $\Omega \setminus M$ is a region of holomorphy with respect to the family $\mathcal{S}_0 := \mathcal{S} \cup \mathcal{O}(\Omega)$. Moreover, it is easily seen that $\Omega_a = \Omega_{(a,\cdot)}$, $a \in \mathbb{C}^k$ (where Ω_a is taken in the sense of Theorem 9.1.2 and $\Omega_{(a,\cdot)} = \{w \in G : (a, w) \in \Omega\}$). By Theorem 9.1.2, there exists a pluripolar set $P \subset \mathbb{C}^k$ such that for any $a \in \mathrm{pr}_{\mathbb{C}^k}(\Omega) \setminus P$, the fiber $\Omega_{(a,\cdot)} \setminus M_{(a,\cdot)}$ is a region of holomorphy with respect to the family $(\mathcal{S}_0)_a$. In particular, for any $a \in \mathrm{pr}_{\mathbb{C}^k}(\Omega) \setminus P$, the fiber $M_{(a,\cdot)}$ is singular with respect to $\mathcal{S}_a$.

In the general case write $D = \bigcup_{j=1}^{\infty} D_j$, where each $D_j \subset D$ is a univalent pseudoconvex domain. Let $\Omega_j := \Omega \cap (D_j \times G)$, $\mathcal{S}_j := \mathcal{S}|_{\Omega_j}$, $j \in \mathbb{N}$. By the first part of the proof, for each j there exists a pluripolar set $P_j \subset D_j$ such that for any $a \in \mathrm{pr}_D(\Omega_j) \setminus P_j$, the fiber $M_{(a,\cdot)} \cap (\Omega_j)_{(a,\cdot)}$ is singular with respect to the family $\mathcal{S}_a|_{(\Omega_j)_{(a,\cdot)}}$. It remains to put $P := \bigcup_{j=1}^{\infty} P_j$. □

Lemma 9.1.5. *Let $D \subset \mathbb{C}^k$ be a domain of holomorphy, let $(G, \pi) \in \mathfrak{R}_c(\mathbb{C}^\ell)$ be a Riemann domain of holomorphy, let G_0 be a subdomain of G, and let $A \subset D$. Assume that for every $a \in A$ we are given a relatively closed pluripolar set $M(a) \subset G$. Let*

$$\mathcal{S} := \{f \in \mathcal{O}(D \times G_0) : \forall_{a \in A}\ \exists_{\hat{f}_a \in \mathcal{O}(G \setminus M(a))} :\ \hat{f}_a = f(a, \cdot) \text{ on } G_0\}.$$

Assume that for every $a \in A$ the set $M(a)$ is singular with respect to the family $\widehat{\mathcal{S}}_a := \{\hat{f}_a : f \in \mathcal{S}\}$. Then there exists a pluripolar set $P \subset A$ such that if we put $A_0 := A \setminus P$, then the set

$$M(A_0) := \bigcup_{a \in A_0} \{a\} \times M(a)$$

is relatively closed in $A_0 \times G$.

Proof. Put $G(a) := G \setminus M(a)$, $a \in A$. We have to prove that there exists a pluripolar set $P \subset A$ such that the set

$$G(A_0) := (A_0 \times G) \setminus M(A_0) = \bigcup_{a \in A_0} \{a\} \times G(a)$$

is open in $A_0 \times G$. In particular, it would be enough to prove that $G(A_0) = (A_0 \times G) \cap \psi(Y)$, where $Y \subset X$ is open and $\psi : (X, p) \to (D \times G, \mathrm{id} \times \pi)$ is a morphism.

First observe that $(G(a), \pi)$ is an $\widehat{\mathcal{S}}_a$-domain of holomorphy for every $a \in A$ (because $M(a)$ is singular with respect to $\widehat{\mathcal{S}}_a$, and (G, π) is a domain of holomorphy).

Let $\varphi : (D \times G_0, \mathrm{id} \times \pi) \to (X, p)$ be the maximal $\mathcal{S}$-extension. Observe that if $p = (u, v) : X \to \mathbb{C}^k \times \mathbb{C}^\ell$, then $D = (u \circ \varphi)(D \times G_0) \subset u(X)$. Since (X, p) is an $\mathcal{S}^\varphi$-domain of holomorphy, Theorem 9.1.2 implies that there exists a pluripolar set $P \subset A$ such that (X_a, p_a) is an $(\mathcal{S}^\varphi)_a$-region of holomorphy for every $a \in A \setminus P =: A_0$ (we keep notation from Theorem 9.1.2).

Let $a \in D$. Observe that $\varphi(a, \cdot) : (G_0, \pi) \to (X_a, p_a)$ is a morphism. Let X_a^o be the connected component of X_a that contains $\varphi(\{a\} \times G_0)$. Then

$$\varphi(a, \cdot) : (G_0, \pi) \to (X_a^o, p_a)$$

is an $\mathcal{S}_a$-extension with $\mathcal{S}_a := \{f(a, \cdot) : f \in \mathcal{S}\}$ and $(\mathcal{S}_a)^{\varphi(a,\cdot)} = (\mathcal{S}^\varphi)_a|_{X_a^o}$.

Consequently, for every $a \in A_0$ the Riemann domain $(G(a), \pi)$ is isomorphic with (X_a^o, p_a); let $\sigma_a : (X_a^o, p_a) \to (G(a), \pi)$ be an isomorphism such that $\sigma_a \circ \varphi(a, \cdot) = \mathrm{id}$ (notice that σ_a is uniquely determined).

Since $\mathcal{O}(D \times G)|_{D \times G_0} \subset \mathcal{S}$ and $(D \times G, \mathrm{id} \times \pi)$ is the Riemann domain of holomorphy, the lifting theorem ([Jar-Pfl 2000], Proposition 1.9.2) implies that there exists a lifting of the inclusion $(D \times G_0, \mathrm{id} \times \pi) \to (D \times G, \mathrm{id} \times \pi)$, i.e. a uniquely determined morphism $\psi = (\psi_D, \psi_G) : (X, p) \to (D \times G, \mathrm{id} \times \pi)$ such that $\psi \circ \varphi = \mathrm{id}$.

Then $\psi_G : (X_a, p_a) \to (G, \pi)$ is a morphism such that $\psi_G \circ \varphi(a, \cdot) = \mathrm{id}$. Consequently, $\psi_G = \sigma_a$ on X_a^o for every $a \in A_0$.

Finally,

$$\begin{aligned} G(A_0) &= \bigcup_{a \in A_0} \{a\} \times G(a) = \bigcup_{a \in A_0} \{a\} \times \sigma_a(X_a^o) \\ &= \bigcup_{a \in A_0} \{a\} \times \psi_G(X_a^o) = \psi\Big(\bigcup_{a \in A_0} X_a^o\Big) = \psi(Y) \cap (A_0 \times G), \end{aligned}$$

where $Y := \bigcup_{a \in D} X_a^o$. It remains to observe that Y is open. □

9.2 Chirka–Sadullaev theorem

> §§ 2.3, 2.4, 3.2, 9.1.

9.2.1 The Gonchar class R^o

To be able to present a result of Chirka–Sadullaev (see Theorem 9.2.24) which will be the basis of the following discussion we need some results from classical complex analysis of one complex variable mainly due to G. A. Gonchar (see [Gon 1972] and [Gon 1974]). *This subsection is based on the classical potential theory – all needed tools may be found in* [Ran 1995], *Chapter* 5.

Definition 9.2.1. Let $D \subset \mathbb{C}$ be a domain and let $f \in \mathcal{O}(D)$. We say that f belongs to the class $\boldsymbol{R}^0$ if there is a disc $\mathbb{D}(a, r) \subset\subset D$ such that

$$\lim_{k \to \infty} (\rho_{k, \mathbb{D}(a,r)}(f))^{1/k} = 0,$$

where

$$\rho_{k, \mathbb{D}(a,r)}(f) := \inf\{\|f - p/q\|_{\mathbb{D}(a,r)} : p, q \in \mathcal{P}_k(\mathbb{C})\}.$$

Note that $f \in \boldsymbol{R}^0$, if and only if, there exist a disc $\mathbb{D}(a, r) \subset\subset D$ and a sequence $(p_k/q_k)_k$, $p_k, q_k \in \mathcal{P}_k(\mathbb{C})$, such that $\|f - p_k/q_k\|_{\mathbb{D}(a,r)}^{1/k} \underset{k \to \infty}{\longrightarrow} 0$. We will see that in fact the definition is independent of the disc $\mathbb{D}(a, r)$.

Proposition 9.2.2. *Let $D \subset \mathbb{C}$ be a domain with $\overline{\mathbb{D}}(a,r) \cup \overline{\mathbb{D}}(b,s) \subset D$, $f \in \mathcal{O}(D)$. Assume that $\|f - p_k/q_k\|_{\mathbb{D}(a,r)}^{1/k} \to 0$, where $p_k, q_k \in \mathcal{P}_k(\mathbb{C})$, $k \in \mathbb{N}$. Then there exist polynomials $\tilde{p}_k, \tilde{q}_k \in \mathcal{P}_k(\mathbb{C})$, $k \in \mathbb{N}$, such that*

$$\|f - \tilde{p}_k/\tilde{q}_k\|_{\mathbb{D}(a,r)\cup\mathbb{D}(b,s)}^{1/k} \underset{k\to\infty}{\longrightarrow} 0.$$

In particular, if $(\rho_{k,\mathbb{D}(a,r)}(f))^{1/k} \to 0$, then $(\rho_{k,\mathbb{D}(b,s)}(f))^{1/k} \to 0$.

Proof. We may assume that $a = 0$. Let $r' > r$ be such that $\mathbb{D}(r') \subset\subset D$. Choose smooth domains $D_2 \subset\subset D_1 \subset\subset D$ such that $\overline{\mathbb{D}}(r') \cup \overline{\mathbb{D}}(b,s) \subset D_2$. Recall that such domains are regular with respect to the Dirichlet problem.

By assumption there are polynomials $p_k, q_k \in \mathcal{P}_k(\mathbb{C})$ with

$$\varepsilon_k^{1/k} := \|f - p_k/q_k\|_{\mathbb{D}(r)}^{1/k} \to 0.$$

We may assume that p_k, q_k do not have common zeros and that $\varepsilon_k < 1$ for $k \geq k_1$. Put $\mathfrak{r}_k := p_k/q_k$. Then $\mathfrak{r}_k$ is holomorphic in a neighborhood of $\overline{\mathbb{D}}(r)$; in particular, q_k has no zeros on this neighborhood.

Fix a $k \geq k_1$. Denote by $\beta_{k,1}, \dots, \beta_{k,\varkappa_k}$ ($\varkappa_k \leq k$) all the zeros of q_k (counted with multiplicities) which lie inside of $\mathbb{D}(2R)$ with $R := \max\{1, \sup_{z\in D_1} |z|\}$. Put $\omega_k(z) := \prod_{j=1}^{\varkappa_k}(z - \beta_{k,j})$. Then $\mathfrak{s}_k := \mathfrak{r}_k\omega_k$ is a rational function of degree less than or equal to k which has no poles inside of $\mathbb{D}(2R)$. Then

$$\|\mathfrak{s}_k\|_{\mathbb{D}(r)} \leq \|\mathfrak{r}_k\|_{\mathbb{D}(r)}\|\omega_k\|_{\mathbb{D}(r)} \leq (\|f\|_{\mathbb{D}(r)} + 1)^k(3R)^k =: A_1^k.$$

Moreover, there is a $t > 1$, independent of k, such that $\|\mathfrak{s}_k\|_{\mathbb{D}(R)} \leq t^k\|\mathfrak{s}_k\|_{\mathbb{D}(r)}$. Indeed, let $G := \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}(r)$. If $\beta \in G$, then $g_G(\beta, \cdot)$ means the classical Green function with logarithmic pole β, i.e. $g_G(\beta, \cdot)$ is harmonic on $G \setminus \{\beta\}$, bounded outside of a neighborhood of β, $g_G(\beta, z) = -\log|z - \beta| + O(1)$, and $g_G(\beta, z) \underset{z\to\zeta}{\longrightarrow} 0$ for all $\zeta \in \partial\mathbb{D}(r)$. In particular, we have $g_G(\beta, z) = \log\frac{|\bar{\beta}z - r^2|}{r|\beta - z|}$.

Denote by $\alpha_{k,1}, \dots, \alpha_{k,\ell_k}$ all poles of $\mathfrak{s}_k$. Recall that they lie outside of $\mathbb{D}(2R)$. Then

$$u_k := \log\frac{|\mathfrak{s}_k|}{\|\mathfrak{s}_k\|_{\mathbb{D}(r)}} - \sum_{j=1}^{\ell_k} g_G(\alpha_{k,j}, \cdot)$$

is subharmonic on G with non-positive boundary values. By the maximum principle it follows that

$$|\mathfrak{s}_k(z)| \leq \|\mathfrak{s}_k\|_{\mathbb{D}(r)} \exp\Big(\sum_{j=1}^{\ell_k} g_G(\alpha_{k,j}, z)\Big), \quad z \in G.$$

Using the concrete formulas for the involved Green functions, we receive

$$|\mathfrak{s}_k(z)| \leq \|\mathfrak{s}_k\|_{\mathbb{D}(r)}\Big(\frac{3R}{r}\Big)^k, \quad z \in \mathbb{D}(R) \setminus \mathbb{D}(r).$$

It remains to put $t := 3R/r$.

Put $F_k := (f - \mathfrak{r}_k)\omega_k = f\omega_k - \mathfrak{s}_k$. Note that $F_k \in \mathcal{O}(D_1)$. Then, from the considerations before, the following inequalities become true:

$$\|\mathfrak{s}_k\|_{\bar{D}_1} \le t^k \|\mathfrak{s}_k\|_{\mathbb{D}(r)} \le (tA_1)^k =: A_2^k,$$
$$\|F_k\|_{\mathbb{D}(r)} \le \varepsilon_k (2R + r)^k =: \varepsilon_k A_3^k,$$
$$\|F_k\|_{\bar{D}_1} \le \|f\|_{\bar{D}_1} \|\omega_k\|_{\mathbb{D}(2R)} + \|\mathfrak{s}_k\|_{\bar{D}_1} \le (1 + \|f\|_{\bar{D}_1})^k (1 + 3R)^k + A_2^k \le A_4^k.$$

Applying the Two Constants Theorem with respect to $D_1 \setminus \bar{\mathbb{D}}(r)$ it follows that

$$|F_k(z)| \le (\varepsilon_k A_3^k)^{\gamma(z)} A_4^{k(1-\gamma(z))} \le \varepsilon_k^{\gamma(z)} A_5^k, \quad z \in D_1,$$

where $\gamma(z) := 1$, $|z| \le r$, and γ is the solution of the Dirichlet problem on $D_1 \setminus \bar{\mathbb{D}}(r)$ with the inner boundary values identically one and the outer ones identically zero. Recall that γ is positive on $\bar{D}_2$. Put $c := \inf_{z \in \bar{D}_2} \gamma(z) > 0$. Then we have

$$|f(z) - \mathfrak{r}_k(z)| \le \frac{\varepsilon_k^c A_5^k}{|\omega_k(z)|}, \quad z \in \bar{D}_2 \setminus \{\beta_{k,j} : j = 1, \dots, \varkappa_k\}.$$

It remains to estimate ω_k from below. Fix $s' > s$ with $\bar{\mathbb{D}}(b, s') \subset D_2$. Let $d := \min\{\operatorname{dist}(\bar{\mathbb{D}}(r'), \partial D_2), \operatorname{dist}(\bar{\mathbb{D}}(b, s'), \partial D_2), 1\}$. By the Cartan–Boutroux lemma (see [Ber-Gay 1991], Lemma 4.5.13), there is a finite number of discs $K_1, \dots, K_\sigma$ with $\sum_{j=1}^{\sigma} \operatorname{diam} K_j \le d$ such that

$$|\omega_k(z)| > \left(\frac{d}{2e}\right)^{\varkappa_k} \ge \left(\frac{d}{12}\right)^k, \quad z \in \mathbb{C} \setminus \bigcup_{j=1}^{\sigma} K_j.$$

Put $G'_k := \mathbb{D}(r') \cup \bigcup K_j$ and $G''_k := \mathbb{D}(b, s') \cup \bigcup K_j$, where the union is taken over those K_j's which intersect $\mathbb{D}(r')$, respectively $\mathbb{D}(b, s')$. Then $G_k := G'_k \cup G''_k \subset\subset D_2$. Therefore,

$$|f(z) - \mathfrak{r}_k(z)| \le \varepsilon_k^c A_5^k (12/d)^k =: \varepsilon_k^c A_6^k, \quad z \in \partial G'_k \cup \partial G''_k.$$

Note that $\mathfrak{r}_k$ has no poles on ∂G_k.

Now let $\alpha_{k,1}, \dots, \alpha_{k,\mu(k)}$ denote all poles of $\mathfrak{r}_k$ which are inside of G_k. Then $\mathfrak{r}_k$ may be written as its Laurent expansion around $\alpha_{k,m}$ i.e. $\mathfrak{r}_k = h_{k,m} + g_{k,m}$, where $h_{k,m}$ means the principal part. Set $\mathfrak{r}_k^* := \mathfrak{r}_k - \sum_{j=1}^{\mu(k)} h_{k,j}$. Then $\mathfrak{r}_k^*$ is holomorphic in G_k. Moreover, $\mathfrak{r}_k - \mathfrak{r}_k^*$ is holomorphic outside of G_k and vanishes at ∞. Hence,

$$f(z) - \mathfrak{r}_k^*(z) = \frac{1}{2\pi i} \int_{\partial G'_k} \frac{f(\zeta) - \mathfrak{r}_k^*(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{\partial G'_k} \frac{f(\zeta) - \mathfrak{r}_k(\zeta)}{\zeta - z} d\zeta, \quad z \in \mathbb{D}(r),$$

and

$$f(z) - \mathfrak{r}_k^*(z) = \frac{1}{2\pi i}\int_{\partial G_k''} \frac{f(\zeta) - \mathfrak{r}_k^*(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i}\int_{\partial G_k''} \frac{f(\zeta) - \mathfrak{r}_k(\zeta)}{\zeta - z} d\zeta, \quad z \in \mathbb{D}(b, s).$$

Taking the estimates from above one gets

$$\|f - \mathfrak{r}_k^*\|_{\mathbb{D}(r)\cup\mathbb{D}(b,s)} \le \frac{\varepsilon_k^c A_6^k \max\{2\pi s' + \pi d, 2\pi r' + \pi d\}}{2\pi \min\{s' - s, r' - r\}} \le \varepsilon_k^c A_7^k.$$

Since the constants c and A_7 are independent of k we have $\|f - \mathfrak{r}_k^*\|_{\mathbb{D}(r)\cup\mathbb{D}(b,s)}^{1/k} \underset{k\to\infty}{\longrightarrow} 0$ with $\mathfrak{r}_k^* := \mathfrak{r}_k$ if $k < k_1$ □

The main result of A. Gonchar compares the property described in the former definition and the single-valuedness of the domain of existence W_f of f. Recall that, in general, W_f may be a Riemann domain over $\mathbb{C}$. For the proof of this result we need the following max-min comparison result for a rational function.

Lemma 9.2.3 ([Gon 1969]). *Let K_1, K_2 be disjoint closed discs in $\mathbb{C}$. Then there is a number $\rho = \rho(K_1, K_2) > 1$ such that*

$$\min\{|\mathfrak{r}_k(z)| : z \in K_2\} \le \rho^k \|\mathfrak{r}_k\|_{K_1}$$

for each rational function $\mathfrak{r}_k = p_k/q_k$ with $\deg \mathfrak{r}_k := \max\{\deg q_k, \deg p_k\} \le k$.

Other versions of this estimate may be found in [Gon 1967], [Gon 1968], and [Gon 1969].

Proof. We may assume that $K_1 = \overline{\mathbb{D}}(r_1)$, $K_2 = \overline{\mathbb{D}}(a_2, r_2)$, and $a_2 > 0$. Choose $p \in (0, r_1)$, $q \in (a_2 - r_2, a_2)$ such that $pq = r_1^2$, $(a_2 - p)(a_2 - q) = r_2^2$. Then $\partial K_j = \{z \in \mathbb{C} : |\frac{z-p}{z-q}| = \lambda_j\}$ with $\lambda_1 := p/r_1$ and $\lambda_2 := (a_2 - p)/r_2$ (Exercise). Put $h(z) := \frac{1}{\lambda_1}\frac{z-p}{z-q}$, $\rho := \lambda_2/\lambda_1 > 1$. Then $h(K_1) = \overline{\mathbb{D}}$ and $h(K_2) = \overline{\mathbb{C}} \setminus \mathbb{D}(\rho)$.

Let $\mathfrak{r}_k$ be a rational function of degree equal to k with $\|\mathfrak{r}_k\|_{K_1} = 1$. Put $\widetilde{\mathfrak{r}}_k := \mathfrak{r}_k \circ h^{-1}$. Then $\widetilde{\mathfrak{r}}_k$ is a rational function with $\|\widetilde{\mathfrak{r}}_k\|_{\mathbb{D}} = 1$, $\deg \widetilde{\mathfrak{r}}_k = k$. We have to show that $\min_{\overline{\mathbb{C}}\setminus\mathbb{D}(\rho)} |\widetilde{\mathfrak{r}}_k| \le \rho^k$. We may assume that $\widetilde{\mathfrak{r}}_k$ has no zeros on $\overline{\mathbb{C}} \setminus \mathbb{D}(\rho)$, i.e.

$$\widetilde{\mathfrak{r}}_k(z) = A\frac{\prod_{j=1}^{\ell}(z - \alpha_j)}{\prod_{j=1}^{m}(z - \beta_j)}, \quad A \ne 0,$$

where $k = \max\{\ell, m\}$ (the α_j's and the β_j's are written as often as their multiplicities and it is assumed that $\{\alpha_j : j = 1, \dots, \ell\} \cap \{\beta_j : j = 1, \dots, m\} = \varnothing$). Note that

$|\beta_j| > 1$, $j = 1, \dots, m$. Moreover, we have $k = \ell \geq m$, since ∞ is by assumption not a zero of $\widetilde{\mathfrak{r}}_k$. Put $s := \ell - m$. Then $\widetilde{\mathfrak{r}}_k = z^s \widehat{\mathfrak{r}}_k$, where

$$\widehat{\mathfrak{r}}_k = A \frac{\prod_{j=1}^{\ell}(z - \alpha_j)}{z^s \prod_{j=1}^{m}(z - \beta_j)}$$

is a rational function that is holomorphic at ∞. Observe that $\deg \widehat{\mathfrak{r}}_k = k = \ell$. We have $\min_{\overline{\mathbb{C}} \setminus \mathbb{D}(\rho)} |\widetilde{\mathfrak{r}}_k| = \lambda^s \min_{\overline{\mathbb{C}} \setminus \mathbb{D}(\rho)} |\widehat{\mathfrak{r}}_k|$. Thus the problem reduces to estimate $\widehat{\mathfrak{r}}_k$.

Put $\mathfrak{s}_m(z) := \prod_{j=1}^{m} \frac{1 - \bar{\beta}_j z}{z - \beta_j}$. Then $\widehat{\mathfrak{r}}_k / \mathfrak{s}_m$ is holomorphic on $\overline{\mathbb{C}} \setminus \mathbb{D}(\rho)$ and we have $\frac{|\widehat{\mathfrak{r}}_k(z)|}{|\mathfrak{s}_m(z)|} \leq 1$, $z \in \partial\mathbb{D}$. Therefore, $|\widehat{\mathfrak{r}}_k| \leq |\mathfrak{s}_m|$ on $\overline{\mathbb{C}} \setminus \mathbb{D}(\rho)$. To finish the proof we only need to verify that $\min_{\overline{\mathbb{C}} \setminus \mathbb{D}(\rho)} |\mathfrak{s}_m| \leq \rho^m$. Put $\sigma := 1/\rho$, $a_j := 1/\bar{\beta}_j \in \mathbb{D}_*$, $j = 1, \dots, m$, and $\widehat{\mathfrak{s}}_m(z) := \prod_{j=1}^{m} \frac{z - a_j}{1 - \bar{a}_j z}$. Then

$$\min_{\overline{\mathbb{C}} \setminus \mathbb{D}(\rho)} |\mathfrak{s}_m(z)| = \frac{1}{\max_{\partial\mathbb{D}(\sigma)} |\widehat{\mathfrak{s}}_m(z)|}.$$

It remains to show that $\max_{\partial\mathbb{D}(\sigma)} |\widehat{\mathfrak{s}}_m| \geq \sigma^m$. Note that it is enough to prove this inequality for those σ with $\sigma \neq |a_j|$, $j = 1, \dots, m$.

Using the mean value equation for harmonic functions, we have for an $a \in \mathbb{D}_*$

$$\frac{1}{2\pi} \int_0^{2\pi} \log |\sigma e^{it} - a| dt = \begin{cases} \log |a| & \text{if } |a| > \sigma, \\ \log \sigma & \text{if } |a| < \sigma. \end{cases}$$

Therefore,

$$\frac{1}{2\pi} \sum_{j=1}^{m} \int_0^{2\pi} \left(\log |\sigma e^{it} - a_j| - \log |1 - \bar{a}_j \sigma e^{it}| \right) dt$$
$$= \frac{1}{2\pi} \sum_{j=1}^{m} \int_0^{2\pi} \log |\sigma e^{it} - a_j| dt = \sum_{j:\, |a_j| < \sigma} \log \sigma + \sum_{j:\, |a_j| > \sigma} \log |a_j| \geq m \log \sigma.$$

Thus, $\max_{\partial\mathbb{D}(\sigma)} |\widehat{\mathfrak{s}}_m| \geq \sigma^m$. □

Remark 9.2.4. The number ρ in this lemma is nothing other than the so-called *modulus of the doubly connected domain* $\overline{\mathbb{C}} \setminus (K_1 \cup K_2)$.

Theorem 9.2.5 ([Gon 1972]). *Let $D \subset \mathbb{C}$ be a domain with $0 \in D$ and let $f \in \mathcal{O}(D)$. Assume that $f \in \boldsymbol{R}^0$. Then the domain of existence W_f of f is univalent, i.e. W_f is a domain in $\mathbb{C}$.*

Proof. It remains to show the following statement: if D_1, D_2 be two domains with $\mathbb{D}(r) \subset D_1 \cap D_2$ and $f_j \in \mathcal{O}(D_j)$ with $f = f_1 = f_2$ on $\mathbb{D}(r)$, then $f_1 = f_2$ on $D_1 \cap D_2$.

Take rational approximants $\mathfrak{r}_k = p_k/q_k$ of f, i.e. $\|f - \mathfrak{r}_k\|_{\mathbb{D}(r)}^{1/k} \to 0$ as $k \to \infty$. Fix an $a \in D_1 \cap D_2$ and a disc $\mathbb{D}(a, s) \subset D_1 \cap D_2$. Then, using Proposition 9.2.2, there are rational approximants $\mathfrak{r}_{j,k} = p_{j,k}/q_{j,k}$ with $\|f_j - \mathfrak{r}_{j,k}\|_{\mathbb{D}(a,s)\cup\mathbb{D}(r)}^{1/k} \to 0$, $j = 1, 2$. In particular, $\mathfrak{r}_{j,k}|_{\mathbb{D}(a,s)} \to f_j$ uniformly on $\mathbb{D}(a, s)$. Take now disjoint discs $\overline{\mathbb{D}}(r') \subset \mathbb{D}(r)$ and $\overline{\mathbb{D}}(b, s') \subset \mathbb{D}(a, s)$.

Put $\widetilde{\mathfrak{r}}_k := \mathfrak{r}_{1,k} - \mathfrak{r}_{2,k}$; $\widetilde{\mathfrak{r}}_k$ is then a rational function of degree less than or equal to $2k$ converging to $f_1 - f_2$ uniformly on $\mathbb{D}(a, s)$. Then

$$\|\widetilde{\mathfrak{r}}_k\|_{\mathbb{D}(r')} \leq \|f - \mathfrak{r}_{1,k}\|_{\mathbb{D}(r)} + \|f - \mathfrak{r}_{2,k}\|_{\mathbb{D}(r)} \leq 2\varepsilon_k,$$

where $\varepsilon_k := \max\{\|f - \mathfrak{r}_{j,k}\|_{\mathbb{D}(r)}^{1/k}, j = 1, 2\}$. Applying Lemma 9.2.3 we see that

$$\min\{|\widetilde{\mathfrak{r}}_k(z)| : z \in \mathbb{D}(b, s')\} \leq 2(\rho\varepsilon_k)^k \underset{k\to\infty}{\longrightarrow} 0.$$

Therefore, $f_1 - f_2$ has at least one zero on $\overline{\mathbb{D}}(b, s')$. Since s' could be arbitrarily small, it follows that $(f_1 - f_2)(b) = 0$ and, since $a, b \neq a$ were arbitrary, both functions coincide in a and so on $D_1 \cap D_2$. □

Remark 9.2.6. (a) To conclude that W_f is univalent it is sufficient to know that $\liminf_{k\to+\infty}(\rho_{k,\mathbb{D}(r')}(f))^{1/k} = 0$ (see [Gon 1972]).

(b) A similar result remains true in higher dimensions; see [Gon 1974].

A large class of functions belonging to $\boldsymbol{R}^0$ is given in the next theorem.

Theorem 9.2.7. *If $S \subset \mathbb{C}$ is closed and polar, then $\mathcal{O}(\mathbb{C} \setminus S) \subset \boldsymbol{R}^0$.*

Theorem 9.2.7 is already mentioned in [Gon 1972] saying that it is a consequence of results by J. L. Walsh. Here we present a complete proof which is based on a paper of J. Karlsson (see [Kar 1976]).

First let us repeat some simple facts from interpolation theory. Let $f \in \mathcal{O}(D)$ and fix $2n + 1$ pairwise different points $a_1, \dots, a_{2n+1} \in D$. Then there are polynomials $P_n, Q_n \in \mathcal{P}_n(\mathbb{C})$, Q_n normalized, such that $(fQ_n - P_n)(a_j) = 0$, $j = 1, \dots, 2n+1$. In fact, here we have $2n + 1$ equations with $2n$ unknowns which can be solved, and so P_n and Q_n exist. Observe that the rational function $\mathfrak{r}_n = P_n/Q_n$ is uniquely determined (Exercise). We say that P_n/Q_n is *the rational interpolant for f and the points a_j, $j = 1, \dots, 2n + 1$.*

As preparation for Theorem 9.2.7 we prove an approximation result up to small exceptional sets, small in the sense of the *logarithmic capacity* cap. To be precise we have the following statement.

Proposition 9.2.8. *Let $S \subset \mathbb{C}$ be a closed polar set with $0 \notin S$ and $f \in \mathcal{O}(\mathbb{C} \setminus S)$. For each n, fix $2n + 1$ pairwise different points $a_{n,j} \in \mathbb{D}(\rho)$, $j = 1, \dots, 2n + 1$, where $\mathbb{D}(3\rho) \cap S = \varnothing$. Denote by P_n/Q_n the rational interpolants for f and the $a_{n,j}$'s.*

Then, for any $\varepsilon > 0$, $\delta > 0$, *and* $R > 0$, *there exists an* n_0 *such that for each* $n \in \mathbb{N}$, $n \geq n_0$, *there is a set* $A_n \subset \overline{\mathbb{D}}(R)$ *with* $\operatorname{cap}(A_n) < \delta$ *such that*

$$|f(z) - (P_n/Q_n)(z)| \leq \varepsilon^n, \quad z \in \overline{\mathbb{D}}(R) \setminus A_n.$$

Proof. Step 1^0: In a first step we assume that:

- S is compact and polar (not necessarily $0 \notin S$),
- f is bounded at infinity,
- all points $a_{n,j}$ belong to an arbitrary compact set E which is disjoint to S (e.g. $E = \overline{\mathbb{D}}(\rho)$ as in the proposition).

Now fix ε, δ, and R as in the proposition. Put $2\delta_1 := \min\{1, \frac{\delta^2}{2R}\}$ and $\gamma := \operatorname{dist}(E, S)$. Moreover, take a positive $\eta < \delta_1$ sufficiently small (the precise choice of η will be made later). Then, since $\operatorname{cap}(S) = 0$, there is a normalized (monic) polynomial $h(z) = z^k +$lower degree terms ($k = k(\eta)$) such that

$$S \subset \{z \in \mathbb{C} : |h(z)| \leq \eta^k\} =: D_\eta = D$$

(see [Ran 1995], Chapter 5.5). Let $n > k$ and choose $\ell = \ell(k, n) \in \mathbb{N}$ such that $n - k < k\ell \leq n$. Put $\omega_n(z) := \prod_{j=1}^{2n+1}(z - a_{n,j})$. Then

$$g := h^\ell \frac{fQ_n - P_n}{\omega_n} \in \mathcal{O}(\mathbb{C} \setminus S).$$

Note that $|g(z)| \underset{|z|\to\infty}{\longrightarrow} 0$. Chose an integration cycle $\Gamma \subset \operatorname{int} D$ such that

$$\operatorname{dist}(\Gamma, S) < \gamma/2, \quad \operatorname{ind}_\Gamma(a) = -1,\ a \in S, \text{ and } \operatorname{ind}_\Gamma(a) = 0,\ a \notin D,$$

where $\operatorname{ind}_\gamma$ denotes the winding number. Then the Cauchy integral formula gives

$$g(z) = \frac{1}{2\pi i} \int_\Gamma \frac{h^\ell(\zeta)(fQ_n - P_n)(\zeta)}{\omega_n(\zeta)(\zeta - z)} d\zeta = \frac{1}{2\pi} \int_\Gamma \frac{(h^\ell fQ_n)(\zeta)}{\omega_n(\zeta)(\zeta - z)} d\zeta, \quad z \in \mathbb{C} \setminus D.$$

The second equation is true because the winding number is zero for points outside of D.

Therefore,

$$|g(z)| \leq M\eta^{k\ell} \frac{\|Q_n\|_\Gamma}{\inf_\Gamma |\omega_n(\zeta)|}, \quad z \in \mathbb{C} \setminus D,$$

where

$$M := 1 + \frac{l(\Gamma)\|f\|_\Gamma}{2\pi \operatorname{dist}(\Gamma, \mathbb{C} \setminus D)}$$

and $l(\Gamma)$ equals the length of Γ. So we have for $z \notin D$, $Q_n(z) \neq 0$:

$$\left| h^\ell(z)\left(f - \frac{P_n}{Q_n}\right)(z) \right| \leq M\eta^{k\ell} \frac{\|Q_n\|_\Gamma}{|Q_n(z)|} \frac{|\omega_n(z)|}{\inf_\Gamma |\omega_n(\zeta)|}.$$

We may assume that $R > 1$, $\Gamma \subset \mathbb{D}(R)$, and $E \subset \mathbb{D}(r)$ for a suitable positive r. Then

$$\frac{|\omega_n(z)|}{\inf_\Gamma |\omega_n(\zeta)|} \le \left(\frac{2(R+r)}{\gamma}\right)^{2n+1} =: M_1^{2n+1}, \quad z \in \overline{\mathbb{D}}(R) \setminus D,\ Q_n(z) \neq 0.$$

Let $Q_n(z) = \prod_{j=1}^m (z - b_{n,j})$, $m \le n$. Assume that $b_{n,j} \in \overline{\mathbb{D}}(2R)$, $j = 1, \dots m^*$, and $|b_{n,j}| > 2R$ for the remaining zeros. Then

$$\frac{|\zeta - b_{n,j}|}{|z - b_{n,j}|} \le \frac{3R}{|z - b_{n,j}|}, \quad j = 1, \dots, m^*.$$

If $j > m^*$, then

$$\frac{|\zeta - b_{n,j}|}{|z - b_{n,j}|} \le \frac{|\zeta| + |b_{n,j}|}{|b_{n,j}| - |z|} \le \frac{|\zeta| + 2R}{2R - |z|} \le 3.$$

Hence,

$$\frac{|Q_n(\zeta)|}{|Q_n(z)|} \le (3R)^m / |Q^*(z)|, \quad |z| \le R,\ \zeta \in \Gamma,\ Q_n(z) \neq 0,$$

where $Q_n^*(z) := \prod_{j=1}^{m^*} (z - b_{n,j})$.

What remains is to find a lower estimate for $(h^\ell Q_n^*)(z)$. Put

$$B_n := \{w \in \mathbb{C} : |(h^\ell Q_n^*)(w)| \le \delta_1^{k\ell + m^*}\}.$$

Recall that $h^\ell Q_n^*$ is a normalized polynomial of degree $kl + m^* \le 2n$. Therefore, $\operatorname{cap}(B_n) \le \delta_1$ (see [Ran 1995], Table 5.1). Consequently,

$$\left|\left(f - \frac{P_n}{Q_n}\right)(z)\right| \le \eta^{k\ell} M (3R)^n M_1^{2n+1} \left(\frac{1}{\delta_1}\right)^{2n} < \varepsilon^n,$$
$$z \in \overline{\mathbb{D}}(R) \setminus (B_n \cup D),\ Q_n(z) \neq 0,$$

when η is correctly chosen.

Put $A_n := (B_n \cup D \cup \{b_{n,j} : j = 1, \dots, m\}) \cap \overline{\mathbb{D}}(R)$. Recall that $\operatorname{cap}(D) \le \eta < \delta_1$. Using [Ran 1995], Theorem 5.1.4(a), it follows that

$$\frac{1}{\log \frac{2R}{\operatorname{cap}(A_n)}} \le \frac{1}{\log \frac{2R}{\operatorname{cap}(B_n)}} + \frac{1}{\log \frac{2R}{\operatorname{cap}(D)}} + \sum_{j=1}^m \frac{1}{\log \frac{2R}{\operatorname{cap}(\{b_{n,j}\})}}.$$

Since the last sum in the above estimate vanishes, we have $\operatorname{cap}(A_n) \le (\delta_1 2R)^{1/2} \le \delta$, which finishes the proof of the compact case.

Note that the set A_n may be chosen to be closed.

Step 2^0. Now we assume that

- S is an unbounded closed polar set,
- all other conditions are as in Proposition 9.2.8.

Fix a point a outside of S with $|a| > 2\rho$ and denote by L the map

$$L(z) := (z-a)^{-1}, \quad z \in \mathbb{C} \setminus \{a\}.$$

Put $S' := L(S) \cup \{0\}$. Then S' is compact. Set $g := f \circ L^{-1}$. Obviously, g is a holomorphic function on $\mathbb{C} \setminus S'$. Note that g is bounded at infinity. Finally, let $S_k := S \cap \overline{\mathbb{D}}(k)$, $k \in \mathbb{N}$. Then, using [Ran 1995], Theorem 5.3.1, we conclude that

$$\operatorname{cap}(L(S_k')) \leq \frac{1}{\operatorname{dist}(a,S)^2} \operatorname{cap}(S_k) \leq \frac{1}{\operatorname{dist}(a,S)^2} \operatorname{cap}(S) = 0.$$

Therefore, $L(S)$ is polar and so is S'.

Put $\tilde{a}_{n,j} := L(a_{n,j})$. Observe that all these points are sitting in the compact set $E := L(\mathbb{D}(\rho))$ which is disjoint to S'. Hence we can apply the first step for the data S', $\tilde{a}_{n,j}$, E, and g.

By p_n/q_n we denote the rational interpolants for g and the $\tilde{a}_{n,j}$'s. Put $Q_n(z) := (z-a)^n q_n(L(z))$ and $P_n(z) := (z-a)^n p_n(L(z))$. Then $Q_n, P_n \in \mathcal{P}_n(\mathbb{C})$. Moreover we have

$$(fQ_n - P_n)(a_{n,j}) = (a_{n,j}-a)^n (gq_n - p_n)(\tilde{a}_{n,j}) = 0, \quad j = 1, \dots, 2n+1;$$

i.e. the P_n/Q_n (after normalizing Q_n) are the rational interpolants for f and the $a_{n,j}$'s.

Now fix positive numbers ε, δ, and R. Let $\mathbb{D}(R) \subset \mathbb{D}(a,r)$ for a suitable positive r and choose

- $\tilde{\delta} < \min\{r, \frac{\sqrt{\delta}}{|a|+r}\}$,
- $\delta' < \tilde{\delta}/r^2$,
- $\varepsilon' = \varepsilon$ and $R' > 1/\tilde{\delta}$.

Then, applying the compact case, there is an n_0 such that for all $n \geq n_0$ there exists a closed set $A_n' \subset \overline{\mathbb{D}}(R')$, $\operatorname{cap}(A_n') < \delta'$, such that for all $w \in \overline{\mathbb{D}}(R') \setminus A_n'$ the following inequality is true:

$$\left|\left(g - \frac{p_n}{q_n}\right)(w)\right| \leq \varepsilon'^n.$$

Fix such an n. Then

$$|(f - P_n/Q_n)(z)| \leq \varepsilon^n, \quad L(z) \in \overline{\mathbb{D}}(R') \setminus A_n',\ z \in \overline{\mathbb{D}}(R) \setminus \mathbb{D}(a,\tilde{\delta}).$$

Observe that for these z we have $1/r \leq |L(z)| \leq 1/\tilde{\delta} = R'$.

Put $A_n := \overline{\mathbb{D}}(a,\tilde{\delta}) \cup L^{-1}(B_n)$, where $B_n := A_n' \cap \overline{\mathbb{A}}(1/r, 1/\tilde{\delta})$ is compact. Using [Ran 1995], Theorem 5.3.1, for $L^{-1}\colon B_n \to \mathbb{C}$ we get that

$$\operatorname{cap}(L^{-1}(B_n)) \leq r^2 \operatorname{cap}(B_n) \leq r^2 \delta' < \tilde{\delta}.$$

Recall that $\operatorname{cap}(\overline{\mathbb{D}}(a, \tilde{\delta})) = \tilde{\delta}$. Moreover, note that $\operatorname{diam}(A_n) \le |a| + r$. Then using [Ran 1995], Theorem 5.1.4(a), it follows that

$$\operatorname{cap}(A_n) \le ((|a| + r)\tilde{\delta})^2 < \delta,$$

which finishes the proof. □

Proof of Theorem 9.2.7. We may assume that $0 \notin S$. Fix an $f \in \mathcal{O}(\mathbb{C} \setminus S)$ and $r > 0$ with $S \cap \overline{\mathbb{D}}(r) = \varnothing$. For each $n \in \mathbb{N}$ choose $2n+1$ pairwise different points inside of $\mathbb{D}(r/3)$. Let P_n/Q_n be the associated rational interpolants. Moreover, let $r_1 > r$ such that $\overline{\mathbb{D}}(r_1)$ does not intersect S. Put $\rho := (r_1 - r)/2$, $s := r + \rho$, and $\Gamma := \partial\mathbb{D}(s)$.

Write $Q_n = Q_n' Q_n''$ with polynomials Q_n', Q_n'', where Q_n' contains all linear factors of Q_n whose zeros have absolute value less then $r + \rho/2$. Using power series expansion around the origin we may write $fQ_n'' = p_n + z^{n+1} g_n$, where $p_n \in \mathcal{P}_n(\mathbb{C})$ and $g_n \in \mathcal{O}(\mathbb{D}(r_1))$. Then, using the Cauchy integral formula, we get

$$g_n(z) = \frac{1}{2\pi i}\int_\Gamma \frac{(fQ_n - P_n)(\zeta)}{Q_n'(\zeta)\zeta^{n+1}(\zeta - z)}d\zeta + \frac{1}{2\pi i}\int_\Gamma \frac{(P_n - Q_n' p_n)(\zeta)}{Q_n'(\zeta)\zeta^{n+1}(\zeta - z)}d\zeta, \quad z \in \overline{\mathbb{D}}(r).$$

Note that the integrand in the second integral is holomorphic outside of $\mathbb{D}(r + \rho/2)$ and vanishes at infinity. Therefore, this integral is equal to 0.

Now fix $k \in \mathbb{N}$ and choose $\varepsilon_k > 0$ (sufficiently small). Put $\delta = \rho/16$ and $R = r_1$. Then, using Proposition 9.2.8, we find an n_k such that for each $n \ge n_k$ there exists a set $A_n \subset \overline{\mathbb{D}}(r_1)$, $\operatorname{cap}(A_n) < \delta$, such that

$$|(fQ_n - P_n)(z)| \le \varepsilon_k^n |Q_n(z)|, \quad z \in \overline{\mathbb{D}}(r_1) \setminus A_n.$$

Put $\widetilde{A}_n := \{z \in \overline{\mathbb{A}} : |(fQ_n - P_n)(z)| > \varepsilon_k^n \|Q_n\|_{\overline{\mathbb{A}}}\}$, where $\mathbb{A} := \mathbb{A}(s - \rho/2, s + \rho/2)$. Then $\widetilde{A}_n \subset A_n$ and $\widetilde{A}_n$ is open in $\overline{\mathbb{A}}$. Let $\widetilde{A}_n'$ be a connected component of $\widetilde{A}_n$. Then, by the maximum principle, we see that it has to cut $\partial\mathbb{A}$. Recall that (see [Ran 1995], Theorem 5.3.2(a))

$$\operatorname{diam} \widetilde{A}_n' \le 4 \operatorname{cap} \widetilde{A}_n' \le 4 \operatorname{cap} A_n < 4\delta = \rho/4;$$

therefore, A_n and Γ are disjoint.

Now we continue our estimates for $z \in \mathbb{D}(r)$ (note that Q_n'' is without zeros in $\mathbb{D}(r)$):

$$\left| f(z) - \frac{p_n(z)}{Q_n''(z)} \right| = \left| \frac{g(z)z^{n+1}}{Q_n''(z)} \right| \le \frac{1}{2\pi} 2\pi s \left(\frac{r}{s}\right)^{n+1} \frac{1}{s - r} \varepsilon_k^n \frac{\|Q_n\|_{\overline{\mathbb{A}}}}{|Q_n''(z)| \inf_{|\zeta|=s} |Q_n'(\zeta)|}.$$

Let $Q(z) = (z - a_1)\cdots(z - a_m)(z - a_{m+1})\cdots(z - a_\ell)$ with $m \le \ell \le n$ and $|a_j| < r + \rho/2$, $j = 1, \ldots, m$, and $|a_j| \ge r + \rho/2$ for the remaining zeros. If $z \in \mathbb{D}(r)$, then

$$\frac{\|Q_n''\|_{\mathbb{A}}}{|Q_n''(z)|} \le \prod_{j=m+1}^{\ell} \frac{s + \rho/2 + |a_j|}{|a_j| - r} \le \left(\frac{2s + \rho}{\rho/2}\right)^{\ell - m},$$

and if $|\zeta| = s$, then

$$\frac{\|Q_n'\|_{\mathbb{A}}}{|Q_n'(\zeta)|} \le \prod_{j=1}^{m} \frac{2s+\rho}{s-r-\rho/2} \le \left(\frac{2s+\rho}{\rho/2}\right)^m.$$

Finally, we end up with

$$\left| f(z) - \frac{p_n(z)}{Q_n''(z)} \right| \le \frac{r}{s-r} \left(\frac{\varepsilon_k 2r(2s+\rho)}{s\rho} \right)^n, \quad z \in \mathbb{D}(r),\ n \ge n_k,$$

or

$$\left| f(z) - \frac{p_n(z)}{Q_n''(z)} \right| \le M(M')^n \varepsilon_k^n, \quad z \in \mathbb{D}(r),\ n \ge n_k,$$

where $M := 1 + \frac{r}{s-r}$ and $M' := \frac{2r(2s+\rho)}{s\rho}$. Without loss of generality, we may choose $n_k < n_{k+1}$ for all k's. Then

$$(\rho_{n,\mathbb{D}(r)}(f))^{1/n} \le MM'\varepsilon_k \le 1/k, \quad n_k \le n < n_{k+1},$$

if the ε_k's were correctly chosen (observe that all other constants depend only on the geometric situation). Hence, $(\rho_{n,\mathbb{D}(r)}(f))^{1/n} \underset{n\to\infty}{\longrightarrow} 0$. □

On the other hand we have the following partial converse result (see [Kar 1976]); the full answer will be given in Theorem 9.2.14.

Proposition 9.2.9. *If $D \subset \mathbb{C}$ is a domain such that $K := \mathbb{C} \setminus D$ is compact and $\mathcal{O}(D) \subset \boldsymbol{R}^0$, then K is polar.*

To be able to present the proof of this proposition we need the following information about the behavior of Padé approximants.

Lemma 9.2.10. *Let $f \in \mathcal{O}(\mathbb{D}(R))$. For each $n \in \mathbb{N}$ denote by P_n/Q_n the* ***n-th Padé approximant of f at zero of order n****, i.e. $P_n, Q_n \in \mathcal{P}_n(\mathbb{C})$ with*

$$Q_n f - P_n = c_{2n+1} z^{2n+1} + \textit{higher terms}.$$

Assume that $\deg Q_n = n$*, $Q_n(0) \ne 0$, Q_n normalized and without zeros in $\mathbb{D}(R)$. Moreover, let $\mathfrak{r}_n = p_n/q_n$ be a rational function without poles in $\mathbb{D}(R)$. If $r \in (0, R)$, then*

$$|(f - P_n/Q_n)(z)| \le \frac{|z|^{2n+1}}{r^{2n}} \left(\frac{R+r}{R-r}\right)^{2n} \frac{\|f - \mathfrak{r}_n\|_{\mathbb{D}(r)}}{r-|z|}, \quad z \in \mathbb{D}(r).$$

Proof. We may assume that q_n has no zeros in $\mathbb{D}(R)$. Using the Cauchy integral formula we see that, for $z \in \mathbb{D}(r)$,

$$\begin{aligned} q_n(z)\frac{(fQ_n - P_n)(z)}{z^{2n+1}} &= \frac{1}{2\pi i} \int_{\partial\mathbb{D}(r)} \frac{q_n(\zeta)(fQ_n - P_n)(\zeta)}{\zeta^{2n+1}(\zeta - z)} d\zeta \\ &= \frac{1}{2\pi i} \int_{\partial\mathbb{D}(r)} \frac{(q_n f Q_n)(\zeta)}{\zeta^{2n+1}(\zeta - z)} d\zeta \end{aligned}$$

(the integrand of the remaining integral tends to zero if z tends to infinity and hence this integral vanishes).

Therefore (with the same argument),

$$(f - P_n/Q_n)(z) = \frac{z^{2n+1}}{Q_n(z)q_n(z)} \frac{1}{2\pi i} \int_{\partial\mathbb{D}(r)} \frac{Q_n(\zeta)(fq_n - p_n)(\zeta)}{\zeta^{2n+1}(\zeta - z)} d\zeta, \quad z \in \mathbb{D}(r).$$

Then

$$|(f - P_n/Q_n)(z)| \le |z| \left(\frac{|z|}{r}\right)^{2n} \frac{\|Q_n p_n\|_{\mathbb{D}(r)}}{|(Q_n q_n)(z)|} \frac{\|f - p_n/q_n\|_{\mathbb{D}(r)}}{r - |z|}, \quad z \in \mathbb{D}(r).$$

It remains to observe that

$$\frac{\|Q_n q_n\|_{\mathbb{D}(r)}}{\min\{|(Q_n q_n)(z)| : z \in \mathbb{D}(r)\}} \le \left(\frac{R+r}{R-r}\right)^{2n},$$

which proves the lemma. □

Proof of Theorem 9.2.9. We may assume that $0 \notin K$. Suppose that K is not polar. Put $K' := \{z \in \mathbb{C} : 1/z \in K\}$. Then, using [Ran 1995], Theorem 5.3.1, it is clear that K' is a compact set which is not polar. Put $D' := \{z \in \overline{\mathbb{C}} : 1/z \in D\}$. Then D' is connected and $K' = \overline{\mathbb{C}} \setminus D'$.

In what follows we need the Bernstein Lemma (see [Ran 1995], Theorem 5.5.7) telling us that for a Fekete polynomial q_n for K' of degree $n \ge 2$, the following inequalities hold:

$$e^{g_{D'}(z,\infty)} \left(\frac{\operatorname{cap}(K')}{\delta_n(K')}\right)^{\tau_{D'}(z,\infty)} \le \left(\frac{|q_n(z)|}{\|q_n\|_{K'}}\right)^{1/n} \le e^{g_{D'}(z\infty)}, \quad z \in D' \setminus \{\infty\}.$$

Here δ_n denotes the n-th transfinite diameter (see [Ran 1995], Definition 5.5.1) and $\tau_{D'}$ is the Harnack distance (see [Ran 1995], Definition 1.3.4) of D'. Recall that $1 \le \tau_{D'}(\cdot,\infty) \underset{|z|\to\infty}{\longrightarrow} 1$, $\tau_{D'}(\cdot,\infty)$ is continuous, and that $\operatorname{cap}(K') \le \delta_n(K') \underset{n\to\infty}{\longrightarrow} 1$ (see [Ran 1995], Fekete–Szegö Theorem 5.5.2).

Put $n_1 = 1$ and $n_{k+1} := 2n_k + 1$. Now choose numbers A_k with $|A_k| = ((1-\varepsilon)\operatorname{cap}(K'))^{n_k}$, where $\varepsilon \in (0,1)$. Fix an arbitrary point $a \in D' \cap \mathbb{C}$ and choose a closed disc $\overline{\mathbb{D}}(a, s_a) \subset D'$. Then, $\tau_{D'}(z,\infty) \le M_a$, $z \in \mathbb{D}(a, s_a)$, for a certain positive M_a. Applying [Ran 1995], Theorem 5.5.4, we conclude that

$$\left|\frac{A_k}{q_{n_k}}\right| \le \frac{|A_k|}{\operatorname{cap}(K')^{n_k}} \left(\frac{\delta_{n_k}(K')}{\operatorname{cap}(K')}\right)^{M_a}, \quad z \in \mathbb{D}(a, s_a).$$

For large k, $k \ge k_\varepsilon$, we have $1 \le \left(\delta_{n_k}(K')/\operatorname{cap}(K')\right)^{M_a} \le (1+\varepsilon)$. Thus $\frac{|A_k|}{|q_{n_k}(z)|} \le (1-\varepsilon^2)^{n_k}$, $k \ge k_\varepsilon$. Therefore, the series

$$g(z) := \sum_{k=1}^{\infty} A_k/q_{n_k}(z), \quad z \in D',$$

is uniformly convergent on $\mathbb{D}(a, s_a)$. Since a was arbitrary, we conclude that g is holomorphic on D'. Moreover, we know that $\tau_{D'}(z, \infty) \le 2$ whenever $|z|$ is sufficiently large. Using the former estimates gives that g remains bounded at infinity.

Put $f(z) := g(1/z)$, $z \in D \setminus \{0\}$. Then f is holomorphic on $D \setminus \{0\}$ and it extends holomorphically to the origin, i.e. $f \in \mathcal{O}(D)$. Observe that

$$f(z) = \sum_{k=1}^{\infty} \frac{(-1)^{n_k} A_k z^{n_k}}{w_{k,1} \dots w_{k,n_k} (z - 1/w_{k,1}) \cdots (z - 1/w_{k,n_k})}, \quad z \in D.$$

Put

$$(P_k/Q_k)(z) := \sum_{j=1}^{k} \frac{(-1)^{n_j} A_k z^{n_j}}{w_{j,1} \dots w_{j,n_j} (z - 1/w_{j,1}) \cdots (z - 1/w_{j,n_j})}.$$

Obviously, $Q_k, P_k \in \mathcal{P}_{N_k}(\mathbb{C})$, and $\deg Q_k = n_k$, Q_k is normalized and without zeros in $\mathbb{D}(\operatorname{dist}(0, K))$, where $N_k := \sum_{j=1}^{k} n_j$. Moreover, we have

$$f(z) - \frac{P_k(z)}{Q_k(z)} = \sum_{j=k+1}^{\infty} \frac{(-1)^{n_j} A_k z^{n_j}}{w_{j,1} \dots w_{j,n_j} (z - 1/w_{j,1}) \cdots (z - 1/w_{j,n_j})}$$

which has a zero of order at least $n_{k+1} = 2N_k + 1$ at the origin, i.e. P_k/Q_k is the N_k-th Padé approximant of f at the origin.

Fix now a point $z_0 \in \mathbb{D}(r/4) \setminus \{0\}$, where $r := \operatorname{dist}(0, K)$, and choose the A_k in such a way that $A_k/q_{n_k}(z_0) > 0$ for all k. Then

$$\begin{aligned} |f(z_0) - (P_k/Q_k)(z_0)| &= \left| g(1/z_0) - \sum_{j=1}^{k} \frac{A_j}{q_{n_j}(1/z_0)} \right| \\ &\ge \frac{A_{k+1}}{q_{n_{k+1}}(1/z_0)} = \frac{\big((1-\varepsilon)\operatorname{cap}(K')\big)^{n_{k+1}}}{|q_{n_{k+1}}(1/z_0)|}. \end{aligned}$$

Applying the right-hand side of Bernstein's lemma and [Ran 1995], Theorem 5.5.4, we get

$$|q_{n_{k+1}}(1/z_0)|^{1/n_{k+1}} \le e^{g_{D'}(1/z_0,\infty)} \|q_{n_{k+1}}\|_{K'}^{1/n_{k+1}} \le e^{g_{D'}(1/z_0,\infty)} \delta_{n_{k+1}}.$$

In particular, we end up with

$$|f(z_0) - (P_k/Q_k)(z_0)|^{1/N_k} \ge \left(\frac{(1-\varepsilon)\operatorname{cap}(K')}{e^{g_{D'}(1/z_0,\infty)} \delta_{n_{k+1}}} \right)^{2+1/N_k}.$$

Obviously, the last sequence does not converge to zero.

It now remains to show that $f|_{\mathbb{D}(r)}$ does not belong to $\boldsymbol{R}^0$. Assume the contrary. Then we have rational approximants $\tilde{p}_n/\tilde{q}_n$ without poles inside of $\mathbb{D}(r/2)$ such that $\|f - \tilde{p}_n/\tilde{q}_n\|_{\mathbb{D}(r/2)}^{1/n} \to 0$ as $n \to 0$. Then, using Lemma 9.2.10, it follows that

$$|f(z_0) - (P_k/Q_k)(z_0)|^{1/N_k} \le \text{const}\, \|f - \tilde{p}_{N_k}/\tilde{q}_{N_k}\|_{\mathbb{D}(r/2)}^{1/N_k} \to 0;$$

a contradiction. □

There is another way to characterize those functions f which belong to $\boldsymbol{R}^0$.

Theorem 9.2.11. *Let $f \in \mathcal{O}(\overline{\mathbb{D}})$, $f \not\equiv 0$. Put*

$$V_k(f) = V_k := \sup_{j_1,\dots,j_k} \left| \det \begin{pmatrix} a_{j_1} & \cdots & a_{j_1+k-1} \\ \vdots & \dots & \vdots \\ a_{j_k} & \cdots & a_{j_k+k-1} \end{pmatrix} \right|, \quad k \in \mathbb{N},$$

where $f(z) = \sum_{k=0}^{\infty} a_j z^j$ denotes the power series expansion of f. Then:

$$f \in \boldsymbol{R}^0 \quad \textit{if and only if} \quad \lim_{k\to\infty} V_k^{1/k^2} = 0.$$

To be able to prove this result various preparations are necessary. Put

$$\eta_{k+1} = \eta_{k+1}(f) := \inf_{(c_k,\dots,c_0)\in\mathbb{C}^k\times\{1\}} \big(\sup_{j\ge 1} |a_j c_k + \cdots + a_{j+k} c_0|\big), \quad k \ge 0.$$

Note that $\eta_k \ge \eta_{k+1}$, $k \ge 1$. Indeed, take $(c_{k-1},\dots,c_0) \in \mathbb{C}^{k-1} \times \{1\}$. Then

$$\sup_{j\ge 2} |a_j c_{k-1} + \cdots + a_{j+(k-1)} c_0| = \sup_{j\ge 2} |a_{j-1} 0 + a_j c_{k-1} + \cdots + a_{j+(k-1)} c_0| \ge \eta_{k+1}.$$

Since $(c_{k-1},\dots,c_0)$ was arbitrarily chosen, it follows that $\eta_k \ge \eta_{k+1}$.

First we express the property that $f \in \boldsymbol{R}^0$ in terms of the numbers η_k.

Lemma 9.2.12. *If f is as in Theorem 9.2.11, then:*

$$f \in \boldsymbol{R}^0 \quad \textit{if and only if} \quad \lim_{k\to\infty} \eta_k^{1/k} = 0.$$

Proof. Assume first that $f \in \boldsymbol{R}^0$. Using Lemma 9.2.2 it follows that

$$\lim_{k\to+\infty} (\rho_{k,\mathbb{D}}(f))^{1/k} = 0.$$

Therefore,

$$\left\| f - \frac{p_k}{q_k} \right\|_{\overline{\mathbb{D}}}^{1/k} \to 0.$$

for suitable $p_k, q_k \in \mathcal{P}_k(\mathbb{C})$.

Fix a k. We may assume that any zero of q_k is not a zero of p_k and $q_k(0) = 1$. Then all zeros of q_k lie outside of $\overline{\mathbb{D}}$. We denote the zeros of q_k by $\alpha_{k,1},\dots,\alpha_{k,s_k}$ (counted as often as their multiplicities), i.e. $s_k \le k$. Then

$$q_k(z) = \Big(\frac{z}{-\alpha_{k,1}} + 1\Big) \cdots \Big(\frac{z}{-\alpha_{k,s_k}} + 1\Big).$$

Hence, for $z \in \overline{\mathbb{D}}$, $|q_k(z)| \le \big(\frac{1}{|\alpha_{k,1}|}+1\big)\cdots\big(\frac{1}{|\alpha_{k,s_k}|}+1\big) \le 2^k$.

So we reach the following estimate:

$$\|q_k f - p_k\|_{\overline{\mathbb{D}}}^{1/k} \le \|q_k\|_{\overline{\mathbb{D}}}^{1/k} \Big\| f - \frac{p_k}{q_k}\Big\|_{\overline{\mathbb{D}}}^{1/k} \le 2\Big\| f - \frac{p_k}{q_k}\Big\|_{\overline{\mathbb{D}}}^{1/k} \underset{k\to\infty}{\to} 0.$$

Write $p_k(z) = \sum_{\sigma=0}^k d_\sigma z^\sigma$ and $q_k(z) = \sum_{\sigma=0}^k c_\sigma z^\sigma$. Recall that $c_0 = 1$ and $|c_j| \le 2^k$. Then

$$q_k(z)f(z) - p_k(z) = \sum_{m=1}^{\infty}\Big(\sum_{s+\sigma=m} a_s c_\sigma\Big)z^m - \sum_{m=0}^{k} d_m z^m, \quad z \in \mathbb{D}.$$

Let $m > k$. Then, by the Cauchy inequalities, it follows that

$$|a_{m-k}c_k + \cdots + a_m c_0| \le \|q_k f - p_k\|_{\overline{\mathbb{D}}}.$$

So we get $\eta_{k+1}^{1/(k+1)} \le \|q_k f - p_k\|_{\overline{\mathbb{D}}}^{1/(k+1)} \to 0$. Hence, $\lim_{k\to\infty} \eta_k^{1/k} = 0$.

To prove the converse implication let us assume that $\lim \eta_k^{1/k} = 0$. Obviously, we may assume that $\|f\|_{\mathbb{D}} \le 1$. Then we find a polynomial $q_k = \sum_{m=0}^k c_{k,m} z^m \in \mathcal{P}_k(\mathbb{C})$ with $c_{k,0} = 1$ such that

$$\eta_{k+1} \le \sup_{j\ge 1} |a_j c_{k,k} + \cdots + a_{j+k}c_{k,0}| \le \varepsilon_k,$$

where $\varepsilon_k := 2\eta_{k+1}$ if $\eta_{k+1} \ne 0$, otherwise $\varepsilon_k := e^{-k^2}$. Then $q_k(z)f(z) = p_k(z) + \sum_{m=k+1}^{\infty}\big(\sum_{s+\sigma=m} a_s c_{k,\sigma}\big)z^m$ with $p_k \in \mathcal{P}_k(\mathbb{C})$. For $|z| \le 1/2$ we get:

$$|q_k(z)f(z) - p_k(z)| \le \sum_{m=k+1}^{\infty} \varepsilon_k (1/2)^m = \varepsilon_k 2^{-k};$$

thus $\|q_k f - p_k\|_{\mathbb{D}(1/2)}^{1/k} \le (1/2)(\varepsilon_k)^{1/k} \to 0$.

Write $q_k = q_k' q_k''$, where $q_k', q_k'' \in \mathcal{P}_k(\mathbb{C})$ with $q_k'(0) = q_k''(0) = 1$ such that the zeros of the first polynomial are not in $\overline{\mathbb{D}}$ while all zeros of q_k'' belong to $\overline{\mathbb{D}}$. It is easy to see that $|q_k'(z)| \ge 2^{-k}$, $|z| \le 1/2$. Then

$$|q_k''(z)f(z) - p_k(z)/q_k'(z)| \le \varepsilon_k, \quad |z| \le 1/2.$$

What remains is to handle the factor q_k''. Fix a $\delta \in (0, 1/40)$ and put $E_k := \{z \in \mathbb{D}(1/2) : |q_k''(z)| \le \delta^k\}$. Then

$$\begin{aligned}
\operatorname{cap} E_k &\le \operatorname{cap}\{z \in \mathbb{C} : |q_k''(z)| \le \delta^k\}\\
&= \operatorname{cap}\{z \in \mathbb{C} : |\alpha_{k,1}\cdots\alpha_{k,s_k} q_k''(z)| \le \delta^k |\alpha_{k,1}\cdots\alpha_{k,s_k}|\}\\
&\le \delta |\alpha_{k,1}\cdots\alpha_{k,s_k}|^{1/k} \le \delta,
\end{aligned}$$

where the $\alpha_{k,j}$ denote the zeros of q_k''. Set $A_k := \{r \in (3/8, 1/2) : \exists_{\theta_r} : re^{\theta_r} \in E_k\}$. Using [Ran 1995], Chapter 5, Exercise 3, one has $\operatorname{cap} E_k \geq \mathcal{L}^1(A_k)/4$. Therefore, we find a radius $\rho_k \in (3/8, 1/2)$ with $E_k \cap \partial\mathbb{D}(\rho_k) = \varnothing$.

Put

$$\mathrm{r}^*(z) := \frac{1}{2\pi i} \int_{\partial\mathbb{D}(\rho_k)} \frac{r_k(\zeta)}{\zeta - z} d\zeta.$$

Note that r_k^* is a new rational function of degree at most k. If $z \in \mathbb{D}(1/4)$, then

$$|f(z) - \mathrm{r}_k^*(z)| \leq \frac{\rho_k \varepsilon_k}{\delta^k(\rho_k - 1/4)} \leq 4\frac{\varepsilon_k}{\delta^k}.$$

Hence, $\|f - \mathrm{r}^*\|_{\mathbb{D}(1/4)}^{1/k} \leq (4\varepsilon_k)^{1/k}/\delta \to 0$, i.e. $f \in \boldsymbol{R}^0$. □

Moreover, we have the following lemma.

Lemma 9.2.13. *Let f be as in Theorem 9.2.11. Then*

$$\eta_{k+1} V_k \leq V_{k+1} \leq (k+1)\eta_{k+1} V_k, \quad k \geq 1.$$

Proof. Note that $V_k = 0$ implies $V_{k+1} = 0$. Fix a k with $V_k \neq 0$. For an $\varepsilon \in (0, 1)$ choose indices $j_1, \dots, j_k$ such that

$$\Delta_k := \left|\det \begin{pmatrix} a_{j_1} & \cdots & a_{j_1+k-1} \\ \vdots & \cdots & \vdots \\ a_{j_k} & \cdots & a_{j_k+k-1} \end{pmatrix}\right| \geq V_k(1 - \varepsilon).$$

For an arbitrary j we conclude that

$$V_{k+1} \geq \left|\det \begin{pmatrix} a_{j_1} & \cdots & a_{j_1+k-1} & a_{j_1+k} \\ \vdots & \cdots & \vdots & \vdots \\ a_{j_k} & \cdots & a_{j_k+k-1} & a_{j_k+k} \\ a_j & \cdots & a_{j+k-1} & a_{j+k} \end{pmatrix}\right|.$$

Thus,

$$V_{k+1} \geq \Delta_k \left| a_j \frac{\Delta_1}{\Delta_k} + \cdots + a_{j+k} \frac{\Delta_k}{\Delta_k} \right| \geq V_k(1 - \varepsilon)\eta_{k+1}.$$

Since ε is arbitrary chosen, the left inequality is proved.

For the right inequality let $\varepsilon > 0$ be given. Then there exists $(c_k, \dots, c_0) \in \mathbb{C}^{k+1}$, $c_0 = 1$, such that

$$\sup_{j \geq 1} |a_j c_k + \cdots a_{j+k} c_0| < \eta_{k+1} + \varepsilon.$$

Then we start with arbitrary indices $j_1, \dots, j_{k+1}$. We get

$$\left|\det\begin{pmatrix} a_{j_1} & \cdots & a_{j_1+k} \\ \vdots & \dots & \vdots \\ a_{j_{k+1}} & \cdots & a_{j_{k+1}+k}\end{pmatrix}\right|$$
$$= \left|\det\begin{pmatrix} a_{j_1} & \cdots & a_{j_1+k-1} & a_{j_1+k}c_0 + \cdots + a_{j_1}c_k \\ \vdots & \dots & \vdots & \vdots \\ a_{j_{k+1}} & \cdots & a_{j_k+k-1} & a_{j_{k+1}+k}c_0 + \cdots + a_{j_{k+1}}c_k\end{pmatrix}\right|$$
$$\leq (k+1)V_k \sup_{j\geq 1} |a_j c_k + \cdots + a_{j+k}c_0| \leq (k+1)V_k(\eta_{k+1} + \varepsilon),$$

from where the left inequality immediately follows with $\varepsilon \to 0$. □

Proof of Theorem 9.2.11. Lemma 9.2.13 leads to $\eta_1 \cdots \eta_k \leq V_k \leq k!\eta_1 \cdots \eta_k$, $k \geq 1$.

If $V_k^{1/k^2} \to 0$, then $V_k^{1/k^2} \geq (\eta_1 \cdots \eta_k)^{1/k^2} \geq (\eta_k^k)^{1/k^2}$, which gives the claim.

Now assume that $f \in \boldsymbol{R}^0$. By Lemma 9.2.12 it follows that $\eta_k^{1/k} \to 0$. Then $V_k^{1/k^2} \leq (k!\eta_1 \cdots \eta_k)^{1/k^2}$. Fix an arbitrary ε. Then there is a k_ε such that for all $k \geq k_\varepsilon$ we have $\eta_k^{1/k} \leq \varepsilon$. Thus, if $k > k_\varepsilon$, then:

$$V_k^{1/k^2} \leq (k!)^{1/k^2} (\eta_1 \cdots \eta_{k_\varepsilon})^{1/k^2} \varepsilon^{((k_\varepsilon+1)+\cdots+k)/k^2}$$

which implies that $\limsup V_k^{1/k^2} \leq \sqrt{\varepsilon}$, i.e. this sequence converges to 0.

Hence the theorem is completely proved. □

Using the former result we get

Theorem 9.2.14. *If $D \subset \mathbb{C}$ is a domain with $\mathcal{O}(D) \subset \boldsymbol{R}^0$, then $\mathbb{C} \setminus D$ is polar.*

Proof. We may assume that $0 \in D$. The case when $K := \mathbb{C} \setminus D$ is compact follows from Proposition 9.2.9.

Now let K be arbitrary and suppose it is not polar. Put $K_j := K \cap \overline{\mathbb{D}}(j)$. Then K_j is not polar for all j, $j \geq j_0 > 1$. Put $D_j := \mathbb{C} \setminus K_j$. Two cases have to be discussed.

Case 1: $(\mathbb{C} \setminus \overline{\mathbb{D}}(j_0)) \setminus K \neq \varnothing$.

Fix two points $a, b \in D_{j_0}$. If both points are not contained in K, then they can be connected by a path lying in $D \subset D_{j_0}$. If a, b are outside of $\overline{K}_{j_0}$, then there is a curve outside of $\overline{K}_{j_0}$ connecting them. Moreover, if $|a| > j$ and $|b| \leq j$, one may connect them inside of D_{j_0} via a point $c \in (\mathbb{C} \setminus \overline{\mathbb{D}}(j_0)) \setminus K$. Hence D_{j_0} is connected. Then the compact case applies and leads to an $f \in \mathcal{O}(D_{j_0}) \subset \mathcal{O}(D)$ with $f \notin \boldsymbol{R}^0$; a contradiction.

Case 2: $\mathbb{C} \setminus \overline{\mathbb{D}}(j_0) \subset K$.

Then $D_{j_0} \subset \mathbb{D}(j_0)$. Put

$$f(z) = \sum_{k=1}^{\infty} a_k z^k := \sum_{k=1}^{\infty} (z/j_0)^{2^k}, \quad |z| < j_0,$$

and

$$A_n := \begin{pmatrix} a_1 & \cdots & a_n \\ \vdots & \cdots & \vdots \\ a_n & \cdots & a_{2n-1} \end{pmatrix}, \quad n \in \mathbb{N}.$$

Obviously, $f \in \mathcal{O}(\mathbb{D}(j_0))$. Suppose that $f \in \boldsymbol{R}^0$. Then the former criterion leads to

$$0 = \lim_{k\to+\infty} V_k(f)^{1/k^2} \ge \limsup_{k\to+\infty} |A_k|^{1/k^2} \ge \lim_{k\to+\infty} |A_{2^k}|^{1/2^{2k}} = \lim_{k\to+\infty} j_0^{1/2^k} = 1;$$

a contradiction. Hence, $f \notin \boldsymbol{R}^0$, which gives the desired contradiction. □

Remark 9.2.15. A similar result in higher dimensions was proved in [Sad 1984].

The last criterion may be used to discuss the fiberwise behavior of functions of several complex variables. Namely, we have the following result, due to Sadullaev.

Theorem 9.2.16 ([Sad 1984]). *Let $f \in \mathcal{O}(\mathbb{D}^{p+1})$ and let $A \subset \mathbb{D}^p$ be not pluripolar. Assume that $f(z', \cdot) \in \boldsymbol{R}^0$ for all $z' \in A$. Then $f(z', \cdot) \in \boldsymbol{R}^0$ for all $z' \in \mathbb{D}^p$.*

Proof. Without loss of generality we may assume that $f \in \mathcal{O}(\mathbb{D}^p \times \overline{\mathbb{D}})$ and $|f| \le 1$ on $\mathbb{D}^{p+1}$. Let $f(z) = f(z', z_{p+1}) = \sum_{j=0}^{\infty} a_j(z') z_{p+1}^j$ be the Hartogs series of f. The a_j are holomorphic functions on $\mathbb{D}^p$ with $|a_j| \le 1$. Put

$$V_k(f)(z') = V_k(z') := \sup_{j_1,\dots,j_k} \left| \det \begin{pmatrix} a_{j_1}(z') & \cdots & a_{j_1+k-1}(z') \\ \vdots & \cdots & \vdots \\ a_{j_k}(z') & \cdots & a_{j_k+k-1}(z') \end{pmatrix} \right|, \quad k \in \mathbb{N},\ z' \in \mathbb{D}^p.$$

Observe that $\log V_k$ is the upper semicontinuous supremum of a family of plurisubharmonic functions, bounded from above. Hence, $V_k \in \mathcal{PSH}(\mathbb{D}^p)$.

Put $v := \limsup_{k\to+\infty} (1/k^2) \log V_k$. Thus Theorem 9.2.11 implies that $v = -\infty$ on A. Using Proposition 2.3.12 we have $v^* \in \mathcal{PSH}(\mathbb{D}^p)$. Moreover, $v^* = v = -\infty$ on $A \setminus P$ for a suitable pluripolar set P (see Proposition 2.3.22). Note that $A \setminus P$ is not pluripolar which implies that $v^* \equiv -\infty$ on $\mathbb{D}^p$. In particular, $v \equiv -\infty$ on $\mathbb{D}^p$, which proves that $f(z', \cdot) \in \boldsymbol{R}^0$, $z' \in \mathbb{D}^p$. □

9.2.2 Oka–Nishino theorem

Definition 9.2.17. Let $D \subset \mathbb{C}^n$ be a domain and let $S \subset D$ be relatively closed. We say that S is *pseudoconcave* if any point $a \in S$ has a neighborhood $U_a \subset D$ such that the open set $U_a \setminus S$ is a region of holomorphy.

Proposition 9.2.18. *Assume that D is a domain of holomorphy and $S \subset D$ is relatively closed. Then S is pseudoconcave in D iff $D \setminus S$ is a region of holomorphy.*

Proof. Assume that S is pseudoconcave. Let $(D_k)_{k=1}^{\infty}$ be an exhaustion of D by domains of holomorphy. It suffices to show that $D_k \setminus S$ is pseudoconvex, $k \in \mathbb{N}$ (cf. Theorem 2.5.5 (a)). Since the notion of pseudoconvexity may be localized (cf. Theorem 2.5.5 (f)), we only need to show that each point $a \in \partial(D_k \setminus S)$ has a neighborhood U_a such that $(D_k \setminus S) \cap U_a$ is pseudoconvex. The situation is trivial if $a \notin S$. If $a \in S$ and U_a is as in Definition 9.2.17, then $(D_k \setminus S) \cap U_a = D_k \cap (U_a \setminus S)$ is also pseudoconvex. □

The following important result describing certain pseudoconcave sets in an analytic way will be used in the proof of Theorem 9.2.24.

Theorem 9.2.19 (Oka–Nishino theorem). *Let $D \subset \mathbb{C}^p$ be a domain of holomorphy and let $\varnothing \neq S \subset D \times \mathbb{C}$ be a pseudoconcave set such that*

(a) *for every $a \in D$ there exist $r, R > 0$ with $S \cap (\mathbb{P}(a,r) \times \mathbb{C}) \subset \mathbb{P}(a,r) \times \mathbb{D}(R)$,*

(b) *the set $\{a \in D : \#S_{(a,\cdot)} < +\infty\}$ is not pluripolar.*

Then there exist $s \in \mathbb{N}$ and $g_1, \dots, g_s \in \mathcal{O}(D)$ such that

$$S = \Big\{(z,w) \in D \times \mathbb{C} : w^s + \textstyle\sum_{j=1}^{s} g_j(z) w^{s-j} = 0\Big\}.$$

In particular, if the set $\{a \in D : \#S_{(a,\cdot)} = 1\}$ is not pluripolar, then

$$S = \{(z,w) \in D \times \mathbb{C} : w = f(z)\}$$

is a graph with $f \in \mathcal{O}(D)$.

Proof. Let $A_k := \{a \in D : \#S_{(a,\cdot)} \le k\}$, $k \in \mathbb{Z}_+$. First observe that A_0 is empty.

Indeed, suppose that $a \in A_0$. Let r, R be as in (a). Since $S \cap (\{a\} \times \overline{\mathbb{D}}(R)) = \varnothing$, we may assume that $r > 0$ is so small that $\mathbb{P}(a,r) \subset\subset D$ and $S \cap (\mathbb{P}(a,r) \times \mathbb{D}(R)) = \varnothing$. Thus $S \cap (\mathbb{P}(a,r) \times \mathbb{C}) = \varnothing$. Take an arbitrary domain of holomorphy $U_0 \subset\subset D$ with $\mathbb{P}(a,r) \subset U_0$. Using (a), we find an $R_0 > 0$ such that $S \cap (U_0 \times \mathbb{C}) \subset U_0 \times \mathbb{D}(R_0)$. Now, by the cross theorem (Theorem 5.4.1) applied to the cross

$$(\mathbb{P}(a,r) \times \mathbb{C}) \cup (U_0 \times \mathbb{A}(R_0, +\infty)),$$

we conclude that every function $f \in \mathcal{O}((U_0 \times \mathbb{C}) \setminus S)$ extends to $U_0 \times \mathbb{C}$. Thus $S = \varnothing$; a contradiction.

We have assumed that $\bigcup_{k=0}^{\infty} A_k$ is not pluripolar. Let $s \in \mathbb{N}$ be the minimal number such that A_{s-1} is pluripolar and A_s is not pluripolar. Define

$$\boldsymbol{D}_k(z) := \max\Big\{\Big(\prod_{\mu<\nu} |w_\mu - w_\nu|\Big)^{\frac{2}{k(k-1)}} : w_1, \dots, w_k \in S_{(z,\cdot)}\Big\},$$
$$z \in D,\ k \in \mathbb{N},\ k \ge 2.$$

It is known ([Nis 2001], Theorem 4.8) that $\log \boldsymbol{D}_k \in \mathcal{PSH}(D)$. Since $\boldsymbol{D}_{s+1} = 0$ on A_s and A_s is not pluripolar, we conclude that $\boldsymbol{D}_{s+1} \equiv 0$. Thus $A_s = D$. For $z \in D_0 := D \setminus A_{s-1}$ let $S_{(z,\cdot)} = \{\xi_1(z), \dots, \xi_s(z)\}$,

$$P(z,w) := \prod_{j=1}^{s}(w - \xi_j(z)) = w^s + \sum_{j=1}^{s} g_j(z) w^{s-j}, \quad (z,w) \in D_0 \times \mathbb{C}.$$

Take an $a_0 \in D_0$ and let $S_{(a_0,\cdot)} = \{b_1, \dots, b_s\}$. Let $\delta > 0$ be so small that $\overline{\mathbb{D}}(b_\mu, \delta) \cap \overline{\mathbb{D}}(b_\nu, \delta) = \varnothing$, $\mu \neq \nu$. Fix a $\delta' \in (0, \delta)$ and let $r > 0$ such that $S \cap (\mathbb{P}(a_0, r) \times \overline{\mathbb{A}}(b_j, \delta', \delta)) = \varnothing$, $j = 1, \dots, s$. Suppose that $\mathbb{P}(a_0, r) \not\subset D_0$ and let $a \in \mathbb{P}(a_0, r) \cap A_{s-1}$. We may assume that $S \cap (\{a\} \times \overline{\mathbb{D}}(b_1, \delta)) = \varnothing$. Let $\rho > 0$ be such that $\mathbb{P}(a, \rho) \subset \mathbb{P}(a_0, r)$ and $S \cap (\mathbb{P}(a, \rho) \times \overline{\mathbb{D}}(b_1, \delta)) = \varnothing$. Applying the cross theorem (Theorem 5.4.1) to the cross

$$(\mathbb{P}(a_0, r) \times \mathbb{A}(b_1, \delta', \delta)) \cup (\mathbb{P}(a, \rho) \times \mathbb{D}(b_1, \delta)),$$

we conclude that every function $f \in \mathcal{O}((\mathbb{P}(a_0, r) \times \mathbb{D}(b_1, \delta)) \setminus S)$ extends to $\mathbb{P}(a_0, r) \times \mathbb{D}(b_1, \delta)$; a contradiction.

Thus $\mathbb{P}(a_0, r) \subset D_0$. In particular, D_0 is open and, consequently, a domain (because A_{s-1} is pluripolar). Moreover, we may assume that $\xi_j(z) \in \overline{\mathbb{D}}(b_j, \delta)$, $z \in \mathbb{P}(a_0, r)$, $j = 1, \dots, s$. Using the Hartogs theorem ([Nis 2001], Theorem 4.7), we conclude that $\xi_j \in \mathcal{O}(\mathbb{P}(a_0, r))$, $j = 1, \dots, s$. Thus $g_1, \dots, g_s \in \mathcal{O}(D_0)$. In view of (b), the functions $g_1, \dots, g_s$ are locally bounded in D. Thus, by the Riemann theorem, they extend holomorphically to D. In particular, the function P extends to $D \times \mathbb{C}$. Let $S_0 := \{(z,w) \in D \times \mathbb{C} : P(z,w) = 0\}$. We have $S \cap (D_0 \times \mathbb{C}) = S_0 \cap (D_0 \times \mathbb{C})$. It remains to show that $S = S_0$. Since A_{s-1} is pluripolar, for each point $(a_0, b_0) \in S_0$ there exists a sequence $((a_k, b_k))_{k=1}^{\infty} \subset S_0 \cap (D_0 \times \mathbb{C})$ with $(a_k, b_k) \to (a_0, b_0)$. Thus $S_0 \subset S$. Conversely, let $(a_0, b_0) \in S \cap (A_{s-1} \times \mathbb{C})$. It suffices to find a sequence $((a_k, b_k))_{k=1}^{\infty} \subset S \cap (D_0 \times \mathbb{C})$ with $(a_k, b_k) \to (a_0, b_0)$. First, by the same methods as above, we find $0 < \delta' < \delta$ and $r > 0$ such that $\overline{\mathbb{D}}(b_0, \delta) \cap S_{(a_0,\cdot)} = \{b_0\}$ and $S \cap (\mathbb{P}(a_0, r) \times \overline{\mathbb{A}}(b_0, \delta', \delta)) = \varnothing$. Let $a \in D_0 \cap \mathbb{P}(a_0, r)$. We only need to observe that $\overline{\mathbb{D}}(b_0, \delta) \cap S_{(a,\cdot)} \neq \varnothing$ because otherwise, using the above argument, we would conclude that every function $f \in \mathcal{O}((\mathbb{P}(a_0, r) \times \mathbb{D}(b_0, \delta)) \setminus S)$ extends to $\mathbb{P}(a_0, r) \times \mathbb{D}(b_0, \delta)$. □

Corollary 9.2.20. *Let $D \subset \mathbb{C}^p$ be a domain of holomorphy and let $\varnothing \neq S \subset D \times \mathbb{C}$ be a pseudoconcave set such that*

(a) *$S \cap (D \times \Delta_0) = \varnothing$, where $\Delta_0 \subset \mathbb{C}$ is a domain with $0 \in \Delta_0$,*

(b) *the set $\{a \in D : \#S_{(a,\cdot)} < +\infty\}$ is not pluripolar.*

Then there exist $t \in \mathbb{N}$ and $h_1, \dots, h_t \in \mathcal{O}(D)$ such that

$$S = \Big\{(z,w) \in D \times \mathbb{C} : \textstyle\sum_{j=1}^{t} h_j(z) w^j = 1\Big\}.$$

In particular, if the set $\{a \in D : \#S_{(a,\cdot)} = 1\}$ *is not pluripolar, then*

$$S = \{(z, w) \in D \times \mathbb{C} : h(z)w = 1\}$$

with $h \in \mathcal{O}(D)$, $h \not\equiv 0$.

Proof. Let $S_1 := \{(z, 1/w) : (z, w) \in S\}$. Notice that S_1 is pseudoconcave in $D \times \mathbb{C}_*$. Consequently, the set $S_0 := S_1 \cup (D \times \{0\})$ is pseudoconcave in $D \times \mathbb{C}$. It is clear that S_0 satisfies both assumptions of Theorem 9.2.19. Thus there exist $s \in \mathbb{N}$, $s \geq 2$, and $g_1, \dots, g_{s-1} \in \mathcal{O}(D)$ such that

$$S_0 = \Big\{(z, w) \in D \times \mathbb{C} : w^s + \textstyle\sum_{j=1}^{s-1} g_j(z)w^{s-j} = 0\Big\},$$

which, after the inverse transformation, gives

$$S = \Big\{(z, w) \in D \times \mathbb{C} : \textstyle\sum_{j=1}^{s-1} g_j(z)w^j = -1\Big\}. \qquad \square$$

Remark 9.2.21. (1) Notice that in [Nis 2001] (Theorem 4.9) the Oka–Nishino theorem is formulated under weaker assumptions (a′)+(b), where

(a′) for each $a \in D$ the fiber $S_{(a,\cdot)}$ is bounded.

Unfortunately, the proof presented in [Nis 2001] contains gaps.

(2) Moreover, Corollary 4.1 in [Nis 2001] (which follows from Theorem 4.9) says that if for each $a \in D$ the fiber $S_{(a,\cdot)}$ is bounded and the set $\{a \in D : \#S_{(a,\cdot)} = 1\}$ is not pluripolar, then S is a holomorphic graph over D.

Observe that this result is false – as a counterexample one can take $D := \mathbb{C}$ and the pseudoconcave set

$$S := \{(z, w) \in \mathbb{C} \times \mathbb{C} : zw = 1\} \subset \mathbb{C} \times \mathbb{C},$$

which obviously is not a graph over the whole $\mathbb{C}$.

9.2.3 Chirka–Sadullaev theorem

The following deep result of Sadullaev will play an important role in the sequel.

Theorem* 9.2.22 ([Sad 1982], see the proof of Theorem 1). *Let* $M \subset \mathbb{D}^p \times \mathbb{D}$ *be a pseudoconcave set with* $M \cap (\mathbb{D}^p \times \mathbb{A}(1-\varepsilon, 1)) = \varnothing$ *(for some* $0 < \varepsilon < 1$*). If for almost all* $z' \in \mathbb{D}^p$ *the fiber* $M_{(z',\cdot)}$ *is polar, then* M *is a pluripolar set.*

In fact we will use the following corollary of the above result.

Theorem 9.2.23. *Let* $M \subset \mathbb{D}^p \times \mathbb{C}$ *be a pseudoconcave set with* $M \cap \mathbb{D}^{p+1} = \varnothing$. *If for almost all* $z' \in \mathbb{D}^p$ *the fiber* $M_{(z',\cdot)}$ *is polar, then* M *is a pluripolar set.*

Proof. The result has local character. Take a point $(a, b) \in M$. Then, by Proposition 2.3.21 there exists an $R > |b|$ such that $M_{(a,\cdot)} \cap \partial\mathbb{D}(R) = \varnothing$. Consequently, $(\mathbb{P}(a, r) \times \mathbb{A}(R-\varepsilon, R+\varepsilon)) \cap M = \varnothing$ and we are locally in the situation of Theorem 9.2.22. $\square$

Notice that Theorem 9.2.22 has been generalized in [Sad 1984].

Finally we arrive at the main result in this section dealing with "extension" of singularities.

Theorem 9.2.24 (Cf. [Chi-Sad 1987]). *Let $D \subset \mathbb{C}^p$ be a domain of holomorphy, let $A \subset D$ be non-pluripolar, and let $\varnothing \neq \Delta_0 \subset \mathbb{C}$ be a domain. For $a \in A$, let $M(a) \subset \mathbb{C}$ be a closed polar set with $\Delta_0 \cap M(a) = \varnothing$. Define*

$$\mathcal{S} := \{f \in \mathcal{O}(D \times \Delta_0) : \forall_{a \in A}\ \exists_{\tilde{f}_a \in \mathcal{O}(\mathbb{C} \setminus M(a))} : \ \tilde{f}_a(w) = f(a, w),\ w \in \Delta_0\}.$$

Then the $\mathcal{S}$-envelope of holomorphy of $D \times \Delta_0$ is of the form $(D \times \mathbb{C}) \setminus \widehat{M}$, where $\widehat{M}$ is a relatively closed pluripolar set such that

- *$\widehat{M}_{(a,\cdot)}$ is polar, $a \in D$,*
- *$\widehat{M}_{(a,\cdot)} \subset M(a)$, $a \in A$,*
- *if all the sets $M(a)$, $a \in A$, are discrete, then $\widehat{M}$ is analytic.*

Proof. Define

$$\mathcal{S}_a := \{f(a, \cdot) : f \in \mathcal{S}\}, \quad a \in D.$$

Theorem 9.2.7 implies that $\bigcup_{a \in A} \mathcal{S}_a \subset \boldsymbol{R}^0$. Consequently, by Theorem 9.2.16, $\bigcup_{a \in D} \mathcal{S}_a \subset \boldsymbol{R}^0$. Hence, by Theorem 9.2.5 (a), for any $g \in \mathcal{S}_a$, $a \in D$, the domain of existence $\widehat{G}_g \subset \mathbb{C}$ of g is univalent. Let $\widehat{G}_a$ denote the connected component of the open set $\operatorname{int} \bigcap_{g \in \mathcal{S}_a} \widehat{G}_g$ that contains Δ_0. Then $\widehat{G}_a$ is the $\mathcal{S}_a$-envelope of holomorphy of Δ_0 (cf. [Jar-Pfl 2000], Proposition 1.8.3). Observe that $\mathbb{C} \setminus M(a) \subset \widehat{G}_a$, $a \in A$. Put $B := \bigcup_{a \in D} \{a\} \times \widehat{G}_a$.

Let

$$\varphi \colon (D \times \Delta_0, \mathrm{id}) \to (X, p)$$

be the maximal $\mathcal{S}$-extension. Put $p = (u, v) \colon X \to \mathbb{C}^p \times \mathbb{C}$. Since $p \circ \varphi = \mathrm{id}$, we get $D \times \Delta_0 \subset p(X)$. Since $\varphi \colon (D \times \Delta_0, \mathrm{id}) \to (X, p)$ is also an $\mathcal{O}(D \times \mathbb{C})|_{D \times \Delta_0}$-extension, we get $p(X) \subset D \times \Delta_0$. In particular, $u(X) = D$. Since (X, p) is an $\mathcal{S}^\varphi$-domain of holomorphy, Theorem 9.1.2 implies that there exists a pluripolar set $P \subset D$ such that (X_a, p_a) is an $(\mathcal{S}^\varphi)_a$-region of holomorphy for every $a \in D \setminus P$. Observe that $\varphi(a, \cdot) \colon (\Delta_0, \mathrm{id}) \to (X_a, p_a)$ is a morphism. Let X_a^o be the connected component of X_a that contains $\varphi(\{a\} \times \Delta_0)$. Then $\varphi(a, \cdot) \colon (\Delta_0, \mathrm{id}) \to (X_a^o, p_a)$ is an $\mathcal{S}_a$-extension and $(\mathcal{S}_a)^{\varphi(a,\cdot)} = (\mathcal{S}^\varphi)_a|_{X_a^o}$. Hence, $p_a(X_a^o) \subset \widehat{G}_a$, $a \in D$. Put $Y := \bigcup_{a \in D} X_a^o$. Observe that Y is open. We have proved that $p(Y) \subset B$. Moreover, since $\widehat{G}_a$ is the $\mathcal{S}_a$-envelope of holomorphy of Δ_0, we conclude that $(X_a^o, p_a) \simeq (\widehat{G}_a, \mathrm{id})$, $a \in D \setminus P$. In particular, $p_a(X_a^o) = \widehat{G}_a$, $a \in D \setminus P$. Thus, p is injective on $Y_0 := \bigcup_{a \in D \setminus P} X_a^o$ and $p(Y_0) = \bigcup_{a \in D \setminus P} \{a\} \times \widehat{G}_a =: B_0$. Since P is nowhere dense, we easily conclude that p is injective on the whole of Y.

Take an $a_0 \in D$ and a $g \in \mathcal{O}(p(X_{a_0}^o))$. Since X is Stein and $X_{a_0}^o$ is a connected component of the analytic set $X_{a_0} = \{x \in X : u(x) = a_0\}$, the function $g \circ p$ extends to a $\tilde{g} \in \mathcal{O}(X)$ (cf. [Jar-Pfl 2000], Proposition 2.5.10).

Since $p_a(X_a^o) = \widehat{G}_a \supset \mathbb{C} \setminus M(a)$ for $a \in A \setminus P$, Theorems 9.2.5 and 9.2.16 imply (as at the beginning of the present proof) that $\tilde{g} \circ (p|_Y)^{-1}(a, \cdot) \in \boldsymbol{R}^0$, $a \in D$. In particular, $g = \tilde{g} \circ (p_Y)^{-1}(a_0, \cdot) \in \boldsymbol{R}^0$. Consequently, by Theorem 9.2.14, $\mathbb{C} \setminus p(X_a^o)$ is polar, $a \in D$.

Let $\widetilde{G} := p(Y)$, $\widetilde{M} := (D \times \mathbb{C}) \setminus \widetilde{G}$. We know that

- $D \times \Delta_0 \subset \widetilde{G}$, $B_0 \subset \widetilde{G} \subset B$,
- every function $f \in \mathcal{S}$ extends to the function $\tilde{f} := f^{\varphi} \circ (p|_Y)^{-1} \in \mathcal{O}(\widetilde{G})$,
- every fiber $\widetilde{M}_{(a,\cdot)}$ is polar, $a \in D$.

In particular, $\widetilde{M}$ does not separate domains.

Indeed, suppose that $(U \times V) \setminus \widetilde{M}$ is disconnected for a domain $U \times V \subset D \times \mathbb{C}$. Write $(U \times V) \setminus \widetilde{M} = \Omega_1 \cup \Omega_2$, where Ω_1, Ω_2 are open non-empty and disjoint. Put $W_j := \{a \in U : (V \setminus \widetilde{M}_{(a,\cdot)}) \cap (\Omega_j)_{(a,\cdot)} \neq \varnothing\}$, $j = 1, 2$. Observe that W_1, W_2 are open and non-empty. Since $V \setminus \widetilde{M}_{(a,\cdot)}$ is connected for every $a \in U$, we conclude that $W_1 \cap W_2 = \varnothing$; a contradiction.

Let $\widehat{M} := \widetilde{M}_{s,\tilde{\mathcal{S}}}$ with $\tilde{\mathcal{S}} := \{\tilde{f} : f \in \mathcal{S}\}$ (cf. Remark 2.4.3). We have obtained a relatively closed subset such that

- every function $f \in \mathcal{S}$ extends to an $\hat{f} \in \mathcal{O}((D \times \mathbb{C}) \setminus \widehat{M})$,
- $(D \times \mathbb{C}) \setminus \widehat{M}$ is an $\{\hat{f} : f \in \mathcal{S}\}$-domain of holomorphy (in particular, $\widehat{M}$ is pseudoconcave),
- $\widehat{M}_{(a,\cdot)} \subset \widetilde{M}_{(a,\cdot)}$ is polar, $a \in D$,
- $\widehat{M}_{(a,\cdot)} \subset \widetilde{M}_{(a,\cdot)} \subset M(a)$, $a \in A$.

It remains to use Theorem 9.2.23 to prove that $\widehat{M}$ is pluripolar.

We move to the case where all the fibers $\widehat{M}_{(a,\cdot)}$, $a \in A$, are discrete. We may assume that $0 \in \Delta_0$ and let $\rho_0 > 0$ be such that $\mathbb{D}(\rho_0) \subset \Delta_0$. It suffices to show that $\widehat{M} \cap (D \times \mathbb{D}(R_0))$ is analytic for arbitrary $R_0 > 0$. Fix an $R_0 > \rho_0$ and let

$$D_0 := \{z \in D : \widehat{M} \cap (U \times \mathbb{D}(R_0)) \text{ is analytic for an open neighborhood } U \text{ of } z\}.$$

Obviously, D_0 is open. First we check that it is non-empty. Take an $a \in A^*$ (cf. Definition 3.2.8). In particular, $A \cap U$ is not pluripolar for any arbitrary open neighborhood U of a. Since $\widehat{M}_{(a,\cdot)}$ is discrete, there exists an $R > R_0$ such that $\widehat{M}_{(a,\cdot)} \cap \partial\mathbb{D}(R) = \varnothing$. Since $\widehat{M}$ is relatively closed, there exists an $r > 0$ such that $\mathbb{P}(a, r) \subset\subset D$ and $\widehat{M} \cap (\mathbb{P}(a, r) \times \partial\mathbb{D}(R)) = \varnothing$. Observe that $A \cap \mathbb{P}(a, r) \subset \{z \in \mathbb{P}(a, r) : \widehat{M}_{(z,\cdot)} \text{ is finite}\}$. Consequently, by Theorem 9.2.19, applied to $S := \widehat{M} \cap (\mathbb{P}(a, r) \times \mathbb{D}(R))$, we conclude $\widehat{M} \cap (\mathbb{P}(a, r) \times \mathbb{D}(R))$ is analytic. In particular, $\widehat{M} \cap (\mathbb{P}(a, r) \times \mathbb{D}(R_0))$ is analytic.

Now we check that D_0 is closed in D. Take an accumulation point $a \in D$ of D_0. Recall that the set $C := \widehat{M}_{(a,\cdot)}$ is polar. Let $C' := \{1/w : w \in C\}$. Then $C' \subset \mathbb{D}(1/\rho_0)$ is also polar and, therefore there exists an $R > R_0$ such that $C' \cap \partial\mathbb{D}(1/R) = \varnothing$ (cf. Proposition 2.3.21). In other words, $\widehat{M}_{(a,\cdot)} \cap \partial\mathbb{D}(R) = \varnothing$. Then, as above, there exists an $r > 0$ such that $\mathbb{P}(a, r) \subset\subset D$ and $\widehat{M} \cap (\mathbb{P}(a, r) \times \partial\mathbb{D}(R)) = \varnothing$. Observe that

$D_0 \cap \mathbb{P}(a,r)$ is open, non-empty, and $D_0 \cap \mathbb{P}(a,r) \subset \{z \in \mathbb{P}(a,r) : \widehat{M}_{(z,\cdot)}$ is finite$\}$. Thus, using once again Theorem 9.2.19, we conclude that $\widehat{M} \cap (\mathbb{P}(a,r) \times \mathbb{D}(R_0))$ is analytic.

Finally, $D_0 = D$ and the proof is completed. □

9.3 Separately pluriharmonic functions II

Theorem 9.2.23 implies the following extension theorem for pluriharmonic functions.

Proposition 9.3.1 (Cf. [Sad 2005], [Sad-Imo 2006b]). *Let $D \subset \mathbb{C}^p$ be a domain of holomorphy, let $A \subset D$ be non-pluripolar, and let $\Delta_0 \subset \mathbb{C}$ be a domain with $0 \in \Delta_0$. For $a \in A$, let $M(a) \subset \mathbb{C}$, $\#M(a) \leq 1$, $\Delta_0 \cap M(a) = \varnothing$. Define*

$$\mathcal{F} := \{u \in \mathcal{PH}(D \times \Delta_0) : \forall_{a \in A}\ \exists_{\tilde{u}_a \in \mathcal{H}(\mathbb{C} \setminus M(a))} : \ \tilde{u}_a(w) = u(a,w),\ w \in \Delta_0\}.$$

Then there exists an analytic set $\widehat{M} \subset D \times \mathbb{C}$ such that

- *$\widehat{M}_{(a,\cdot)} \subset M(a)$, $a \in A$,*
- *$\widehat{M} = \{(z,w) \in D \times \mathbb{C} : h(z)w = 1\}$, where $h \in \mathcal{O}(D)$,*
- *every function $u \in \mathcal{F}$ extends to a multivalued pluriharmonic function $\hat{u}$ on $(D \times \mathbb{C}) \setminus \widehat{M}$.*

Moreover, in the case when $(*)$ A is not contained in any set of the form $\bigcup_{k=1}^{\infty} A_k$, where $A_k = \{z \in U_k : \varphi_k(z) = 0\}$, $U_k \subset D$ a domain, $\varphi_k \in \mathcal{PH}(U_k)$, $\varphi_k \not\equiv 0$, the above extension $\hat{u}$ is univalent.

The result will be partially extended in Proposition 10.5.1.

Proof. Let $f := \frac{\partial u}{\partial w}$. Then $f \in \mathcal{O}(D \times \Delta_0)$ and for each $a \in A$ the function $\tilde{f}_a := \frac{\partial \tilde{u}_a}{\partial w}$ gives a holomorphic extension of $f(a,\cdot)$ to $\mathbb{C} \setminus M(a)$. Thus, we may apply Theorem 9.2.24 and we get an analytic set $\widehat{M} \subset D \times \mathbb{C}$ such that

- $\widehat{M}_{(a,\cdot)} \subset M(a)$, $a \in A$,
- f extends to an $\hat{f} \in \mathcal{O}((D \times \mathbb{C}) \setminus \widehat{M})$.

Define

$$F(z,w) := \int_0^w \hat{f}(z,\zeta)d\zeta, \quad (z,w) \in (D \times \mathbb{C}) \setminus \widehat{M},$$

where the integration is over an arbitrary piecewise $\mathcal{C}^1$ curve $\gamma\colon [0,1] \to \mathbb{C} \setminus \widehat{M}_{(z,\cdot)}$ with $\gamma(0) = 0$, $\gamma(1) = w$. Then F defines a multivalued holomorphic function in $(D \times \mathbb{C}) \setminus \widehat{M}$. Thus $\hat{u}(z,w) := 2\,\mathrm{Re}(F(z,w)) + u(z,0)$ defines a multivalued pluriharmonic function on $(D \times \mathbb{C}) \setminus \widehat{M}$. Observe that if $a \in A$, then

$$\begin{aligned}\hat{u}(a,w) &= 2\,\mathrm{Re}\Big(\int_0^w \frac{\partial \tilde{u}_a}{\partial w}(\zeta)d\zeta\Big) + u(a,0)\\ &\overset{\zeta=\xi+i\eta}{=} \mathrm{Re}\Big(\int_0^w \Big(\frac{\partial \tilde{u}_a}{\partial \xi}d\xi + \frac{\partial \tilde{u}_a}{\partial \eta}d\eta\Big) - i\int_0^w \Big(\frac{\partial \tilde{u}_a}{\partial \eta}d\xi - \frac{\partial \tilde{u}_a}{\partial \xi}d\eta\Big)\Big)\\ &\qquad + u(a,0) = \tilde{u}_a(w).\end{aligned}$$

Consequently, $\hat{u}$ is an extension of u.

Let $P := \{a \in D : \widehat{M}_{(a,\cdot)} = \varnothing\}$. Note that if $a \in P$, then the function $\hat{u}(a,\cdot)$ is univalent. In particular, if $\widehat{M} = \varnothing$, then $\hat{u}$ is univalent. Thus, we may assume that $\widehat{M} \neq \varnothing$.

Using Corollary 9.2.20, we easily conclude that

$$\widehat{M} = \{(z,w) \in D \times \mathbb{C} : h(z)w = 1\}$$

with $h \in \mathcal{O}(D)$, $h \not\equiv 0$. Consequently, $P = h^{-1}(0)$ is an analytic subset of D. In particular, $D_0 := D \setminus P$ is a domain. Put $\sigma := 1/h$ on D_0,

$$\varphi(z) := \mathrm{Re}\Big(\int_{|\zeta-\sigma(z)|=\delta} \hat{f}(z,\zeta)d\zeta\Big) = \mathrm{Re}\Big(\int_{|\zeta|=\delta} \hat{f}(z,\sigma(z)+\zeta)d\zeta\Big), \quad z \in D_0.$$

Observe that

- if $a \in D_0$, then the function $\hat{u}(a,\cdot)$ is univalent iff $\varphi(a) = 0$; in particular, $\varphi = 0$ on $A \setminus P$,
- $\varphi \in \mathcal{PH}(D_0)$.

Thus, if A satisfies $(*)$, then we must have $\varphi \equiv 0$, which implies that $\hat{u}$ is univalent. □

Example 9.3.2 ([Sad 2005]). Put in Proposition 9.3.1, $D := \mathbb{D}$, $A := (-1,1) \subset \mathbb{R}$, $M(a) := \{-1\}$, $a \in (-1,1)$, $u(z,w) := \mathrm{Re}(z\,\mathrm{Log}(w+1))$, $(z,w) \in \mathbb{D} \times \mathbb{D}$ (where Log stands for the principal branch of logarithm). Observe that if $a \in (-1,1)$, then $u(a,w) = a\log|w+1|$, $w \in \mathbb{C} \setminus \{-1\}$. Thus u satisfies all the assumptions of Proposition 9.3.1 (cf. Example 3.2.20 (c)). Notice that A is the set of zeroes of the harmonic function $\mathbb{D} \ni z \mapsto \mathrm{Im}\, z$.

Taking $\widehat{M} := \mathbb{D} \times \{-1\}$ we see that $\hat{u}(z,w) = \mathrm{Re}(z\log(w+1))$, $(z,w) \in (\mathbb{D} \times \mathbb{C}) \setminus \widehat{M}$, is not univalent.

9.4 Grauert–Remmert, Dloussky, and Chirka theorems

> §§ 2.1, 2.3, 2.4, 2.5, 2.7, 2.8, 9.2.3.

The following extension theorems with singularities, which are nowadays standard tools in complex analysis, will play an important role in the discussion on cross theorems with singularities.

Theorem 9.4.1. *Let (X, p) be a Riemann domain over $\mathbb{C}^n$ such that $\mathcal{O}(X)$ separates points in X and let $(\widehat{X}, \hat{p})$ be its envelope of holomorphy. We assume that X is a subdomain of $\widehat{X}$; in particular, $\hat{p}|_X = p$.*

(a) (Grauert–Remmert; [Gra-Rem 1956]) *Let $M \subset \widehat{X}$ be an analytic subset of pure dimension $(n-1)$. Then $\widehat{X} \setminus M$ is the envelope of holomorphy of $X \setminus M$.*

(b) (Dloussky; [Dlo 1977], see also [Por 2002]) *Let $M \subset X$ be a relatively closed thin subset. Then there exists an analytic subset $\widehat{M}$ of $\widehat{X}$ such that $\widehat{M} \cap X \subset M$ and $\widehat{X} \setminus \widehat{M}$ is the envelope of holomorphy of $X \setminus M$.*

Roughly speaking, the above results say that if $M \subset X$ is analytic, then $\widehat{X \setminus M} = \widehat{X} \setminus \widehat{M}$ where $\widehat{M} \subset \widehat{X}$ is analytic and $\widehat{M} \cap X \subset M$. Observe that in general $\widehat{M} \cap X \subsetneq M$, e.g. if M is analytic and $\dim M \le n-2$, then $\widehat{M} = \varnothing$ (cf. [Jar-Pfl 2008], Propositions 1.9.11, 1.9.14).

Proofs for (a) and (b) may be found in [Jar-Pfl 2000]. Here we present a proof for (b) given by Porten (see [Por 2002]). We emphasize that (b) is also true in the case when $\mathcal{O}(X)$ does not separate points of X.

Proof. (a) For the proof see the one of Theorem 2.5.9 in [Jar-Pfl 2000]).

(b) From the very beginning we may assume that M is an analytic set of pure dimension $(n-1)$ (use Remark 2.4.3 (e) and Proposition 2.4.6) and that all points of M are singular with respect to $\mathcal{O}(X \setminus M)$ (see Proposition 2.4.4), i.e. $M = M_{s,\mathcal{O}(X \setminus M)}$.

To prove (b) we have to find "an extension" of M as an analytic subset $\widehat{M}$ of pure dimension $(n-1)$ of $\widehat{X}$ and to apply (a). In fact, let

$$\alpha\colon (X \setminus M, p|_{X \setminus M}) \to (W, q)$$

be the envelope of holomorphy of $X \setminus M$. Recall that α is injective. Obviously, $d_{X \setminus M} \le d_W \circ \alpha$ and therefore, $\mathcal{F} := \mathcal{O}^{(6n+1)}(W) \circ \alpha \subset \mathcal{O}^{(6n+1)}(X \setminus M)$ (cf. Definition 2.1.8). Applying that W is an $\mathcal{O}^{(6n+1)}(W)$-domain of holomorphy (see Theorem 2.5.10) and that $\mathcal{O}^{(6n+1)}(W)$ is a natural Fréchet space of holomorphic functions, there exists a $g \in \mathcal{O}^{(6n+1)}(W)$ such that W is a $\{g\}$-domain of holomorphy (see Proposition 2.1.27 and Remark 2.1.28). Put $f_0 := g \circ \alpha$. Recall that $\mathcal{O}^{(6n+1)}(X \setminus M)$ may be thought as a subset of $\mathcal{M}(X)$, i.e. there is an $\tilde{f}_0 \in \mathcal{M}(X)$ with $\tilde{f}_0 = f$ on $X \setminus M$ (see Proposition 2.8.3). Applying Theorem 2.8.4 gives a meromorphic extension $\hat{f}_0 \in \mathcal{M}(\widehat{X})$ of $\tilde{f}_0$, i.e. $\hat{f}_0 = \tilde{f}_0$ on X. In particular, $\hat{f}_0|_{X \setminus M}$ is holomorphic. Therefore, $\mathcal{S}(\hat{f}_0) \cap X \subset M$.

Note that if $\mathcal{S}(\hat{f}_0) \cap X = M$, then $\widehat{M} := \mathcal{S}(\hat{f}_0)$ is the analytic set we are looking for. So it remains to apply (a). Assume that $\widehat{M} \cap X \subsetneq M$. Then there exists a point $x_0 \in M$ which is non-singular for $\{f_0\}$. Hence there is a connected open neighborhood $U = U(x_0) \subset X$ and f_0 extends holomorphically to the whole of U. Put $\widetilde{X} := X \cup U$ and note that $\mathrm{id}_{X \setminus M, \widetilde{X}} \colon X \setminus M \to \widetilde{X}$ is an $\{f_0\}$-extension. Observe that g is the holomorphic extension of f_0 via α. By Proposition 2.1.23, it follows that $\alpha\colon X \setminus M \to W$ is a maximal $\{f_0\}$-extension. Therefore, we find a morphism

$\psi\colon (\widetilde{X}, p|_{\widetilde{X}}) \to (W, q)$ with $\psi \circ \mathrm{id}_{X\setminus M,\widetilde{X}} = \alpha$. Fix now an arbitrary $h \in \mathcal{O}(X \setminus M)$. Take $\widehat{h} \in \mathcal{O}(W)$ with $\widehat{h} \circ \alpha = h$ and put $\tilde{h} := \widehat{h} \circ \psi$ on $\widetilde{X}$. Then $\tilde{h} \in \mathcal{O}(\widetilde{X})$ and $\tilde{h} \circ \mathrm{id}_{X\setminus M,\widetilde{X}} = \widehat{h} \circ \psi \circ \mathrm{id}_{X\setminus M,\widetilde{X}} = \widehat{h} \circ \alpha = h$, i.e. $\tilde{h}$ is a holomorphic extension of h to $\widetilde{X}$. Since h was arbitrarily chosen we get a contradiction to the fact that $M = M_{s,\mathcal{O}(X\setminus M)}$. □

A similar result is true when the exceptional set M is assumed to be pluripolar.

Theorem 9.4.2 (Chirka; [Chi 1993]). *Let (X, p) and $\widehat{X}$ be as in Theorem 9.4.1 and let $M \subset X$ be a relatively closed pluripolar set. Then there exists a relatively closed pluripolar set $\widehat{M} \subset \widehat{X}$ such that $\widehat{M} \cap X \subset M$ and $\widehat{X} \setminus \widehat{M}$ is the envelope of holomorphy of $X \setminus M$.*

To prove this theorem we have to use a local argument based on Theorem 9.2.24 as it was done in the original proof by Dloussky for Theorem 9.4.1 (see [Jar-Pfl 2000]).

The main step is the following lemma.

Lemma 9.4.3. *Put $T := (\mathbb{D}^{n-1} \times \mathbb{A}(\rho, 1)) \cup (\mathbb{P}_{n-1}(r) \times \mathbb{D})$ $(r, \rho \in (0, 1))$ and let $M \subset T$ be a relatively closed pluripolar set with $M \subset \mathbb{P}_{n-1}(r) \times \mathbb{D}(\rho)$. Then there exists a relatively closed pluripolar set $\widehat{M} \subset \mathbb{D}^n$ such that*

- *$\widehat{M} \cap T \subset M$ and $\widehat{M} \subset \mathbb{D}^{n-1} \times \mathbb{D}(\rho)$,*
- *for any $f \in \mathcal{O}(T \setminus M)$ there exists an $\hat{f} \in \mathcal{O}(\mathbb{D}^n \setminus \widehat{M})$ with $f = \hat{f}$ on $T \setminus M$,*
- *$\mathbb{D}^n \setminus \widehat{M}$ is a domain of holomorphy.*

In particular, $\mathbb{D}^n \setminus \widehat{M}$ is the envelope of holomorphy of $T \setminus M$.

Proof. We may assume that $M = M_{s,\mathcal{O}(T\setminus M)}$. Put

$$\widetilde{M} = \{(z', z_n) \in \mathbb{P}_{n-1}(r) \times \mathbb{C}_* : (z', 1/z_n) \in M\}.$$

Then $\widetilde{M} \in \mathcal{PLP}$ and $\widetilde{M} \cap (\mathbb{D}^{n-1} \times \mathbb{D}(1/\rho)) = \varnothing$. Put

$$B := \{z' \in \mathbb{P}_{n-1}(r) : \widetilde{M}_{(z',\cdot)} \notin \mathcal{PLP}\}.$$

Proposition 2.3.31 (a) implies that $B \in \mathcal{PLP}$. Finally, set $A := \mathbb{P}_{n-1}(r) \setminus B$. Then A is locally pluriregular. Applying Theorem 9.2.24 (with $D = \mathbb{D}^{n-1}$, $\varDelta_0 = \mathbb{D}(1/\rho)$, $M(a) = \widetilde{M}_{(a,0)}$, $a \in A$) leads to a relatively closed pluripolar set $\widehat{\widetilde{M}}$ in $\mathbb{D}^{n-1} \times \mathbb{C}$ such that

- $\widehat{\widetilde{M}}_{(a,\cdot)}$ is pluripolar, $a \in \mathbb{D}^{n-1}$,
- $\widehat{\widetilde{M}}_{(a,\cdot)} \subset \widetilde{M}_{(a,\cdot)}$, $a \in A$,
- $\widehat{\widetilde{M}} \cap (\mathbb{D}^{n-1} \times \mathbb{D}(1/\rho)) = \varnothing$,
- $(\mathbb{D}^{n-1} \times \mathbb{C}) \setminus \widehat{\widetilde{M}}$ is a domain of holomorphy,

- $(*)$ for every $h \in \mathcal{O}(\mathbb{D}^{n-1} \times \mathbb{D}(1/\rho))$ such that $h(a,\cdot)$ extends holomorphically to $\mathbb{C} \setminus \widetilde{M}_{(a,\cdot)}$, $a \in A$, there exists an $\widehat{h} \in \mathcal{O}((\mathbb{D}^{n-1} \times \mathbb{C}) \setminus \widehat{\widetilde{M}})$ such that $\widehat{h} = h$ on $\mathbb{D}^{n-1} \times \mathbb{D}(1/\rho)$.

Now fix an $f \in \mathcal{O}(T \setminus M)$. Note that $f(z',\cdot) \in \mathcal{O}(\mathbb{A}(\rho,1))$, $z' \in \mathbb{D}^{n-1}$. Therefore, f can be written as its Laurent expansion

$$f(z',z_n) = \sum_{j\in\mathbb{Z}} a_j(z')z_n^j = \sum_{j\in\mathbb{Z}_+} a_j(z')z_n^j + \sum_{j\in\mathbb{Z}\setminus\mathbb{Z}_+} a_j(z')z_n^j =: f^+(z) + f^-(z).$$

Note that $f^+ \in \mathcal{O}(\mathbb{D}^n)$ and $f^- \in \mathcal{O}(\mathbb{D}^{n-1} \times \mathbb{A}(\rho,\infty))$. Moreover, f^- is bounded at infinity. Hence, $f^- = f - f^+$ on $\mathbb{D}^{n-1} \times \mathbb{A}(\rho,1)$. Let

$$\tilde{f}(z) := \begin{cases} f(z) - f^+(z) & \text{if } z \in (\mathbb{P}_{n-1}(r) \times \mathbb{D}) \setminus M, \\ f^-(z) & \text{if } z \in \mathbb{D}^{n-1} \times \mathbb{A}(\rho,\infty). \end{cases}$$

Note that $\tilde{f} \in \mathcal{O}(((\mathbb{P}_{n-1}(r) \times \mathbb{C}) \setminus M) \cup (\mathbb{D}^{n-1} \times \mathbb{A}(\rho,\infty)))$.

Set $g(z) =: \tilde{f}(z', 1/z_n)$ on $\mathbb{D}^{n-1} \times \mathbb{A}(0,1/\rho)$. Then $g \in \mathcal{O}(\mathbb{D}^{n-1} \times \mathbb{A}(0,1/\rho))$. Observe that g is bounded along $\mathbb{D}^{n-1} \times \{0\}$. Thus, we may even assume that $g \in \mathcal{O}(\mathbb{D}^{n-1} \times \mathbb{D}(1/\rho))$. Summarizing, g satisfies the condition in $(*)$. Therefore, there exists a $\widehat{g} \in \mathcal{O}((\mathbb{D}^{n-1}\times\mathbb{C})\setminus\widehat{\widetilde{M}})$ with $g = \widehat{g}$ on $\mathbb{D}^{n-1}\times\mathbb{D}(1/\rho)$ and on $\{a\}\times(\mathbb{C}\setminus\widetilde{M}_{(a,\cdot)})$, $a \in A$.

Finally, put $\hat{\tilde{f}}(z) := \widehat{g}(z', 1/z_n)$ on $(\mathbb{D}^{n-1} \times \mathbb{D}_*) \setminus \widehat{M}'$, where

$$\widehat{M}' := \{(z,z_n) \in \mathbb{D}^{n-1} \times \mathbb{D}_* : (z', 1/z_n) \in \widehat{\widetilde{M}}\}.$$

Note that $\widehat{M}'$ is a relatively closed pluripolar subset of $\mathbb{D}^{n-1}\times\mathbb{D}_*$, that $(\mathbb{D}^{n-1}\times\mathbb{D}_*)\setminus\widehat{M}'$ is a domain of holomorphy, and that $\widehat{M}'_{(a,\cdot)} \subset M_{(a,\cdot)}$, $a \in A$. Moreover, $\hat{\tilde{f}} = \tilde{f}$ on $\mathbb{D}^{n-1} \times \mathbb{A}(\rho,1)$ and $\hat{\tilde{f}}(a,\cdot) = \tilde{f}(a,\cdot)$ on $\mathbb{D}_* \setminus M_{(a,\cdot)}$, $a \in A$.

Set $\hat{f} := \hat{\tilde{f}} + f^+$. Then $\hat{f} \in \mathcal{O}((\mathbb{D}^{n-1} \times \mathbb{D}_*) \setminus \widehat{M}')$. Observe that $\hat{f} = f$ on $\mathbb{D}^{n-1} \times \mathbb{A}(\rho,1)$ and $\hat{f}(a,\cdot) = f(a,\cdot)$ on $\mathbb{D}_* \setminus M_{(a,\cdot)}$ whenever $a \in A$. So we have assigned to any $f \in \mathcal{O}(T \setminus M)$ a new function $\hat{f} \in \mathcal{O}((\mathbb{D}^{n-1} \times \mathbb{D}_*) \setminus \widehat{M}')$ with all the properties described just before. Then we define $\widehat{M} := (\widehat{M}' \cup (\mathbb{D}^{n-1} \times \{0\}))_{s,\mathcal{F}}$, where $\mathcal{F} := \{\hat{f} : f \in \mathcal{O}(T \setminus M)\}$.

It remains to show that $\widehat{M}_{(a,0)} \subset M_{(a,0)}$, $a \in \mathbb{P}_{n-1}(r)$. Otherwise one may find a point $z^0 = (a^0, a_n^0) \in \widehat{M} \setminus M$ with $a^0 \in \mathbb{P}_{n-1}(r)$. Then, since M is relatively closed, $\mathbb{P}_n(z^0,\varepsilon) \subset (\mathbb{P}_{n-1}(r) \times \mathbb{D}) \setminus M$ for a small ε. Note that f and $\hat{f}$ are holomorphic on the domain $G := \mathbb{P}_n(z^0,\varepsilon) \setminus \widehat{M}$. Moreover, they coincide on the non-pluripolar set $G \cap (A \times \mathbb{D}_*)$. Therefore, $\hat{f} = f$ on $\mathbb{P}_n(z^0,\varepsilon) \setminus \widehat{M}$, which contradicts that $\widehat{M}$ is singular for $\mathcal{F}$. Hence, $\widehat{M} \cap (\mathbb{P}_{n-1} \times \mathbb{D}) \subset M$. With a similar argument it follows that $f = \hat{f}$ on $\mathbb{P}_{n-1}(r) \setminus M$. □

Lemma 9.4.4. *Let T be as in Lemma 9.4.3 and let $M \subset T$ be relatively closed pluripolar. Then there exists a relatively closed pluripolar set $\widehat{M} \subset \mathbb{D}^n$ with*

- $\widehat{M} \cap T \subset M$,
- *for any $f \in \mathcal{O}(T \setminus M)$ there exists an $\hat{f} \in \mathcal{O}(\mathbb{D}^n \setminus \widehat{M})$ with $f = \hat{f}$ on $T \setminus M$,*
- *$\mathbb{D}^n \setminus \widehat{M}$ is a domain of holomorphy.*

In particular, $\mathbb{D}^n \setminus \widehat{M}$ is the envelope of holomorphy of $T \setminus M$.

Proof. As in the previous lemma we may assume that $M = M_{s,\mathcal{O}(T\setminus M)}$.

Fix numbers $\rho < \sigma' < \sigma'' < 1$ and put

$$B := \{z' \in \mathbb{D}^{n-1} : M_{(z',\cdot)} \cap \partial\mathbb{D}(s) \neq \varnothing \text{ for all } s \in [\sigma', \sigma'']\}.$$

Recall that $B \subset \{z' \in \mathbb{D}^{n-1} : M_{(z',\cdot)} \notin \mathcal{PLP}\}$ (see [Ran 1995], Exercise 5.3.3 (i)). Therefore, B is pluripolar (see Proposition 2.3.31 (a)). Moreover, the fact that M is closed in T implies that also B is closed in $\mathbb{D}^{n-1}$. Hence, $G' := \mathbb{D}^{n-1} \setminus B$ is a domain (see Proposition 2.3.29 (c)).

We claim that $(*)$ for any $b' \in G'$ there exist a positive number $r(b')$ with $\mathbb{P}_{n-1}(b', r(b')) \subset G'$ and a relatively closed pluripolar set $\widehat{M}_{b'} \subset \mathbb{P}_{n-1}(b', r(b')) \times \mathbb{D} =: Z_{b'}$ such that

- $\widehat{M}_{b'} \cap T \subset M$;
- for any $f \in \mathcal{O}(T \setminus M)$ there exists a uniquely defined extension $\hat{f}_{b'} \in \mathcal{O}(Z_{b'} \setminus \widehat{M}_{b'})$ such that $f = \hat{f}_{b'}$ on $(T \cap Z_{b'}) \setminus M$;
- $\widehat{M}_{b'} = (\widehat{M}_{b'})_{s,\mathcal{F}_{b'}}$, where $\mathcal{F}_{b'} := \{\hat{f}_{b'} : f \in \mathcal{O}(T \setminus M)\}$.

Obviously, there is nothing to verify when $b' \in \mathbb{P}_{n-1}(r) \cap G'$. Now fix a $b' \in G' \setminus \mathbb{P}_{n-1}(r)$ and take another point $a' \in \mathbb{P}_{n-1}(r) \setminus B$. Choose a continuous curve $\gamma\colon [0,1] \to G'$ connecting a' with b'.

Let us first take an arbitrary $t_0 \in [0,1]$. We find a number $\sigma(t_0) \in [\sigma', \sigma'']$ such that

$$M_{(\gamma(t_0),\cdot)} \cap \partial\mathbb{D}(\sigma(t_0)) = \varnothing.$$

Applying that M is closed in T there are numbers $\rho < \sigma'(t_0) < \sigma(t_0) < \sigma''(t_0) < 1$ and a neighborhood $V(\gamma(t_0)) \subset G'$ with

$$\big(V(\gamma(t_0)) \times \mathbb{A}(\sigma'(t_0), \sigma''(t_0))\big) \cap M = \varnothing.$$

Finally, by a compactness argument, we may choose points $0 = t_0 < t_1 < \cdots < t_N = 1$ and polydiscs $\mathbb{Q}_j := \mathbb{P}_{n-1}(\gamma(t_j), \tilde{r}_j) \subset V(\gamma(t_j))$ such that

- $\mathbb{Q}_0 \subset \mathbb{P}_{n-1}(r)$;
- $\bigcup_{j=0}^{N} \mathbb{Q}_j \supset \gamma([0,1])$;
- $\gamma(t_j) \in \mathbb{Q}_{j-1}, \quad j = 1, \dots, N.$

Take an $r_j > 0$ such that $V_j := \mathbb{P}_{n-1}(\gamma(t_j), r_j) \subset\subset \mathbb{Q}_{j-1} \cap \mathbb{Q}_j$, $j = 1, \dots, N$. Then, for $j = 1, \dots, N$, put

$$T_j := (V_j \times \mathbb{D}(\sigma_j'')) \cup (\mathbb{Q}_j \times \mathbb{A}(\sigma_j', \sigma_j'')),$$

where $\sigma_j' := \sigma'(\gamma(t_j))$ and $\sigma_j'' := \sigma''(\gamma(t_j))$.

We claim that for any $j = 1, \dots, N$ there is a relatively closed pluripolar set $\widehat{M}_j \subset \mathbb{Q}_j \times \mathbb{D}$ satisfying

- $\widehat{M}_j \cap T \subset M$;
- for any $f \in \mathcal{O}(T \setminus M)$ there exists a unique $\hat{f}_j \in \mathcal{O}((\mathbb{Q}_j \times \mathbb{D}) \setminus \widehat{M}_j)$ such that $f = \hat{f}_j$ on $((\mathbb{Q}_j \times \mathbb{D}) \cap T) \setminus M$;
- $(\widehat{M}_j)_{s,\mathcal{G}_j} = \widehat{M}_j$, where $\mathcal{G}_j := \{\hat{f}_j : f \in \mathcal{O}(T \setminus M)\}$.

To verify this statement we use induction. In the first step set $M_1 := T_1 \cap M$. Obviously, M_1 is a relatively closed pluripolar subset of T_1 with $M_1 \subset V_1 \times \mathbb{D}(\sigma_j')$. Therefore, we may apply Lemma 9.4.3. So we get a relatively closed set $\widetilde{M}_1 \subset \mathbb{Q}_1 \times \mathbb{D}(\sigma_1'')$, $\widetilde{M}_1 \in \mathcal{PLP}$, such that

- $\widetilde{M}_1 \cap T_1 \subset M_1$ and $\widetilde{M}_1 \subset \mathbb{Q}_1 \times \mathbb{D}(\sigma_1')$,
- any $g \in \mathcal{O}(T_1 \setminus M_1)$ extends to a uniquely defined function $\tilde{g} \in \mathcal{O}(T_1 \setminus \widetilde{M}_1)$ with $f = \tilde{g}$ on $T_1 \setminus M_1$.

In particular, if $f \in \mathcal{O}(T \setminus M)$, then $f|_{T_1 \setminus M_1}$ extends to $\tilde{f}_1 := \widetilde{f|_{T_1 \setminus M_1}} \in \mathcal{O}((\mathbb{Q}_1 \times \mathbb{D}(\sigma_1'')) \setminus \widetilde{M}_1)$ with $f = \tilde{f}_1$ on $T_1 \setminus M_1$.

Moreover, we may assume that $\widetilde{M}_1 = (\widetilde{M}_1)_{s,\mathcal{F}_1}$, where $\mathcal{F}_1 := \{\tilde{f}_1 : f \in \mathcal{O}(T \setminus M)\}$.

Let $f \in \mathcal{O}(T \setminus M)$. Note that f and $\tilde{f}_1$ are defined on the domain $G_1 := ((\mathbb{Q}_1 \times \mathbb{D}(\sigma_1'')) \cap T) \setminus (M \cup \widetilde{M}_1)$ and $f = \tilde{f}_1$ on $\mathbb{Q}_1 \times \mathbb{A}(\sigma_1', \sigma_1'')$. Then the identity theorem gives that both functions coincide on G_1. Applying that $\widetilde{M}_1 = (\widetilde{M}_1)_{s,\mathcal{F}_1}$ and $M = M_{s,\mathcal{O}(T \setminus M)}$ leads to

$$\widetilde{M}_1 \cap (\mathbb{Q}_1 \times \mathbb{D}(\sigma_1'')) \cap T = M \cap (\mathbb{Q}_1 \times \mathbb{D}(\sigma_1'')) \cap T.$$

Hence, $\tilde{f}_1 = f$ on $(\mathbb{Q}_1 \times \mathbb{D}(\sigma_1'')) \cap T \setminus M$. Put

$$\widehat{M}_1 = \widetilde{M}_1 \cup (M \cap (\mathbb{Q}_1 \times \mathbb{D})), \quad \hat{f}_1 := \begin{cases} f & \text{on } ((\mathbb{Q}_1 \times \mathbb{D}) \cap T) \setminus M, \\ \tilde{f}_1 & \text{on } (\mathbb{Q}_1 \times \mathbb{D}(\sigma_1'')) \setminus \widetilde{M}_1. \end{cases}$$

Then $\widehat{M}_1$ and $\hat{f}_1$ satisfy the condition in the above statement.

Assume now that the statement is proved for a $j < N$, i.e. there is a relatively closed pluripolar set $\widehat{M}_{j-1} \subset \mathbb{Q}_{j-1} \times \mathbb{D}$ with

- $\widehat{M}_{j-1} \cap T \subset M$,

- for any $f \in \mathcal{O}(T \setminus M)$ there exists an $\hat{f}_{j-1} \in \mathcal{O}((\mathbb{Q}_{j-1} \times \mathbb{D}) \setminus \widehat{M}_{j-1})$ with $f = \hat{f}_{j-1}$ on $(T \cap (\mathbb{Q}_{j-1} \times \mathbb{D})) \setminus M$,
- $\widehat{M}_{j-1} = (\widehat{M}_{j-1})_{s,\mathcal{F}_{j-1}}$, where $\mathcal{F}_{j-1} := \{\hat{f}_{j-1} : f \in \mathcal{O}(T \setminus M)\}$.

Observe that $\widehat{M}_{j-1} \cap T \cap (\mathbb{Q}_{j-1} \times \mathbb{D}) = M \cap T \cap (\mathbb{Q}_{j-1} \times \mathbb{D})$.

Put $M_j := \widehat{M}_{j-1} \cap T_j$. Then M_j is a relatively closed pluripolar subset of T_j. Applying again Lemma 9.4.3 we find a relatively closed pluripolar set $\widetilde{M}_j \subset \mathbb{Q}_j \times \mathbb{D}(\sigma_j'')$ such that

- $\widetilde{M}_j \cap T_j \subset M_j$ and $\widetilde{M}_j \subset \mathbb{Q}_j \times \mathbb{D}(\sigma_j')$,
- for any $g \in \mathcal{O}(T_j \setminus M_j)$ there exists a $\tilde{g} \in \mathcal{O}((\mathbb{Q}_j \times \mathbb{D}(\sigma_j'')) \setminus \widetilde{M}_j)$ such that $g = \tilde{g}$ on $T_j \setminus M_j$.

Fix an $f \in \mathcal{O}(T \setminus M)$. By induction, we know that there exists an $\hat{f}_{j-1} \in \mathcal{O}((\mathbb{Q}_{j-1} \times \mathbb{D}) \setminus \widehat{M}_{j-1})$ such that $f = \hat{f}_{j-1}$ on $(\mathbb{Q}_{j-1} \times \mathbb{D}) \setminus M$. Thus, $\hat{f}_{j-1}|_{T_j \setminus M_j} \in \mathcal{O}(T_j \setminus M_j)$. Therefore we find an $\hat{f}_j \in \mathcal{O}((\mathbb{Q}_j \times \mathbb{D}(\sigma_j'')) \setminus \widetilde{M}_j)$ such that $\tilde{f}_j$ and $\hat{f}_{j-1}$ coincide on $T_j \setminus M_j$.

Now we may assume that $\widetilde{M}_j = (\widetilde{M}_j)_{s,\mathcal{F}_j}$, where $\mathcal{F}_j := \{\tilde{f}_j : f \in \mathcal{O}(T \setminus M)\}$. Arguing as above it follows that $\widetilde{M}_j$ and M coincide in $T \cap (\mathbb{Q}_j \times \mathbb{D}(\sigma_j''))$. It remains to set

$$\widehat{M}_j := \widetilde{M}_j \cup ((\mathbb{Q}_j \times \mathbb{D}) \cap M), \quad \hat{f}_j := \begin{cases} f & \text{on } ((\mathbb{Q}_j \times \mathbb{D}) \cap T) \setminus M, \\ \tilde{f}_j & \text{on } (\mathbb{Q}_j \times \mathbb{D}(\sigma_j'')) \setminus \widetilde{M}_j. \end{cases}$$

Hence, $(*)$ is completely verified.

Now let there be two points b' and b'' in G' with $\mathbb{Q}(b', b'') := \mathbb{P}_{n-1}(b', r(b')) \cap \mathbb{P}_{n-1}(b'', r(b'')) \neq \varnothing$ and take $\widehat{M}_{b'}$ and $\widehat{M}_{b''}$ as above. Because of the fact that both sets are singular with respect to $\mathcal{F}_{b'}$ and $\mathcal{F}_{b''}$, respectively, we easily see that $\widehat{M}_{b'} = \widehat{M}_{b''}$ on $\mathbb{Q}(b', b'')$. Therefore, $\widetilde{M} := \bigcup_{b' \in G'} \widehat{M}_{b'} \subset G' \times \mathbb{D}$ is a relatively closed pluripolar set such that

- $\widetilde{M} \cap T \subset M$;
- any $f \in \mathcal{O}(T \setminus M)$ extends holomorphically as a uniquely defined $\tilde{f} \in \mathcal{O}(G' \times \mathbb{D} \setminus \widetilde{M})$.

It remains to put $\widehat{M} := (\widetilde{M} \cup (B \times \mathbb{D}))_{s,\mathcal{F}}$, where $\mathcal{F} := \{\tilde{f} : f \in \mathcal{O}(T \setminus M)\}$. □

Now we are in the position to prove Theorem 9.4.2; in fact, we will follow the argument used in the original proof of Theorem 9.4.1 (see [Jar-Pfl 2000], proof of Proposition 3.4.10).

Proof of Theorem 9.4.2. Again we may assume that $M = M_{s,\mathcal{O}(X \setminus M)}$. Let

$$\alpha\colon (X \setminus M, p) \to (Y, q)$$

denote the envelope of holomorphy of $X \setminus M$; recall that $q \circ \alpha = p$. Put $W := \alpha(X \setminus M)$; W is a subdomain of Y.

By Proposition 1.9.9 from [Jar-Pfl 2000] there exists the lifting of the identity morphism $\mathrm{id}\colon (X \setminus M, p) \to (\widehat{X}, \widehat{p})$ to a morphism $\varphi\colon (Y, q) \to (\widehat{X}, \widehat{p})$ with $\varphi \circ \alpha = \mathrm{id}$. In particular, $\varphi|_W\colon W \to V := X \setminus M \subset \widehat{X}$ is a biholomorphic mapping from W onto V.

With respect to φ we have (see Section 2.6)

- the Hausdorff space $\overset{=\varphi}{Y}$ and the continuous mapping $\overset{=}{\varphi}\colon \overset{=\varphi}{Y} \to \widehat{X}$;
- the Riemann domain $(\overset{*\varphi}{Y}, \overset{=\varphi}{q}|_{\overset{*\varphi}{Y}})$, where $\overset{=\varphi}{q} := \widehat{p} \circ \overset{=}{\varphi}$ and $\overset{*\varphi}{Y} = Y \cup \Sigma$;
- the morphism $\overset{=}{\varphi}|_{\overset{*\varphi}{Y}}\colon (\overset{*\varphi}{Y}, \overset{=\varphi}{q}|_{\overset{*\varphi}{Y}}) \to (\widehat{X}, \widehat{p})$.

Note that $\varphi(W) = X \setminus M$, M is relatively closed pluripolar, and

$$\varphi\colon W \to X \setminus M$$

is a biholomorphic map.

Then Proposition 2.6.4 implies the existence of an open set $U \subset \overset{*\varphi}{Y}$ such that $W \subset U$ and $\overset{=}{\varphi}\colon U \to X$ is biholomorphic. Thus $\psi := (\overset{=}{\varphi}|_U)^{-1}\colon (X, p) \to (\overset{*\varphi}{Y}, \overset{=\varphi}{q})$ is a holomorphic extension of X.

For a moment assume that $(\overset{*\varphi}{Y}, \overset{=\varphi}{q})$ is a domain of holomorphy. Then

$$\overset{=}{\varphi}\colon (\overset{*\varphi}{Y}, \overset{=\varphi}{q}) \to (\widehat{X}, \widehat{p})$$

is an isomorphism. Put $\widehat{M} := \overset{=}{\varphi}(\Sigma)$. Then $\widehat{M}$ is a closed pluripolar subset of $\widehat{X}$ and $\varphi\colon (Y, q) \to (\widehat{X} \setminus \widehat{M}, \widehat{p})$ is an isomorphism. Hence $\widehat{X} \setminus \widehat{M}$ is the envelope of holomorphy of $X \setminus M$, i.e. the theorem is proved.

So it remains to verify that $(\overset{*\varphi}{Y}, \overset{=\varphi}{q})$ is a domain of holomorphy. The proof of this statement is based on Theorem 2.7.1(v). Take the morphism

$$\overset{=}{\varphi}|_{\overset{*\varphi}{Y}}\colon (\overset{*\varphi}{Y}, \overset{=\varphi}{q}|_{\overset{*\varphi}{Y}}) \to (\widehat{X}, p)$$

and recall that $\widehat{X}$ is a domain of holomorphy. Fix r and ρ and let T be as in Lemma 9.4.3. Take a biholomorphic mapping $f\colon T \to f(T) \subset \overset{*\varphi}{Y}$ such that $g := \overset{=}{\varphi} \circ f\colon T \to \widehat{X}$ extends to a biholomorphic mapping $\widehat{g}\colon \mathbb{D}^n \to \widehat{g}(\mathbb{D}^n) \subset \widehat{X}$. We have to show that there is a holomorphic extension $\widehat{f}\colon \mathbb{D}^n \to \overset{*\varphi}{Y}$ (i.e. $\widehat{f} = f$ on T).

In fact, set $P := f^{-1}(f(T) \cap \Sigma)$. Then P is a pluripolar set relatively closed in T. By Lemma 9.4.4 there is a pluripolar set $\widehat{P} \subset \mathbb{D}^n$, relatively closed in $\mathbb{D}^n$, such that $\mathbb{D}^n \setminus \widehat{P}$ is the envelope of holomorphy of $T \setminus P$. Applying the lifting theorem

([Jar-Pfl 2000], Proposition 1.9.9) we see that f extends to a biholomorphic mapping $\tilde{f}: \mathbb{D}^n \setminus \widehat{P} \to \tilde{f}(\mathbb{D}^n \setminus \widehat{P}) \subset Y$ (recall that Y is a domain of holomorphy). Then $\varphi \circ \tilde{f}|_{T \setminus P} = \overline{\overline{\varphi}} \circ f = \widehat{g}$ on $T \setminus P$. Therefore, $\varphi : \tilde{f}(\mathbb{D}^n \setminus \widehat{P}) \to \widehat{g}(\mathbb{D}^n) \setminus \widehat{g}(\widehat{P})$ is biholomorphic. Again using Proposition 2.6.4 we find an open set $U \subset \overset{*\varphi}{Y}$ satisfying $\tilde{f}(\mathbb{D}^n \setminus \widehat{P}) \subset U$ such that $\overline{\overline{\varphi}} : U \to \widehat{g}(\mathbb{D}^n)$ is biholomorphic. Now it remains to define the extension as follows: $\hat{f} := (\overline{\overline{\varphi}}|_U)^{-1} \circ \widehat{g}$, which finishes the proof. □

Chapter 10
Cross theorem with singularities

Summary. Our main aim is to discuss a general version of the Chirka–Sadullaev Theorem 9.2.24. We find a counterpart of the Grauert–Remmert, Dloussky, and Chirka theorems (Theorems 9.4.1, 9.4.2) for crosses; most of the presented results are based on [Jar-Pfl 2001], [Jar-Pfl 2003a], [Jar-Pfl 2003b], [Jar-Pfl 2003c], [Jar-Pfl 2007], [Jar-Pfl 2011]; see also [Ale-Ama 2003].

The main result is Theorem 10.2.6 whose proof will be given in §§ 10.4, 10.6.

10.1 Öktem and Siciak theorems

The next step after Theorem 9.2.24 was taken 10 years later by Öktem who studied the following range problem in *mathematical tomography* (cf. [Ökt 1998], [Ökt 1999]).

For $\omega = (\cos\alpha, \sin\alpha)$ let $\omega^\perp := (-\sin\alpha, \cos\alpha)$. Define

$$\ell_{\omega,p} := \{x = (x_1, x_2) \in \mathbb{R}^2 : \langle x, \omega\rangle = x_1\cos\alpha + x_2\sin\alpha = p\}, \quad p \in \mathbb{R},$$

and let $\mathcal{L}_{\omega,p}$ be the Lebesgue measure on the line $\ell_{\omega,p}$. For $\mu \in \mathbb{R}_*$, the *exponential Radon transform* is given by the mapping

$$\mathcal{C}_0^\infty(\mathbb{R}^2, \mathbb{C}) \ni h \overset{R_\mu}{\mapsto} R_\mu(h), \quad R_\mu(h)\colon \mathbb{T}\times\mathbb{R} \to \mathbb{R},$$
$$R_\mu(h)(\omega, p) := \int_{\ell_{\omega,p}} h(x) e^{\mu\langle x,\omega^\perp\rangle} d\mathcal{L}_{\omega,p}(x).$$

The main problem is to recover h from $R_\mu(h)$ which is measured. So it is important to know the shape of the range of R_μ. Assume that $g = R_\mu(h)$ for some $h \in \mathcal{C}_0^\infty(\mathbb{R}^2, \mathbb{C})$. Let $\widehat{g}\colon \mathbb{T}\times\mathbb{C} \to \mathbb{C}$ be the Fourier transform of g with respect to the second variable, i.e.

$$\widehat{g}(\omega, \zeta) := \int_{\mathbb{R}} g(\omega, p) e^{-i\zeta p} d\mathcal{L}^1(p).$$

Easy calculations lead to $\widehat{g}(\omega, \zeta) = \tilde{h}(\zeta\omega + i\mu\omega^\perp)$, where $\tilde{h}$ denotes the two-dimensional Fourier transform of h. Taking $\zeta = it$, $t \in \mathbb{R}$, gives $\widehat{g}(\omega, it) = \widehat{g}(\sigma, -it)$ whenever $t\omega + \mu\omega^\perp = -t\sigma + \mu\sigma^\perp$, $\omega, \sigma \in \mathbb{T}$. It turns out that this necessary condition is, in fact, also a sufficient one.

Theorem 10.1.1 ([Ökt 1998]). *Let $g\colon \mathbb{T}\times\mathbb{R} \to \mathbb{C}$ and $\mu \neq 0$. Then the following statements are equivalent:*

(i) *there is an $h \in \mathcal{C}_0^\infty(\mathbb{R}^2, \mathbb{C})$ with $g = R_\mu(h)$;*

(ii) *$g \in \mathcal{C}_0^\infty(\mathbb{T} \times \mathbb{R}, \mathbb{C})$ and $\widehat{g}(\omega, it) = \widehat{g}(\sigma, -it)$ whenever $\omega, \sigma \in \mathbb{T}$ and $t \in \mathbb{R}$ are such that $t\omega + \mu\omega^\perp = -t\sigma + \mu\sigma^\perp$.*

To prove this result Öktem used the following extension theorem with singularities.

Theorem 10.1.2 ([Ökt 1998], [Ökt 1999]). *Let*

$$\boldsymbol{X} := \mathbb{X}(\mathbb{R}, \mathbb{R}; \mathbb{C}, \mathbb{C}) = (\mathbb{R} \times \mathbb{C}) \cup (\mathbb{C} \times \mathbb{R})$$

(note that $\widehat{\boldsymbol{X}} = \mathbb{C}^2$ – cf. Proposition 3.2.3). Let $M := \{(z_1, z_2) \in \mathbb{C}^2 : z_1 = z_2\}$. Let $f : \boldsymbol{X} \setminus M \to \mathbb{C}$ be such that for all $a, b \in \mathbb{R}$ the functions

$$\mathbb{C} \setminus \{a\} \ni w \mapsto f(a, w), \quad \mathbb{C} \setminus \{b\} \ni w \mapsto f(z, b)$$

are holomorphic. Then there exists an $\hat{f} \in \mathcal{O}(\mathbb{C}^2 \setminus M)$ with $\hat{f} = f$ on $\boldsymbol{X} \setminus M$.

We should mention that there are also different proofs of Theorem 10.1.1; see, for example, [AEK 1996] and [Bar-Zam 2009].

Sketch of the proof for the sufficiency in Theorem 10.1.1. Assume $\mu > 0$ and let $g \in \mathcal{C}_0^\infty(\mathbb{T} \times \mathbb{R})$ such that $\widehat{g}(\omega, it) = \widehat{g}(\sigma, -it)$ whenever $t\omega + \mu\omega^\perp = t\sigma + \mu\sigma^\perp$, $\omega, \sigma \in \mathbb{T}$. Put $f = \widehat{g}$.

Note that any $x \in \mathbb{R}^2$, $\|x\| > \mu$, lies on exactly two lines $\mathbb{R}\omega + \mu\omega^\perp$ and $\mathbb{R}\sigma + \mu\sigma^\perp$ with $\omega, \sigma \in \mathbb{T}$. Hence, one may define for $x \in \mathbb{R}^2$, $\|x\| \geq 2$,

$$F(ix) := f(\omega, it) \quad \text{if } x = t\omega + \mu\omega^\perp \text{ for some } \omega \in \mathbb{T}, t \in \mathbb{R}.$$

By the necessary condition the function F is well defined on $K_\mu := \mathbb{R}^2 \setminus \mathbb{B}_2^{\mathbb{R}}(\mu)$.

The main goal in the proof of Öktem is to show that

($*$) there exists an entire function $F^* \in \mathcal{O}(\mathbb{C}^2)$ with $F^*|_{iK_\mu} = F$.

It follows that F^* allows an inverse Fourier transform h. Then $h \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{C})$ and $R_\mu(h) = g$. Hence it remains to verify the holomorphic extension ($*$).

To see the main argument for ($*$), let $\xi \in \mathbb{R}$ and put $\omega(\xi) := (\frac{1-\xi^2}{1+\xi^2}, \frac{2\xi}{1+\xi^2}) \in \mathbb{T}$. Then one defines a mapping $\varphi \colon \mathbb{R}^2 \to K_\mu$ by

$$\varphi(\xi, \eta) := ix, \text{ where } x \in (\mathbb{R}\omega(\xi) + \mu\omega(\xi)^\perp) \cap (\mathbb{R}\omega(\eta) + \mu\omega(\eta)^\perp).$$

Calculation gives

$$\varphi(\xi, \eta) = -\frac{i\mu}{1 + \xi\eta}(\xi + \eta, \xi\eta - 1).$$

Note that φ is even defined on $\overline{\mathbb{C}}^2 \setminus \Gamma$, where $\Gamma := \{(\xi, \eta) \in \overline{\mathbb{C}}^2 : \xi = -1/\eta\}$. Finally, put $G = F \circ \varphi$.

One can prove that

- $G(\xi, \cdot)$ extends to a holomorphic function on $\overline{\mathbb{C}} \setminus \{-1/\xi\}$ for $\xi \in \overline{\mathbb{R}}$,

• $G(\cdot, \eta)$ extends to a holomorphic function on $\overline{\mathbb{C}} \setminus \{-1/\eta\}$ for $\eta \in \overline{\mathbb{R}}$.

Note that the transformation $(\xi, \eta) \mapsto (\xi, -1/\eta)$ puts us, in principle, in the situation of Theorem 10.1.2. So it follows that G extends to a holomorphic function $G^* \in \mathcal{O}(\overline{\mathbb{C}}^2 \setminus \Gamma)$.

What remains to verify is that the function F^* defined by the relation $G^* = F^* \circ \varphi$ is the entire function we were looking for. □

Öktem's result (Theorem 10.1.2) was extended by J. Siciak in [Sic 2001].

Theorem 10.1.3 ([Sic 2001]). *Let $A_j \subset \mathbb{C}$ be locally regular, $j = 1, \dots, N$, and let*

$$\boldsymbol{X} := \mathbb{X}((A_j, \mathbb{C})_{j=1}^N) = \bigcup_{j=1}^N A_1 \times \dots \times A_{j-1} \times \mathbb{C} \times A_{j+1} \times \dots \times A_N$$

(*observe that $\widehat{\boldsymbol{X}} = \mathbb{C}^N$*). *Let $M := \{z \in \mathbb{C}^N : P(z) = 0\}$, where P is a non-constant polynomial of N-complex variables. Let $f : \boldsymbol{X} \setminus M \to \mathbb{C}$ be such that for all $j \in \{1, \dots, N\}$ and $(a_j', a_j'') \in A_j' \times A_j''$ with $M_{(a_j', \cdot, a_j'')} \neq \mathbb{C}$, the function*

$$\mathbb{C} \setminus M_{(a_j', \cdot, a_j'')} \ni z_j \mapsto f(a_j', z_j, a_j'')$$

is holomorphic. Then there is an $\hat{f} \in \mathcal{O}(\mathbb{C}^N \setminus M)$ such that $\hat{f} = f$ on $\boldsymbol{X} \setminus M$.

Notice that Öktem's theorem is just the case where $N = 2$, $A_1 = A_2 = \mathbb{R}$, and $P(z_1, z_2) = z_1 - z_2$.

The above theorems have been generalized in [Jar-Pfl 2001], [Jar-Pfl 2003a], [Jar-Pfl 2003b], [Jar-Pfl 2008] to various cross theorems with analytic and pluripolar singularities, which will be presented in the next sections.

10.2 General cross theorem with singularities

[>] §§ 2.1.1, 2.3, 2.4, 3.2, 5.4, 7.1, 9.1, 9.2.3, 9.4.

We begin with the formulation of a very general extension problem for crosses with singularities. Consider the following configuration:

(C1) D_j is a Riemann domain of holomorphy over $\mathbb{C}^{n_j}$,

(C2) $A_j \subset D_j$, A_j is locally pluriregular,

(C3) $\Sigma_j \subset \Sigma_j^0 \subset A_j' \times A_j''$, Σ_j^0 is pluripolar, $j = 1, \dots, N$,

(C4) $\boldsymbol{X} := \mathbb{X}((A_j, D_j)_{j=1}^N)$, $\boldsymbol{T} := \mathbb{T}((A_j, D_j, \Sigma_j)_{j=1}^N)$, $\boldsymbol{T}^0 := \mathbb{T}((A_j, D_j, \Sigma_j^0)_{j=1}^N)$ (note that $\boldsymbol{T}^0 \subset \boldsymbol{T} \subset \boldsymbol{X}$),

(C5) $M \subset \boldsymbol{T}$,

(C6) for any $j \in \{1, \dots, N\}$ and any $(a'_j, a''_j) \in (A'_j \times A''_j) \setminus \Sigma_j$, the fiber $M_{(a'_j,\cdot,a''_j)}$ is closed in D_j,

(C7) for any $j \in \{1, \dots, N\}$ and any $(a'_j, a''_j) \in (A'_j \times A''_j) \setminus \Sigma^0_j$, the fiber $M_{(a'_j,\cdot,a''_j)}$ is pluripolar in D_j.

Remark 10.2.1. (a) Clearly:

- if $\Sigma_j = \varnothing$, $j = 1, \dots, N$, then $\boldsymbol{T} = \boldsymbol{X}$,
- if $\Sigma^0_j = \Sigma_j$, $j = 1, \dots, N$, then $\boldsymbol{T}^0 = \boldsymbol{T}$.

(b) Note that some fibers $M_{(a'_j,\cdot,a''_j)}$ with $(a'_j, a''_j) \in \Sigma^0_j \setminus \Sigma_j$ may be also pluripolar, i.e. the sets $\Sigma^0_1, \dots, \Sigma^0_N$ need not be "minimal".

(c) If M is relatively closed in $\boldsymbol{T}$, then (C6) is automatically satisfied.

(d) If M is pluripolar, then for every $j \in \{1, \dots, N\}$ the set

$$C_j := \{(a'_j, a''_j) \in (A'_j \times A''_j) \setminus \Sigma_j : M_{(a'_j,\cdot,a''_j)} \text{ is not pluripolar}\}$$

is pluripolar (cf. Proposition 2.3.31). Consequently, $\Sigma^0_j := \Sigma_j \cup C_j$, $j = 1, \dots, N$, are minimal sets with (C7).

We will see in Remark 10.7.2 (b) that there are non-pluripolar sets M with (C7).

(e) If M *is analytic*, i.e.

(C5$_a$) $M = \boldsymbol{T} \cap S$, where $S \subset U$ is an analytic subset of an open neighborhood $U \subset \widehat{\boldsymbol{X}}$ of $\boldsymbol{T}$ with $\operatorname{codim} S \geq 1$, then for any $j \in \{1, \dots, N\}$ and any $(a'_j, a''_j) \in (A'_j \times A''_j) \setminus \Sigma_j$, we have:

$$\begin{aligned} M_{(a'_j,\cdot,a''_j)} \text{ is pluripolar } &\iff M_{(a'_j,\cdot,a''_j)} \text{ is thin in } D_j \\ &\iff M_{(a'_j,\cdot,a''_j)} \text{ is a proper analytic set in } D_j \\ &\iff M_{(a'_j,\cdot,a''_j)} \neq D_j. \end{aligned}$$

In particular, if we take Σ^0_j as in (d), $j = 1, \dots, N$, then $\boldsymbol{T}^0 \setminus M = \boldsymbol{T} \setminus M$.

(f) For every locally pluriregular set $B_N \subset D_N$, the set $((A'_N \setminus \Sigma^0_N) \times B_N) \setminus M$ is locally pluriregular (by Proposition 3.2.21) and dense in $((A'_N \setminus \Sigma_N) \times B_N) \setminus M$. In particular:

- $\boldsymbol{c}(\boldsymbol{T}^0) \setminus M$ is locally pluriregular and dense in $\boldsymbol{c}(\boldsymbol{T}) \setminus M$,
- $\boldsymbol{T}^0 \setminus M$ is locally pluriregular and dense in $\boldsymbol{T} \setminus M$.

Indeed, take $(a'_N, a_N) \in ((A'_N \setminus \Sigma_N) \times B_N) \setminus M$. Since A'_N is locally pluriregular and Σ^0_N is pluripolar, there exists a sequence

$$A'_N \setminus \Sigma^0_N \ni a'^k \underset{k \to +\infty}{\longrightarrow} a'_N.$$

The set $P := \bigcup_{k=1}^{\infty} M_{(a'^k,\cdot)}$ is pluripolar. In particular, there exists a sequence $B_N \setminus P \ni a_N^k \to a_N$. Then

$$((A'_N \setminus \Sigma_N^0) \times B_N) \setminus M \ni (a'^k, a_N^k) \underset{k\to+\infty}{\longrightarrow} (a'_N, a_N).$$

Definition 10.2.2. We say that a function $f : \boldsymbol{T} \setminus M \to \mathbb{C}$ is *separately holomorphic* (we write $f \in \mathcal{O}_s(\boldsymbol{T} \setminus M)$) if for any $j \in \{1, \dots, N\}$ and any $(a'_j, a''_j) \in (A'_j \times A''_j) \setminus \Sigma_j$, the function

$$D_j \setminus M_{(a'_j,\cdot,a''_j)} \ni z_j \mapsto f(a'_j, z_j, a''_j)$$

is holomorphic (if $M_{(a'_j,\cdot,a''_j)} = D_j$, then we understand that the above condition is automatically satisfied).

Notice that in the case where $M = \varnothing$ the above definition coincides with that of $\mathcal{O}_s(\boldsymbol{T})$ (Definition 7.1.1).

Remark 10.2.3. With the notion above one may reformulate Theorem 10.1.3:

Let $\boldsymbol{X}$, M be as in Theorem 10.1.3. Then every $f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$ extends to an $\hat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus M)$.

Assume that (C1)–(C7) are satisfied and let $\varnothing \neq \mathcal{F} \subset \mathcal{O}_s(\boldsymbol{T} \setminus M)$. Our aim is to decide whether there exists a new set $\widehat{M} \subset \widehat{\boldsymbol{X}}$ such that every function $f \in \mathcal{F}$ extends holomorphically to $\widehat{\boldsymbol{X}} \setminus \widehat{M}$. We will always assume that the family $\mathcal{F}$ satisfies the following "almost necessary" condition for holomorphic extension:

(C8) for every $a \in \boldsymbol{c}(\boldsymbol{T}) \setminus M$ there exists a polydisc $\widehat{\mathbb{P}}(a, \rho_a)$ such that for every $f \in \mathcal{F}$ there exists an $\tilde{f}_a \in \mathcal{O}(\widehat{\mathbb{P}}(a, \rho_a))$ with $\tilde{f}_a = f$ on $\widehat{\mathbb{P}}(a, \rho_a) \cap \boldsymbol{c}(\boldsymbol{T}) \setminus M$.

The following class of functions will be needed in our results (Theorems 10.2.9, 10.2.12).

Definition 10.2.4. Let $\mathcal{O}_s^c(\boldsymbol{T} \setminus M)$ denote the space of all functions $f \in \mathcal{O}_s(\boldsymbol{T} \setminus M)$ such that for any $j \in \{1, \dots, N\}$ and any $b_j \in D_j$, the function

$$(A'_j \times A''_j) \setminus (\Sigma_j \cup M_{(\cdot,b_j,\cdot)}) \ni (z'_j, z''_j) \mapsto f(z'_j, b_j, z''_j)$$

is continuous (cf. Definition 7.1.1).

Obviously $\mathcal{O}_s(\boldsymbol{T} \setminus M) \cap \mathcal{C}(\boldsymbol{T} \setminus M) \subset \mathcal{O}_s^c(\boldsymbol{T} \setminus M)$.

In order to shorten some long formulas we propose the following useful convention. Suppose that D is a Riemann region, $A \subset D$, $a \in D$, and $\widehat{\mathbb{P}}_D(a, r)$ exists (cf. Definition 2.1.3). Then we put

$$A[a, r] := A \cap \widehat{\mathbb{P}}_D(a, r).$$

Lemma 10.2.5. *Assume that* (C1)–(C7) *are satisfied.*

(a) *If $M \subset \boldsymbol{T}$ is relatively closed, then $\mathcal{F} := \mathcal{O}_s^c(\boldsymbol{T} \setminus M)$ satisfies* (C8).

(b) *If $M \subset \boldsymbol{X}$ is relatively closed, then the family $\mathcal{F} := \mathcal{O}_s(\boldsymbol{X} \setminus M)$ satisfies* (C8) *for $\boldsymbol{T} = \boldsymbol{X}$.*

Proof. (a) Take an $a \in \boldsymbol{c}(\boldsymbol{T}) \setminus M$ and let $r > 0$ be such that $\widehat{\mathbb{P}}(a,r) \cap M = \varnothing$. Put

$$\boldsymbol{X}_a := \boldsymbol{X}[a,r] = \mathbb{X}((A_j[a_j,r], \widehat{\mathbb{P}}(a_j,r))_{j=1}^N),$$
$$\boldsymbol{T}_a := \boldsymbol{T}[a,r] = \mathbb{T}((A_j[a_j,r], \widehat{\mathbb{P}}(a_j,r), \Sigma_j[(a_j',a_j''),r])_{j=1}^N).$$

Observe that $f|_{\boldsymbol{T}_a} \in \mathcal{O}_s^c(\boldsymbol{T}_a)$ for every $f \in \mathcal{F}$. Consequently, by Theorem 7.1.4, we know that each f extends to an $\hat{f}_a \in \mathcal{O}(\widehat{\boldsymbol{X}}_a)$ with $\hat{f}_a = f$ on $\boldsymbol{T}_a$. It remains to take a $\rho_a \in (0,r)$ so small that $\widehat{\mathbb{P}}(a,\rho_a) \subset \widehat{\boldsymbol{X}}_a$.

(b) Use the same method as in (a) with Theorem 5.4.1 instead of Theorem 7.1.4. □

Our main goal is to prove the following extension theorem.

Theorem 10.2.6 (Main extension theorem for generalized crosses with pluripolar singularities). *Under assumptions* (C1)–(C8) *there exists a relatively closed pluripolar set* $\widehat{M} \subset \widehat{\boldsymbol{X}}$ *such that*

(M1) $\widehat{M} \cap \boldsymbol{c}(\boldsymbol{T}^0) \subset M$,

(M2) *for every* $f \in \mathcal{F}$ *there exists an* $\hat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus \widehat{M})$ *such that* $\hat{f} = f$ *on* $\boldsymbol{c}(\boldsymbol{T}^0) \setminus M$,

(M3) *the set* $\widehat{M}$ *is singular with respect to the family* $\widehat{\mathcal{F}} := \{\hat{f} : f \in \mathcal{F}\}$ (*cf. Definition* 2.4.1),

(M4) *if for any* $j \in \{1,\dots,N\}$ *and* $(a_j',a_j'') \in (A_j' \times A_j'') \setminus \Sigma_j^0$, *the fiber* $M_{(a_j',\cdot,a_j'')}$ *is thin in* D_j, *then* $\widehat{M}$ *is analytic in* $\widehat{\boldsymbol{X}}$ (*and in view of* (M3), *either* $\widehat{M} = \varnothing$ *or* $\widehat{M}$ *is of pure codimension* 1 – *cf. Remark* 2.4.3 (e)).

The proof of Theorem 10.2.6 will be given in §§ 10.4, 10.6. First we discuss various consequences. We need the following two lemmas.

Lemma 10.2.7. *Under the notation of Theorem* 10.2.6, *if* $\Sigma_j^0 \subset \Sigma_j' \subset A_j' \times A_j''$, $j = 1,\dots,N$, *are such that* $\widehat{M} \cap \boldsymbol{T}' \subset M$ *for* $\boldsymbol{T}' := \mathbb{T}((A_j, D_j, \Sigma_j')_{j=1}^N)$, *then* $\hat{f} = f$ *on* $\boldsymbol{T}' \setminus M$ *for all* $f \in \mathcal{F}$.

Proof. For any $j \in \{1,\dots,N\}$, $(a_j',a_j'') \in (A_j' \times A_j'') \setminus \Sigma_j'$, and $f \in \mathcal{F}$, the functions $\hat{f}(a_j',\cdot,a_j'')$ and $f(a_j',\cdot,a_j'')$ are holomorphic in the domain $D_j \setminus M_{(a_j',\cdot,a_j'')}$ and equal on the non-pluripolar set $A_j \setminus M_{(a_j',\cdot,a_j'')}$. It remains to use the identity principle. □

Lemma 10.2.8. *Under the notation of Theorem* 10.2.6, *there exist pluripolar sets* $\Sigma_j' \subset A_j' \times A_j''$ *with* $\Sigma_j^0 \subset \Sigma_j'$, $j = 1,\dots,N$, *such that* $\widehat{M} \cap \boldsymbol{T}' \subset M$ *with* $\boldsymbol{T}' := \mathbb{T}((A_j, D_j, \Sigma_j')_{j=1}^N)$. *In particular, by Lemma* 10.2.7, $\hat{f} = f$ *on* $\boldsymbol{T}' \setminus M$ *for all* $f \in \mathcal{F}$.

Proof. By Proposition 9.1.4, for each $j \in \{1,\dots,N\}$ there exists a pluripolar set $\Sigma_j' \subset A_j' \times A_j''$, $\Sigma_j^0 \subset \Sigma_j'$, such that for any $(a_j',a_j'') \in (A_j' \times A_j'') \setminus \Sigma_j'$, the fiber

$\widehat{M}_{(a'_j,\cdot,a''_j)}$ is singular with respect to the family $\{\hat{f}(a'_j,\cdot,a''_j) : f \in \mathcal{F}\}$. In particular, $\widehat{M}_{(a'_j,\cdot,a''_j)} \subset M_{(a'_j,\cdot,a''_j)}$ for any $(a'_j,a''_j) \in (A'_j \times A''_j) \setminus \Sigma'_j$, $j = 1,\dots,N$, which means that $\widehat{M} \cap \boldsymbol{T}' \subset M$. □

Using Lemmas 10.2.5 and 10.2.8, we easily conclude that Theorem 10.2.6 implies the following fundamental result.

Theorem 10.2.9 (Extension theorem for generalized crosses with pluripolar singularities). *Assume that* (C1)–(C7) *are satisfied and $M \subset \boldsymbol{T}$ is relatively closed. Let*

$$\mathcal{F} = \mathcal{F}(\boldsymbol{T} \setminus M) := \begin{cases} \mathcal{O}_s(\boldsymbol{X} \setminus M) & \text{if } \Sigma_1 = \dots = \Sigma_N = \varnothing, \\ \mathcal{O}_s^c(\boldsymbol{T} \setminus M) & \text{otherwise.} \end{cases}$$

Then there exist a relatively closed pluripolar set $\widehat{M} \subset \widehat{\boldsymbol{X}}$ and a generalized N-fold cross $\boldsymbol{T}' := \mathbb{T}((A_j, D_j, \Sigma'_j)_{j=1}^N) \subset \boldsymbol{T}^0$ with $\Sigma_j^0 \subset \Sigma'_j \subset A'_j \times A''_j$, Σ'_j pluripolar, $j = 1,\dots,N$, such that

(P1) $\widehat{M} \cap (\boldsymbol{c}(\boldsymbol{T}^0) \cup \boldsymbol{T}') \subset M$,

(P2) *for any $f \in \mathcal{F}$ there exists an $\hat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus \widehat{M})$ with $\hat{f} = f$ on $(\boldsymbol{c}(\boldsymbol{T}^0) \cup \boldsymbol{T}') \setminus M$,*

(P3) *$\widehat{M}$ is singular with respect to the family $\{\hat{f} : f \in \mathcal{F}\}$,*

(P4) *if for all $j \in \{1,\dots,N\}$ and $(a'_j,a''_j) \in (A'_j \times A''_j) \setminus \Sigma_j^0$, the fiber $M_{(a'_j,\cdot,a''_j)}$ is thin in D_j, then $\widehat{M}$ is analytic.*

(P5) *$\hat{f}(\widehat{\boldsymbol{X}} \setminus \widehat{M}) \subset f(\boldsymbol{T} \setminus M)$; in particular*

- $\|\hat{f}\|_{\widehat{\boldsymbol{X}} \setminus \widehat{M}} \le \|f\|_{\boldsymbol{T} \setminus M}$,
- $|\hat{f}(z)| \le \|f\|_{\boldsymbol{c}(\boldsymbol{T}) \setminus M}^{1 - \sum_{j=1}^N \boldsymbol{h}^*_{A_j,D_j}(z_j)} \|f\|_{\boldsymbol{T} \setminus M}^{\sum_{j=1}^N \boldsymbol{h}^*_{A_j,D_j}(z_j)}$, $z = (z_1,\dots,z_N) \in \widehat{\boldsymbol{X}}$.

Remark 10.2.10. (a) Notice that from the point of view of Theorem 10.2.9 it suffices to consider only the following two configurations:

- (C1)–(C7) with $\Sigma_j = \varnothing$, $j = 1,\dots,N$, $M \subset \boldsymbol{T} = \boldsymbol{X}$, M relatively closed, and $\mathcal{F} = \mathcal{O}_s(\boldsymbol{X} \setminus M)$,
- (C1)–(C7) with $\Sigma_j = \Sigma_j^0$, $j = 1,\dots,N$, $M \subset \boldsymbol{T} = \boldsymbol{T}^0$, M relatively closed, and $\mathcal{F} = \mathcal{O}_s^c(\boldsymbol{T}^0 \setminus M)$.

Observe that condition (P5) follows from Lemma 2.1.14 with

$$(G, D, A_0, A, \mathcal{F}) = (D_1 \times \dots \times D_N, \widehat{\boldsymbol{X}} \setminus \widehat{M}, \boldsymbol{c}(\boldsymbol{T}^0) \setminus M, \boldsymbol{T} \setminus M, \mathcal{F}(\boldsymbol{T} \setminus M)).$$

To get the inequality in (P5) we use Lemma 3.2.5 and Propositions 3.2.14, 3.2.28 $(\boldsymbol{h}^*_{\boldsymbol{c}(\boldsymbol{T}^0)\setminus M, \widehat{\boldsymbol{X}} \setminus \widehat{M}}(z) = \boldsymbol{h}^*_{\boldsymbol{c}(\boldsymbol{T}^0), \widehat{\boldsymbol{X}}}(z) = \boldsymbol{h}^*_{\boldsymbol{c}(\boldsymbol{X}), \widehat{\boldsymbol{X}}}(z) = \sum_{j=1}^N \boldsymbol{h}^*_{A_j,D_j}(z_j))$.

(b) Note that if we were able to prove Theorem 7.1.4 for a class of functions more general than $\mathcal{O}_s^c(\boldsymbol{X})$ (cf. Remark 7.1.5), then we would get a more general version of Lemma 10.2.5 (a) and, consequently, more general versions of Theorems 10.2.9 and 10.2.12.

To understand better the role played by the "test cross" $\boldsymbol{T}'$, let us consider the following example.

Example 10.2.11. Let $N = 2$, $n_1 = n_2 = 1$, $D_1 = D_2 = \mathbb{C}$, $A_1 := \mathbb{D}$, $\Sigma_1 = \Sigma_2 = \varnothing$, $\Sigma_1^0 := \varnothing$, $\Sigma_2^0 := \{0\}$. Then $\boldsymbol{X} := \mathbb{X}(\mathbb{D}, A_2; \mathbb{C}, \mathbb{C}) = \boldsymbol{T}$, $\boldsymbol{T}^0 = \mathbb{X}(\mathbb{D}_*, A_2; \mathbb{C}, \mathbb{C})$. Note that $\widehat{\boldsymbol{X}} = \mathbb{C}^2$.

Assume that $M \subset \{0\}\times\mathbb{C}$ is a closed (pluripolar) set. Then (C1)–(C7) are obviously satisfied. Suppose that $\widehat{M}$ is a solution of the above extension problem with (P1)–(P4).

Put $\boldsymbol{Y} := \mathbb{X}(\mathbb{D}_*, A_2; \mathbb{C}_*, \mathbb{C}) \subset \boldsymbol{X} \setminus M$. Observe that $\widehat{\boldsymbol{Y}} = \mathbb{C}_*\times\mathbb{C}$. If $f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$, then $f|_{\boldsymbol{Y}} \in \mathcal{O}_s(\boldsymbol{Y})$. Thus, every $f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$ extends to an $\tilde{f} \in \mathcal{O}(\mathbb{C}_* \times \mathbb{C})$ with $\tilde{f} = f$ in $\boldsymbol{Y}$ and, consequently, $\tilde{f} = \hat{f}$ on $(\mathbb{C}_* \times \mathbb{C}) \setminus \widehat{M}$. Since $\widehat{M}$ is singular, we conclude that $\widehat{M} \subset \{0\} \times \mathbb{C}$. Consider the following two particular cases.

(a) Let $A_2 = \mathbb{D}$, $M := \{0\} \times \overline{\mathbb{D}}$. Let $f_0 \colon \boldsymbol{X} \setminus M \to \mathbb{C}$,

$$f_0(z, w) := \begin{cases} 1/z & \text{if } z \neq 0, \\ 0 & \text{if } z = 0,\ |w| > 1, \end{cases}$$

and observe that $f_0 \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$. Since f_0 extends to an $\hat{f}_0 \in \mathcal{O}(\mathbb{C}^2 \setminus \widehat{M})$ with $\hat{f}_0 = f_0$ on $\boldsymbol{T}' \setminus M$, we conclude that $\hat{f}_0(z, w) = 1/z$, $(z, w) \in (\mathbb{C}_* \times \mathbb{C}) \setminus \widehat{M}$. Hence $\{0\} \times \mathbb{C} \subset \widehat{M}$. Thus $\widehat{M} = \{0\} \times \mathbb{C}$. Consequently, $\widehat{M} \cap \boldsymbol{X} = \{0\} \times \mathbb{C} \not\subset M$, which shows that, in general, $\widehat{M} \cap \boldsymbol{X} \not\subset M$.

(b) Let $A_2 := \{w \in \mathbb{C} : r < |w| < 1\}$, where $0 < r < 1$,

$$M := \{0\} \times \{w \in \mathbb{C} : |w| = r\}.$$

Now we look at the function $f_0 \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$ defined by

$$f_0(z, w) := \begin{cases} w & \text{if } z \neq 0 \text{ or } (z = 0 \text{ and } |w| > r), \\ 0 & \text{if } z = 0 \text{ and } |w| < r. \end{cases}$$

Obviously, $\hat{f}_0(z, w) = w$, $(z, w) \in \mathbb{C}^2 \setminus \widehat{M}$. Observe that $w = \hat{f}_0(0, w) \neq f_0(0, w) = 0$ for $|w| < r$, which shows that, in general, $\hat{f} \not\equiv f$ on $\boldsymbol{X} \setminus M$ (even if $\hat{f}$ is defined on $\boldsymbol{X} \setminus M$).

In the special case (C5$_a$) we can prove much more, namely we have the following result.

Theorem 10.2.12 (Extension theorem for generalized crosses with analytic singularities). *Assume that* (C1)–(C4), (C5$_a$) *are satisfied. Let*

$$\mathcal{F} = \mathcal{F}(\boldsymbol{T} \setminus M) := \begin{cases} \mathcal{O}_s(\boldsymbol{X} \setminus M) & \textit{if } \Sigma_1 = \dots = \Sigma_N = \varnothing, \\ \mathcal{O}_s^c(\boldsymbol{T} \setminus M) & \textit{otherwise}. \end{cases}$$

Then there exist an analytic set $\widehat{M} \subset \widehat{\boldsymbol{X}}$ *and an open neighborhood* $U_0 \subset U$ *of* $\boldsymbol{T}$ *such that:*

(S1) $\widehat{M} \cap U_0 \subset S$,

(S2) *for any* $f \in \mathcal{F}$ *there exists an* $\hat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus \widehat{M})$ *with* $\hat{f} = f$ *on* $\boldsymbol{T} \setminus M$,

(S3) $\widehat{M}$ *is singular with respect to the family* $\{\hat{f} : f \in \mathcal{F}\}$,

(S4) *if* $U = \widehat{\boldsymbol{X}}$, *then* $\widehat{M}$ *is the union of all one codimensional components of* S,

(S5) $\hat{f}(\widehat{\boldsymbol{X}} \setminus \widehat{M}) \subset f(\boldsymbol{T} \setminus M)$; *in particular:*

- $\|\hat{f}\|_{\widehat{\boldsymbol{X}} \setminus \widehat{M}} = \|f\|_{\boldsymbol{T} \setminus \boldsymbol{M}}$,
- $|\hat{f}(z)| \leq \|f\|_{\boldsymbol{c}(\boldsymbol{T}) \setminus \boldsymbol{M}}^{1-\sum_{j=1}^{N} \boldsymbol{h}^*_{A_j, D_j}(z_j)} \|f\|_{\boldsymbol{T} \setminus \boldsymbol{M}}^{\sum_{j=1}^{N} \boldsymbol{h}^*_{A_j, D_j}(z_j)}$, $z = (z_1, \dots, z_N) \in \widehat{\boldsymbol{X}}$.

Notice that from the point of view of the above theorem it suffices to consider only the following two configurations:

- (C1)–(C4), (C5$_a$), (C7) with $\Sigma_j = \varnothing$, Σ_j^0 minimal (as in Remark 10.2.1 (d)), $j = 1, \dots, N$, and $\mathcal{F} = \mathcal{O}_s(\boldsymbol{X} \setminus M)$,
- (C1)–(C4), (C5$_a$), (C7) with Σ_j^0 minimal, and $\mathcal{F} = \mathcal{O}_s^c(\boldsymbol{T} \setminus M)$.

Observe that condition (S5) follows from Lemma 2.1.14 with

$$(G, D, A_0, A, \mathcal{F}) = (\widehat{\boldsymbol{X}} \setminus \widehat{M}, \widehat{\boldsymbol{X}} \setminus \widehat{M}, \boldsymbol{c}(\boldsymbol{T}) \setminus M, \boldsymbol{T} \setminus M, \mathcal{F}(\boldsymbol{T} \setminus M)).$$

To get the inequality in (S5) we use Lemma 3.2.5 and Propositions 3.2.14, 3.2.28.

The proof that Theorem 10.2.9 implies Theorem 10.2.12 will be given in the next section.

It is clear that Theorem 10.2.12 generalizes Siciak's Theorem 10.1.3.

Remark 10.2.13. Let $S \subsetneq U \subset \widehat{\boldsymbol{X}}$ be an analytic subset of an open connected neighborhood U of $\boldsymbol{X}$. Put $M_X := S \cap \boldsymbol{X}$, $M_T := S \cap \boldsymbol{T}$, and let $\widehat{M}_X$, $\widehat{M}_T$ be constructed according to Theorem 10.2.9 with respect to $\mathcal{F} = \mathcal{O}_s(\boldsymbol{X} \setminus M_X)$ and $\mathcal{F} = \mathcal{O}_s^c(\boldsymbol{T} \setminus M_T)$, respectively. Take an $f \in \mathcal{O}_s(\boldsymbol{X} \setminus M_X)$ and let $\hat{f}_X \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus \widehat{M}_X)$ be such that $\hat{f}_X = f$ on $\boldsymbol{X} \setminus M_X$. In particular, $f|_{\boldsymbol{T} \setminus M_T} = \hat{f}_X|_{\boldsymbol{T} \setminus M_T} \in \mathcal{O}_s^c(\boldsymbol{T} \setminus M_T)$. Let $\hat{f}_T \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus \widehat{M}_T)$ be such that $\hat{f}_T = f$ on $\boldsymbol{T} \setminus M_T$. By the identity principle we have $\hat{f}_T = \hat{f}_X$ on $\widehat{\boldsymbol{X}} \setminus (\widehat{M}_X \cup \widehat{M}_T)$. Since $\widehat{M}_X$ is singular, we conclude that $\widehat{M}_X \subset \widehat{M}_T$. [?] We do not know whether $\widehat{M}_X = \widehat{M}_T$ [?]

Observe that condition (S4) says that the above equality is satisfied if $U = \widehat{\boldsymbol{X}}$.

Remark 10.2.14. (a) [?] It would be interesting to have results similar to Theorems 10.2.6, 10.2.9, 10.2.12 for (N, k)-crosses (cf. § 7.2) [?]

(b) We also mention that there are extension results for boundary crosses with singularities – cf. [NVA-Pfl 2009], [NVA-Pfl 2010] for details.

10.3 Proof of Theorem 10.2.12

> §§ 7.1, 10.2.

The aim of this section is to prove that Theorem 10.2.9 implies Theorem 10.2.12. The proof is based on [Jar-Pfl 2011]. Roughly speaking, the main idea of the proof is the following:

- We apply Theorem 10.2.9 with minimal Σ_j^0, $j = 1, \dots, N$ (as in Remark 10.2.1) and we get a relatively closed pluripolar set $\widehat{M} \subset \widehat{\boldsymbol{X}}$ and a generalized N-fold cross $\boldsymbol{T}'$ with (P1)–(P3). In particular, (S3) is satisfied.
- By Remark 10.2.1(10.2.1), for any $j \in \{1, \dots, N\}$ and $(a_j', a_j'') \in (A_j' \times A_j'') \setminus \Sigma_j^0$, the fiber $M_{(a_j', \cdot, a_j'')}$ is thin in D_j. Thus, by (P4), $\widehat{M}$ is analytic, and consequently, either $\widehat{M} = \varnothing$ or $\widehat{M}$ is of pure codimension 1.
- By (P1) we know that $\widehat{M} \cap \boldsymbol{T}' \subset S$.
- By Remark 10.2.1(10.2.1) and Lemma 10.2.7 we know that if $\widehat{M} \cap \boldsymbol{T} \subset M$, then $\hat{f} = f$ on $\boldsymbol{T}^0 \setminus M = \boldsymbol{T} \setminus M$. Thus (S2) is a consequence of (S1).
- In the first part of the proof (Lemmas 10.3.1, 10.3.2) we show that (S4) is a consequence of (S1)–(S3).
- Next (Lemma 10.3.3) we prove that in fact we may assume that $U = \widehat{\boldsymbol{X}}$.
- Finally, we show that if $\widehat{M} \neq \varnothing$, then for any irreducible component $\widehat{M}_0$ of $\widehat{M}$ we have $\varnothing \neq \Omega \cap \widehat{M}_0 \subset S$ for an open set $\Omega \subset \widehat{\boldsymbol{X}}$. Consequently, the identity principle for analytic sets (cf. [Chi 1989], § 5.3) implies that $\widehat{M}_0 \subset S$, which finishes the proof.

Lemma 10.3.1. *Assume that* (C1) *and* (C2) *are satisfied. Let $Q \subset \widehat{\boldsymbol{X}}$ be an arbitrary analytic set of pure codimension* 1 *and let $\boldsymbol{T}'' = \mathbb{T}((A_j, D_j, \Sigma_j'')_{j=1}^N) \subset \boldsymbol{X}$ be a generalized cross with $\Sigma_1'', \dots, \Sigma_N''$ pluripolar. Then $Q \cap \boldsymbol{T}'' \neq \varnothing$.*

Proof. Suppose that $Q \cap \boldsymbol{T}'' = \varnothing$. Since Q is of pure codimension 1, $\widehat{\boldsymbol{X}} \setminus Q$ is a domain of holomorphy, and therefore, there exists a $g \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus Q)$ such that $\widehat{\boldsymbol{X}} \setminus Q$ is the domain of existence of g. Since $\boldsymbol{T}'' \subset \widehat{\boldsymbol{X}} \setminus Q$, we conclude that $f := g|_{\boldsymbol{T}''} \in \mathcal{O}_s^c(\boldsymbol{T}'')$. By Theorem 7.1.4 there exists an $\hat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}})$ such that $\hat{f} = f$ on $\boldsymbol{T}''$. Consequently, since $\boldsymbol{T}''$ is non-pluripolar, we conclude that $\hat{f} = g$ on $\widehat{\boldsymbol{X}} \setminus Q$. Thus g extends holomorphically to $\widehat{\boldsymbol{X}}$; a contradiction. □

Lemma 10.3.2. *Condition* (S4) *follows from* (S1)–(S3).

Proof. Let S_0 be the union of all irreducible components of S of codimension 1. Consider two cases:

$S_0 \neq \varnothing$: Similarly as in the proof of Lemma 10.3.1, there exists a non-continuable function $g \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus S_0)$. Then $f := g|_{\boldsymbol{T} \setminus M} \in \mathcal{O}_s^c(\boldsymbol{T} \setminus M)$ and, therefore (by (S2)), there exists an $\hat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus \widehat{M})$ with $\hat{f} = f$ on $\boldsymbol{T} \setminus M \subset \widehat{\boldsymbol{X}} \setminus (S_0 \cup \widehat{M})$. The set

$\boldsymbol{T} \setminus M$ is non-pluripolar. Hence $\hat{f} = g$ on $\widehat{\boldsymbol{X}} \setminus (S_0 \cup \widehat{M})$. Since g is non-continuable, we conclude that $S_0 \subset \widehat{M}$.

The set $\widehat{M}$, as a non-empty singular set, must be of pure codimension 1. Since $\widehat{M} \cap U_0 \subset S$ and $Q \cap U_0 \neq \varnothing$ for every irreducible component Q of $\widehat{M}$ (by Lemma 10.3.1), we conclude, using the identity principle for analytic sets, that $\widehat{M} \subset S$. Consequently, $\widehat{M} \subset S_0$.

$S_0 = \varnothing$: Suppose that $\widehat{M} \neq \varnothing$. Then $\widehat{M}$ must be of pure codimension 1. The above proof of the first part shows that $\widehat{M} \subset S$. Since $S_0 = \varnothing$, the codimension of S is ≥ 2; a contradiction. □

Lemma 10.3.3. *If Theorem 10.2.12 is true with $U = \widehat{\boldsymbol{X}}$ (and arbitrary other elements), then it is true in general.*

Proof. It suffices to show that for every $a \in \boldsymbol{T}$ there exists an open neighborhood $U_a \subset U$ such that $\widehat{M} \cap U_a \subset S$. We may assume that $a = (a_1, \dots, a_N) = (a'_N, a_N) \in (A'_N \setminus \Sigma_N) \times D_N$. Let $G_N \subset\subset D_N$ be a domain of holomorphy such that $G_N \cap A_N \neq \varnothing$, $a_N \in G_N$. Since $\{a'_N\} \times G_N \subset \{a'_N\} \times D_N \subset \boldsymbol{T} \subset U$, there exist domains of holomorphy $G_j \subset\subset D_j$, $a_j \in G_j$, $j = 1, \dots, N-1$, such that $G_1 \times \dots \times G_N \subset U$. Consider the N-fold cross $\boldsymbol{X}_a := \mathbb{X}((A_j \cap G_j, G_j)_{j=1}^N) \subset \boldsymbol{X}$. We have $\widehat{\boldsymbol{X}}_a \subset G_1 \times \dots \times G_N \subset U$, $a \in \widehat{\boldsymbol{X}}_a$. Consequently, the analytic set $S \cap \widehat{\boldsymbol{X}}_a$ satisfy all the assumptions of Theorem 10.2.12 with $(U, \boldsymbol{T})$ substituted by

$$(\widehat{\boldsymbol{X}}_a, \mathbb{T}((A_j \cap G_j, G_j, \Sigma_j \cap (G'_j \times G''_j))_{j=1}^N)).$$

Hence, $\widehat{M} \cap \widehat{\boldsymbol{X}}_a \subset S$. □

Proof that Theorem 10.2.9 implies Theorem 10.2.12. Recall that we may assume that $U = \widehat{\boldsymbol{X}}$. We know that $\widehat{M} \cap \boldsymbol{T}' \subset M$. We want to prove that $\widehat{M} \subset S$.

Let $\widehat{M}_0$ be an irreducible component of $\widehat{M}$. By the identity principle for analytic sets we only need to show that $\varnothing \neq \Omega \cap \widehat{M}_0 \subset S$ for an open set $\Omega \subset \widehat{\boldsymbol{X}}$.

For every point $a = (a_1, \dots, a_N) \in \widehat{M}_0$ there exist an open neighborhood U_a and a defining function $g_a \in \mathcal{O}(U_a)$ for $\widehat{M}_0 \cap U_a$ (cf. [Chi 1989], § 2.9). We may assume that $U_a = U^1_{a_1} \times \dots \times U^N_{a_N}$, where $U^j_{a_j} \subset\subset D_j$ is a univalent neighborhood of a_j, $j = 1, \dots, N$. Using the Lindelöf theorem, we find a countable set $I \subset \widehat{M}_0$ such that $\widehat{M}_0 \subset \bigcup_{a \in I} U_a$. Let

$$C_{j,a} = (\mathrm{pr}_{D'_j \times D''_j}(\widehat{M}_0 \cap U_a)) \cap ((A'_j \times A''_j) \setminus \Sigma'_j), \quad j = 1, \dots, N, \ a \in I.$$

Suppose that all the sets $C_{j,a}$ are pluripolar. Put $\Sigma''_j := \Sigma'_j \cup \bigcup_{a \in I} C_{j,a}$. Then Σ''_j is pluripolar, $j = 1, \dots, N$. Let $\boldsymbol{T}'' := \mathbb{T}((A_j, D_j, \Sigma''_j)_{j=1}^N)$. Observe that $\widehat{M}_0 \cap \boldsymbol{T}'' = \varnothing$, which contradicts Lemma 10.3.1. Thus there exists a pair (j, a) such that $C_{j,a}$ is not pluripolar.

Consequently, the proof is reduced to the following lemma.

Lemma 10.3.4. *Let $D \subset \mathbb{C}^p$, $G \subset \mathbb{C}^q$ be domains. Let $S'_0 \subset D \times G$ be an irreducible analytic set of pure codimension 1. Let $g \in \mathcal{O}(D \times G)$ be a defining function for S'_0 and let $S := \{(z,w) \in D \times G : h_1(z,w) = \dots = h_k(z,w) = 0\}$, where $h_1, \dots, h_k \in \mathcal{O}(D \times G)$. Assume that there exists a non-pluripolar set $A \subset \operatorname{pr}_D S'_0$ such that $S'_0 \cap (A \times G) \subset S$. Then there exists an open set $\Omega \subset D \times G$ such that $\varnothing \neq S'_0 \cap \Omega \subset S$.*

Proof. Let $V := \{z \in D : g(z,\cdot) \equiv 0\}$. Then $V \subsetneq D$ is an analytic set. Hence $A_0 := A \setminus V$ is not pluripolar. Fix a pluriregular point $a_0 \in A_0$ and let $(a_0, b_0) \in S'_0$. Write $b_0 = (b'_0, b_{0,q})$. Using a biholomorphic mapping of the form

$$\mathbb{C}^p \times \mathbb{C}^q \ni (z,w) \mapsto (z, b_0 + \Phi(w - b_0)) \in \mathbb{C}^p \times \mathbb{C}^q,$$

where Φ is a suitable unitary transformation, one can easily reduce the problem to the case where $g(a_0, b'_0, \cdot) \not\equiv 0$ in a neighborhood of $b_{0,q}$. Consequently, there exist neighborhoods P of (a_0, b'_0) and Q of $b_{0,q}$ such that $P \times Q \subset D \times G$ and the projection $\operatorname{pr}_P : S'_0 \cap (P \times Q) \to P$ is proper. Thus, there exists an analytic set $\Delta \subsetneq P$ such that

$$\operatorname{pr}_P : (S'_0 \cap (P \times Q)) \setminus \operatorname{pr}_P^{-1}(\Delta) \to P \setminus \Delta$$

is an analytic covering. Observe that the set $B := (A_0 \times \mathbb{C}^{q-1}) \cap (P \setminus \Delta)$ is not pluripolar. Fix a pluriregular point $c \in B$. Then there exists an open set $\Omega \subset P \times Q$, an open connected neighborhood $W \subset P$ of c, and a holomorphic function $\varphi : W \to Q$ such that

$$S'_0 \cap \Omega = \{(z, w', \varphi(z, w')) : (z, w') \in W\}.$$

In particular, $h_j(z, w', \varphi(z, w')) = 0$, $(z, w') \in B \cap W$. Since $B \cap W$ is not pluripolar, we conclude that $h_j(z, w', \varphi(z, w')) = 0$, $(z, w') \in W$, $j = 1, \dots, k$, which implies that $S'_0 \cap \Omega \subset S$. □

10.4 Proof of Theorem 10.2.6 in a special case for $N = 2$

[>] §§ 1.4, 2.3, 2.4, 2.9, 3.2, 4.2, 5.1, 5.4, 9.2.3, 9.4, 10.2.

The aim of this section is to prove Theorem 10.2.6 for $N = 2$ in the case where M is relatively closed in $\boldsymbol{X}$ and $\mathcal{F} = \mathcal{F}(\boldsymbol{X} \setminus M) = \mathcal{O}_s(\boldsymbol{X} \setminus M)$. We should point out that this case is essentially simpler than the general one – recall that in the case $N = 2$, each generalized 2-fold cross is a 2-fold cross.

To simplify notation put $p := n_1$, $q := n_2$, $D := D_1$, $G := D_2$, $A := A_1 \setminus \Sigma_2$, $B := A_2 \setminus \Sigma_1$, $A^0 := A_1 \setminus \Sigma_2^0$, $B^0 := A_2 \setminus \Sigma_1^0$. Recall our main assumptions:

($C_2$1) D, G are Riemann domains of holomorphy.

($C_2$2) A, B are locally pluriregular.

($C_2$3) $A \setminus A^0$, $B \setminus B^0$ are pluripolar.

($C_2$4) $\boldsymbol{X} := (D \times B) \cup (A \times G)$, $\boldsymbol{X}^0 := (D \times B^0) \cup (A^0 \times G)$.
($C_2$5) $M \subset \boldsymbol{X}$.
($C_2$6) For any $(a, b) \in A \times B$, the fibers $M_{(a,\cdot)}$, $M_{(\cdot,b)}$ are closed.
($C_2$7) For any $(a, b) \in A^0 \times B^0$, the fibers $M_{(a,\cdot)}$, $M_{(\cdot,b)}$ are pluripolar.

Our aim is to prove the following theorem.

Theorem 10.4.1 (Cross theorem with singularities). *Let* ($C_2$1)–($C_2$7) *be satisfied. Assume that M is relatively closed. Then there exists a relatively closed pluripolar set $\widehat{M} \subset \widehat{\boldsymbol{X}}$ such that*

(P1) $\widehat{M} \cap \boldsymbol{X}^0 \subset M$,
(P2) *for any $f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$ there exists an $\hat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus \widehat{M})$ with $\hat{f} = f$ on $\boldsymbol{X}^0 \setminus M$,*
(P3) *the set $\widehat{M}$ is singular with respect to the family $\{\hat{f} : f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)\}$,*
(P4) *if for any $(a, b) \in A^0 \times B^0$ the fibers $M_{(a,\cdot)}$, $M_{(\cdot,b)}$ are thin in G and D, respectively, then $\widehat{M}$ is analytic.*

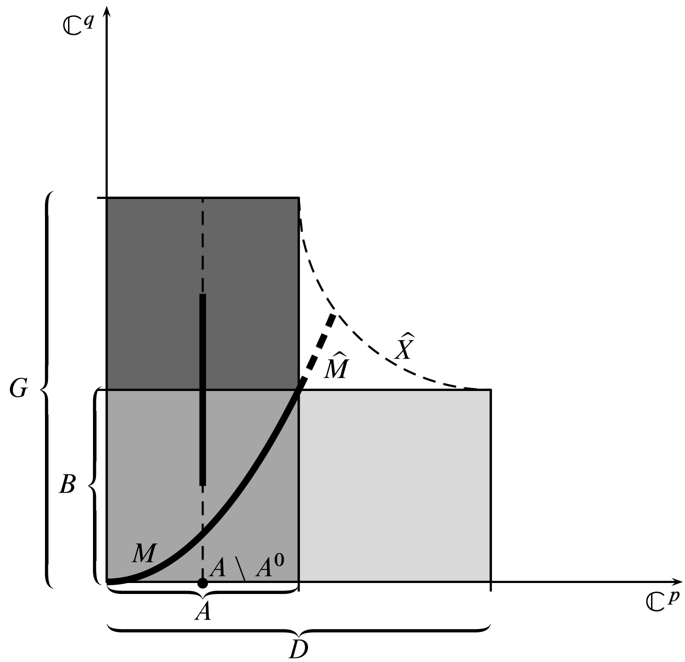


Figure 10.4.1. Extension theorem with singularities.

The main tool for the proof of Theorem 10.4.1 is the following result.

Theorem 10.4.2 (Gluing theorem I). *Let* (C$_2$1)–(C$_2$7) *be satisfied. Fix a family* $\varnothing \neq \mathcal{F} \subset \mathcal{O}_s(\boldsymbol{X} \setminus M)$. *Let* $(D_k)_{k=1}^{\infty}$, $(G_k)_{k=1}^{\infty}$ *be exhaustion sequences of Riemann domains of holomorphy for* D *and* G, *respectively* (*cf. Definition* 1.4.5), *such that*

$$\varnothing \neq A_k^0 := A^0 \cap D_k \subset A \cap D_k =: A_k, \quad \varnothing \neq B_k^0 := B^0 \cap G_k \subset B \cap G_k =: B_k.$$

We assume that for each $k \in \mathbb{N}$ *and* $(a,b) \in \Xi_k := (A_k^0 \times B_k^0) \setminus M$, *there exist*

- *polydiscs* $\widehat{\mathbb{P}}(a, r_{k,a}) \subset D_k$, $\widehat{\mathbb{P}}(b, s_{k,b}) \subset G_k$,
- *relatively closed pluripolar sets*

$$S_{k,a} \subset \widehat{\mathbb{P}}(a, r_{k,a}) \times G_k =: V_{k,a}, \quad S^{k,b} \subset D_k \times \widehat{\mathbb{P}}(b, s_{k,b}) =: V^{k,b},$$

such that

- $S_{k,a} \cap (A^0[a, r_{k,a}] \times G_k) \subset M$, $\quad S^{k,b} \cap (D_k \times B^0[b, s_{k,b}]) \subset M$,
- *for any* $f \in \mathcal{F}$ *there exist* $\tilde{f}_{k,a} \in \mathcal{O}(V_{k,a} \setminus S_{k,a})$, $\tilde{f}^{k,b} \in \mathcal{O}(V^{k,b} \setminus S^{k,b})$ *with*

$$\tilde{f}_{k,a} = f \text{ on } (A^0[a, r_{k,a}] \times G_k) \setminus M, \quad \tilde{f}^{k,b} = f \text{ on } (D_k \times B^0[b, s_{k,b}]) \setminus M.$$

Then there exists a relatively closed pluripolar set $\widehat{M} \subset \widehat{\boldsymbol{X}}$ *such that*

- $\widehat{M} \cap \boldsymbol{X}^0 \subset M$,
- *for any* $f \in \mathcal{F}$ *there exists an* $\hat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus \widehat{M})$ *with* $\hat{f} = f$ *on* $\boldsymbol{X}^0 \setminus M$,
- $\widehat{M}$ *is singular with respect to the family* $\{\hat{f} : f \in \mathcal{F}\}$,
- *if all the sets* $S_{k,a}$, $S^{k,b}$, $(a,b) \in \Xi_k$, $k \in \mathbb{N}$, *are thin, then* $\widehat{M}$ *is analytic.*

Proof. Step 1^0. We may assume that for any $k \in \mathbb{N}$ and $(a,b) \in \Xi_k$ the set $S_{k,a}$ is singular with respect to the family $\{\tilde{f}_{k,a} : f \in \mathcal{F}\}$ and $S^{k,b}$ is singular with respect to $\{\tilde{f}^{k,b} : f \in \mathcal{F}\}$ (cf. Remark 2.4.3). In particular, $S_{k,a}$ (resp. $S^{k,b}$) is thin iff it is analytic (cf. Proposition 2.4.6).

Step 2^0. Fix a $k \in \mathbb{N}$ and define:

$$V_k := \bigcup_{(a,b)\in\Xi_k} V_{k,a} \cup V^{k,b}, \quad S_k := \bigcup_{(a,b)\in\Xi_k} S_{k,a} \cup S^{k,b} \subset V_k,$$
$$\boldsymbol{X}_k := \mathbb{X}(A_k, B_k; D_k, G_k), \quad \boldsymbol{X}_k^0 := \mathbb{X}(A_k^0, B_k^0; D_k, G_k).$$

Step 3^0. Observe that $\boldsymbol{X}_k^0 \subset V_k$.

Indeed, let $(z,w) \in \boldsymbol{X}_k^0$, e.g. $z = a \in A_k^0$, $w \in G_k$. Since $M_{(a,\cdot)}$ is pluripolar, there exists a $b \in B_k^0 \setminus M_{(a,\cdot)}$. Then $(a,b) \in (A_k^0 \times B_k^0) \setminus M = \Xi_k$ and $(z,w) \in \widehat{\mathbb{P}}(a, r_{k,a}) \times G_k = V_{k,a}$.

Step 4^0. Take an $f \in \mathcal{F}$. We want to glue the functions $\{\tilde{f}_{k,a}, \ \tilde{f}^{k,b} : (a,b) \in \Xi_k\}$ to obtain a global holomorphic function $\tilde{f}_k$ on $V_k \setminus S_k$.

Let $(a,b) \in \Xi_k$. Observe $\tilde{f}_{k,a} = f = \tilde{f}^{k,b}$ on the non-pluripolar set

$$(A^0[a, r_{k,a}] \times B^0[b, s_{k,b}]) \setminus M$$

(cf. Remark 10.2.1 (f)). Hence

$$\tilde{f}_{k,a} = \tilde{f}^{k,b} \text{ on } (\widehat{\mathbb{P}}(a, r_{k,a}) \times \widehat{\mathbb{P}}(b, s_{k,b})) \setminus (S_{k,a} \cup S^{k,b}).$$

Since $S_{k,a}$ and $S^{k,b}$ are singular with respect to $\{\tilde{f}_{k,a} : f \in \mathcal{F}\}$ and $\{\tilde{f}^{k,b} : f \in \mathcal{F}\}$, respectively, we conclude that

$$S_{k,a} \cap (\widehat{\mathbb{P}}(a, r_{k,a}) \times \widehat{\mathbb{P}}(b, s_{k,b})) = S^{k,b} \cap (\widehat{\mathbb{P}}(a, r_{k,a}) \times \widehat{\mathbb{P}}(b, s_{k,b})).$$

Now let $a', a'' \in A_k^0$ be such that $C := \widehat{\mathbb{P}}(a', r_{k,a'}) \cap \widehat{\mathbb{P}}(a'', r_{k,a''}) \neq \varnothing$. Fix a $b \in B_k^0 \setminus (M_{(a',\cdot)} \cup M_{(a'',\cdot)})$. We already know that

$$\tilde{f}_{k,a'} = \tilde{f}^{k,b} = \tilde{f}_{k,a''} \text{ on } (C \times \widehat{\mathbb{P}}(b, r_{k,b})) \setminus (S_{k,a'} \cup S^{k,b} \cup S_{k,a''}).$$

Hence, by the identity principle, we conclude that

$$\tilde{f}_{k,a'} = \tilde{f}_{k,a''} \text{ on } (C \times G_k) \setminus (S_{k,a'} \cup S_{k,a''})$$

and

$$S_{k,a'} \cap (C \times G_k) = S_{k,a''} \cap (C \times G_k).$$

The same argument works for $b', b'' \in B_k^0$.

Step 5^0. Let U_k be the connected component of $V_k \cap \widehat{X}_k^0$ with $X_k^0 \subset U_k$. Recall that $\widehat{X}_k^0 = \widehat{X}_k$ (cf. Remark 5.1.8 (f)).

To summarize: we have constructed a relatively closed pluripolar set $S_k \subset U_k$ such that

- $S_k \cap X_k^0 \subset M$,
- for any $f \in \mathcal{F}$ there exists an $\tilde{f}_k \in \mathcal{O}(U_k \setminus S_k)$ with $\tilde{f}_k = f$ on $X_k^0 \setminus M$,
- if all the sets $\{S_{k,a},\ S^{k,b} : (a,b) \in \Xi_k\}$ are thin, then S_k is analytic.

Step 6^0. Recall that $X_k^0 \subset U_k \subset \widehat{X}_k$. Observe that the envelope of holomorphy $\widehat{U}_k$ of U_k coincides with $\widehat{X}_k$.

In fact, let $h \in \mathcal{O}(U_k)$, then $h|_{X_k^0} \in \mathcal{O}_s(X_k^0)$. So, by virtue of Theorem 5.4.1, there exists an $\widehat{h} \in \mathcal{O}(\widehat{X}_k)$ with $\widehat{h} = h$ on X_k^0. Hence $\widehat{h} = h$ on U_k.

Step 7^0. Applying Theorem 9.4.2, we find a relatively closed pluripolar set $\widehat{M}_k \subset \widehat{X}_k$ such that

- $\widehat{M}_k \cap U_k \subset S_k$,
- for any $f \in \mathcal{F}$ there exists a function $\hat{f}_k \in \mathcal{O}(\widehat{X}_k \setminus \widehat{M}_k)$ with $\hat{f}_k = \tilde{f}_k$ on $U_k \setminus S_k$ (in particular, $\hat{f}_k = f$ on $X_k^0 \setminus M$),
- the set $\widehat{M}_k$ is singular with respect to the family $\{\hat{f}_k : f \in \mathcal{F}\}$,
- if all the sets $\{S_{k,a},\ S^{k,b} : (a,b) \in \Xi_k\}$ are analytic, then $\widehat{M}_k$ is analytic.

Step 8^0. Recall that $\boldsymbol{X}_k \nearrow \boldsymbol{X}$ and $\widehat{\boldsymbol{X}}_k \nearrow \widehat{\boldsymbol{X}}$ (cf. Remark 5.1.8 (c)). Since $\widehat{M}_k$ is singular with respect to $\{\hat{f}_k : f \in \mathcal{F}\}$, we get $\widehat{M}_{k+1} \cap \widehat{\boldsymbol{X}}_k = \widehat{M}_k$. Consequently:

- $\widehat{M} := \bigcup_{k=1}^{\infty} \widehat{M}_k$ is a relatively closed pluripolar subset of $\widehat{\boldsymbol{X}}$ with $\widehat{M} \cap \boldsymbol{X}^0 \subset M$,
- for each $f \in \mathcal{F}$, the function $\hat{f} := \bigcup_{k=1}^{\infty} \hat{f}_k$ is holomorphic on $\widehat{\boldsymbol{X}} \setminus \widehat{M}$ with $\hat{f} = f$ on $\boldsymbol{X}^0 \setminus M$,
- $\widehat{M}$ is singular with respect to the family $\{\hat{f} : f \in \mathcal{F}\}$,
- if all the sets $\{S_{k,a},\ S^{k,b} : (a,b) \in \Xi_k,\ k \in \mathbb{N}\}$ are thin, then $\widehat{M}$ is analytic. $\square$

We move to the proof of Theorem 10.4.1. The main idea is to apply Theorem 10.4.2. Thus, in fact, we have to check the following lemma.

Lemma 10.4.3. *Under the assumptions of Theorem 10.4.1, for any $a \in A^0$ and a domain of holomorphy $G' \subset\subset G$ with $B^0 \cap G' \neq \varnothing$ there exist an $r > 0$ and a relatively closed pluripolar set $S \subset \widehat{\mathbb{P}}(a,r) \times G' =: V \subset \widehat{\boldsymbol{X}}$ such that*

- $(A^0[a,r] \times G') \cap S \subset M$,
- *for every function $f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$ there exists an $\tilde{f} \in \mathcal{O}(V \setminus S)$ such that $\tilde{f} = f$ on $(A^0[a,r] \times G') \setminus S$,*
- *if all the fibers $M_{(z,\cdot)}$, $z \in A^0$, are thin, then S is analytic.*

First, we reduce the proof of Lemma 10.4.3 to the following lemma.

Lemma 10.4.4. *Suppose that the assumptions of Theorem 10.4.1 are satisfied. Let $(a,b) \in A^0 \times G$, $\widehat{\mathbb{P}}(a,r_0) \subset\subset D$, $\widehat{\mathbb{P}}(b,R_0) \subset\subset G$ be such that $R_0 > r_0$ and $M \cap \widehat{\mathbb{P}}((a,b),r_0) = \varnothing$. Then for every $0 < R' < R_0$ there exist $0 < r' < r_0$ and a relatively closed pluripolar set $S \subset \widehat{\mathbb{P}}(a,r') \times \widehat{\mathbb{P}}(b,R') =: V \subset \widehat{\boldsymbol{X}}$ such that*

- $(A^0[a,r'] \times \widehat{\mathbb{P}}(b,R')) \cap S \subset M$,
- *for every function $h \in \mathcal{O}_s(\boldsymbol{Y} \setminus M)$, where*

$$\begin{aligned}\boldsymbol{Y} &:= \mathbb{X}(A^0[a,r_0], \widehat{\mathbb{P}}(b,r_0); \widehat{\mathbb{P}}(a,r_0), \widehat{\mathbb{P}}(b,R_0)) \\ &= \widehat{\mathbb{P}}((a,b),r_0) \cup (A^0[a,r_0] \times \widehat{\mathbb{P}}(b,R_0)),\end{aligned}$$

there exists an $\tilde{h} \in \mathcal{O}(V \setminus S)$ such that $\tilde{h} = h$ on $(A^0[a,r'] \times \widehat{\mathbb{P}}(b,R')) \setminus M$,
- *if all the fibers $M_{(z,\cdot)}$, $z \in A^0$, are thin, then S is analytic.*

Notice that, by Terada's theorem (Theorem 4.2.2), the space $\mathcal{O}_s(\boldsymbol{Y} \setminus M)$ consists of all functions $h \in \mathcal{O}(\widehat{\mathbb{P}}((a,b),r_0)$ such that $h(z,\cdot)$ extends holomorphically to $\widehat{\mathbb{P}}(b,R_0)$ for every $z \in A^0[a,r_0]$.

Proof that Lemma 10.4.4 implies Lemma 10.4.3. Let a and G' be as in Lemma 10.4.3. Fix a domain $G'' \subset\subset G$ with $G' \subset\subset G''$. Let Ω be the set of all $w \in G''$ such that there exist $r_w > 0$ with $\widehat{\mathbb{P}}((a,w),r_w) \subset \widehat{\boldsymbol{X}} \cap (\widehat{\mathbb{P}}(a,r_0) \times G''))$, and a relatively closed pluripolar set $S_w \subset \widehat{\mathbb{P}}((a,w),r_w)$ such that

- $S_w \cap (A^0[a, r_w] \times \widehat{\mathbb{P}}(w, r_w)) \subset M$,
- every $f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$ extends to an $\tilde{f}_w \in \mathcal{O}(\widehat{\mathbb{P}}((a, w), r_w) \setminus S_w)$ with $\tilde{f}_w = f$ on $(A^0[a, r_w] \times \widehat{\mathbb{P}}(w, r_w)) \setminus M$,
- S_w is singular with respect to the family $\{\tilde{f}_w : f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)\}$,
- if all the fibers $M_{(z,\cdot)}$, $z \in A^0$, are thin, then S is analytic.

It is clear that Ω is open. Observe that $\Omega \neq \varnothing$.

Indeed, since $B^0 \cap G' \setminus M_{(a,\cdot)} \neq \varnothing$, we find a point $w \in B^0 \cap G' \setminus M_{(a,\cdot)}$. Since M is relatively closed, there is a polydisc $\widehat{\mathbb{P}}((a, w), r') \subset \widehat{\boldsymbol{X}} \setminus M$. Put

$$\boldsymbol{Z} := \mathbb{X}(A^0[a, r'], B^0[w, r']; \widehat{\mathbb{P}}(a, r'), \widehat{\mathbb{P}}(w, r')).$$

Observe that for every $f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$ the function $f|_{\boldsymbol{Z}}$ belongs to $\mathcal{O}_s(\boldsymbol{Z})$. Let $0 < r_w < r'$ be such that $\widehat{\mathbb{P}}((a, w), r_w) \subset \widehat{\boldsymbol{Z}}$. By Theorem 5.4.1, for any $f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$ there exists an $\tilde{f}_w \in \mathcal{O}(\widehat{\mathbb{P}}((a, w), r_w))$ with

$$\tilde{f}_w = f \text{ on } \widehat{\mathbb{P}}((a, w), r_w) \cap \boldsymbol{Z} \supset A^0[a, r_w] \times \widehat{\mathbb{P}}(w, r_w).$$

Consequently, $w \in \Omega$.

Moreover, Ω is relatively closed in G''. Indeed, let c be an accumulation point of Ω in G'' and let $\widehat{\mathbb{P}}(c, 3R) \subset G''$. Take a point $w \in \Omega \cap \widehat{\mathbb{P}}(c, R) \setminus M_{(a,\cdot)}$ and let $0 < \rho < \min\{r_w, 2R\}$ be such that $\widehat{\mathbb{P}}((a, w), \rho) \cap (M \cup S_w) = \varnothing$. Observe that $\tilde{f}_w \in \mathcal{O}(\widehat{\mathbb{P}}((a, w), \rho))$ and

$$\tilde{f}_w(z, \cdot) = f(z, \cdot) \in \mathcal{O}(\widehat{\mathbb{P}}(w, \rho) \setminus M_{(z,\cdot)}), \quad z \in A^0[a, \rho].$$

Define

$$\boldsymbol{Y} := \mathbb{X}(A^0[a, \rho], \widehat{\mathbb{P}}(w, \rho); \widehat{\mathbb{P}}(a, \rho), \widehat{\mathbb{P}}(w, 2R)) = \widehat{\mathbb{P}}((a, w), \rho) \cup (A^0[a, \rho] \times \widehat{\mathbb{P}}(w, 2R))$$

and put $\tilde{\tilde{f}}_w : \boldsymbol{Y} \setminus M \to \mathbb{C}$,

$$\tilde{\tilde{f}}_w := \begin{cases} \tilde{f}_w & \text{on } \widehat{\mathbb{P}}((a, w), \rho), \\ f & \text{on } (A^0[a, \rho] \times \widehat{\mathbb{P}}(w, 2R)) \setminus M. \end{cases}$$

Then $\tilde{\tilde{f}}_w$ is well defined and $\tilde{\tilde{f}}_w \in \mathcal{O}_s(\boldsymbol{Y} \setminus M)$. Now, by Lemma 10.4.4 (with $b := w$, $r_0 := \rho$, $R_0 := 2R$, $R' := R$), there exist $0 < r' < \rho$ and a relatively closed pluripolar set $S \subset \widehat{\mathbb{P}}(a, r') \times \widehat{\mathbb{P}}(w, R)$ such that

- $S \cap (A^0[a, r'] \times \widehat{\mathbb{P}}(w, R)) \subset M$,
- every $f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$ extends to an $\hat{f}_w \in \mathcal{O}((\widehat{\mathbb{P}}(a, r') \times \widehat{\mathbb{P}}(w, R)) \setminus S)$ with $\hat{f}_w = \tilde{\tilde{f}}_w$ on $(A^0[a, r'] \times \widehat{\mathbb{P}}(w, R)) \setminus M$,
- S is singular with respect to the family $\{\hat{f}_w : f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)\}$,
- if all the fibers $M_{(z,\cdot)}$, $z \in A^0$, are thin, then S is analytic.

Take an $r_c > 0$ so small that $\widehat{\mathbb{P}}((a,c),r_c) \subset \widehat{\mathbb{P}}(a,\rho') \times \widehat{\mathbb{P}}(w,R)$ and put

$$S_c := S \cap \widehat{\mathbb{P}}((a,c),r_c), \quad \tilde{f}_c := \hat{f}_w|_{\widehat{\mathbb{P}}((a,c),r_c)\setminus S}.$$

Obviously $\tilde{f}_c = \hat{f}_w = \tilde{\tilde{f}}_w = f$ on $(A^0[a,r_c] \times \widehat{\mathbb{P}}(c,r_c)) \setminus M$. Hence $c \in \Omega$.

Thus $\Omega = G''$.

There exists a finite set $T \subset \bar{G}'$ such that

$$\bar{G}' \subset \bigcup_{w \in T} \widehat{\mathbb{P}}(w,r_w).$$

Define $r := \min\{r_w : w \in T\}$. Take $w', w'' \in T$ with

$$C := \widehat{\mathbb{P}}(w',r_{w'}) \cap \widehat{\mathbb{P}}(w'',r_{w''}) \neq \varnothing.$$

Then $\tilde{f}_{w'} = f = \tilde{f}_{w''}$ on $(A^0[a,r] \times C) \setminus M$. Consequently, $\tilde{f}_{w'} = \tilde{f}_{w''}$ on $(\widehat{\mathbb{P}}(a,r) \times C) \setminus (S_{w'} \cup S_{w''})$. Since $S_{w'}$ and $S_{w''}$ are singular, we conclude that they coincide on $\widehat{\mathbb{P}}(a,r) \times C$ and that the functions $\tilde{f}_{w'}$ and $\tilde{f}_{w''}$ glue together.

Thus we get a relatively closed pluripolar set $S \subset \widehat{\mathbb{P}}(a,r) \times G' =: V$ such that $S \cap (A^0[a,r]) \times G') \subset M$ and any function $f \in \mathcal{O}_s(\boldsymbol{X} \setminus M)$ extends holomorphically to an $\tilde{f} \in \mathcal{O}(V \setminus S)$ with $\tilde{f} = f$ on $(A^0[a,r] \times G') \setminus M$. □

In the next step we reduce the proof of Lemma 10.4.4 to the following lemma.

Lemma 10.4.5. *Let $A \subset \mathbb{P}(r_0) \subset \mathbb{C}^p$ be locally pluriregular and let M be a relatively closed subset of the cross $\boldsymbol{Z} := \mathbb{X}(A, \mathbb{D}(r_0); \mathbb{P}(r_0), \mathbb{D}(R_0))$ with $R_0 > r_0$ such that*

- *the fiber $M_{(z,\cdot)}$ is polar for all $z \in A^0 \subset A$, where $A \setminus A^0$ is pluripolar,*
- *$M \cap (\mathbb{P}(r_0) \times \mathbb{D}(r_0))$ is pluripolar,*
- *$B^0 := \{w \in \mathbb{D}(r_0) : M_{(\cdot,w)} \in \mathcal{PLP}\}$ (note that the set $\mathbb{D}(r_0) \setminus B^0$ is polar).*

Then there exists a relatively closed pluripolar set $\widehat{M} \subset \widehat{\boldsymbol{Z}}$ such that

- *$\widehat{M} \cap \boldsymbol{Z}^0 \subset M$ with $\boldsymbol{Z}^0 := \mathbb{X}(A^0, B^0; \mathbb{P}(r_0), \mathbb{D}(R_0))$,*
- *for every $f \in \mathcal{F} := \mathcal{O}((\mathbb{P}(r_0) \times \mathbb{D}(r_0)) \setminus M) \cap \mathcal{O}_s(\boldsymbol{Z} \setminus M)$ there exists an $\hat{f} \in \mathcal{O}(\widehat{\boldsymbol{Z}} \setminus \widehat{M})$ such that $\hat{f} = f$ on $\boldsymbol{Z}^0 \setminus M$,*
- *if $M \cap (\mathbb{P}(r_0) \times \mathbb{D}(r_0))$ is analytic and all the fibers $M_{(z,\cdot)}$, $z \in A^0$, are thin, then $\widehat{M}$ is analytic.*

Remark 10.4.6. Observe that if $M \cap (\mathbb{P}(r_0) \times \mathbb{D}(r_0)) \neq \varnothing$ and $f \in \mathcal{O}_s(\boldsymbol{Z} \setminus M)$, then f need not belong to $\mathcal{O}((\mathbb{P}(r_0) \times \mathbb{D}(r_0)) \setminus M)$. For example: take $p = 1$, $r_0 = 1$, assume additionally that $A \subsetneq \mathbb{D}$ is closed in $\mathbb{D}$, and let $M := A \times \{0\}$. Define $f : \boldsymbol{Z} \setminus M \to \mathbb{C}$, $f(z, w) := 0$ if $w \neq 0$, $f(z, 0) := 1$. Then $f \in \mathcal{O}_s(\boldsymbol{Z} \setminus M) \setminus \mathcal{O}(\mathbb{D}^2 \setminus M)$.

Proof that Lemma 10.4.5 implies Lemma 10.4.4. See for example [Jar-Pfl 2003a] and [Jar-Pfl 2003b]. Consider a configuration such as in Lemma 10.4.4. We may assume that $\widehat{\mathbb{P}}(a, r_0) = \mathbb{P}_p(r_0) \subset \mathbb{C}^p$, $\widehat{\mathbb{P}}(b, R_0) = \mathbb{P}_q(R_0) \subset \mathbb{C}^q$. Put

$$\boldsymbol{Y} := \mathbb{X}(A^0[0, r_0], \mathbb{P}_q(r_0); \mathbb{P}_p(r_0), \mathbb{P}_q(R_0)) = \mathbb{P}_{p+q}(r_0) \cup (A^0[0, r_0] \times \mathbb{P}_q(R_0)).$$

Let R_0' be the supremum of all $0 < R' < R_0$ such that there exist an $r = r_{R'} \in (0, r_0)$, and a relatively closed pluripolar set $S = S_{R'} \subset V := \mathbb{P}_p(r) \times \mathbb{P}_q(R')$ for which:

- $S \cap (A^0[0, r] \times \mathbb{P}_q(R_0)) \subset M$,
- for any function $h \in \mathcal{O}_s(\boldsymbol{Y} \setminus M)$ there exists an $\tilde{h} = \tilde{\boldsymbol{h}}_{R'} \in \mathcal{O}(V \setminus S)$ such that $\tilde{h} = h$ on $(A^0[0, r] \times \mathbb{P}_q(R')) \setminus M$,
- the set S is singular with respect to the family $\{\tilde{h} : h \in \mathcal{O}_s(\boldsymbol{Y} \setminus M)\}$ (in particular, $S \cap \mathbb{P}_{p+q}(r_0) = \varnothing$),
- if all the fibers $M_{(z,\cdot)}$, $z \in A^0$, are thin, then S is analytic.

It suffices to show that $R_0' = R_0$. Suppose that $R_0' < R_0$. Fix $R_0' < R'' < R_0$ and choose $0 < R' < \tilde{R}' < R_0'$ such that $\sqrt[q]{R'^{q-1} R''} > R_0'$. Let $r := r_{\tilde{R}'}$, $S := S_{\tilde{R}'}$, $\tilde{h} := \tilde{\boldsymbol{h}}_{\tilde{R}'}$. Fix an R''' with $R_0' < R''' < \sqrt[q]{R'^{q-1} R''}$. Put

$$M_q := \big(S \cap (\mathbb{P}_p(r) \times \overline{\mathbb{P}}_q(R'))\big) \cup \big(M \setminus (\mathbb{P}_p(r) \times \mathbb{P}_q(R'))\big).$$

Observe that

- the set $M_q \cap (\mathbb{P}_p(r) \times \mathbb{P}_q(R')) = S \cap (\mathbb{P}_p(r) \times \mathbb{P}_q(R'))$ is pluripolar,
- $\tilde{h} \in \mathcal{O}((\mathbb{P}_p(r) \times \mathbb{P}_q(R')) \setminus M_q)$ for every $h \in \mathcal{O}_s(\boldsymbol{Y} \setminus M)$,
- $M_q \cap \boldsymbol{Y} \subset M$,
- if all the fibers $M_{(z,\cdot)}$, $z \in A^0$, are thin, then the set $M_q \cap (\mathbb{P}_p(r) \times \mathbb{P}_q(R')) = S \cap (\mathbb{P}_p(r) \times \mathbb{P}_q(R'))$ is analytic and all the fibers $(M_q)_{(z,\cdot)} \subset M_{(z,\cdot)}$, $z \in A^0[0, r]$, are thin.

Write $w = (w', w_q) \in \mathbb{C}^q = \mathbb{C}^{q-1} \times \mathbb{C}$. Let

$$C := \{(z, w') \in A^0[0, r] \times \mathbb{P}_{q-1}(R') : (M_q)_{(z,w',\cdot)} \text{ is polar}\}.$$

In the case where all the fibers $M_{(z,\cdot)}$, $z \in A^0$, are thin we put

$$C := \{(z, w') \in A^0[0, r] \times \mathbb{P}_{q-1}(R') : (M_q)_{(z,w',\cdot)} \text{ is discrete}\}.$$

By Proposition 3.2.21, C is locally pluriregular. Observe that for every $c \in C$ and for every $h \in \mathcal{O}_s(\boldsymbol{Y} \setminus M)$, the function $\tilde{h}(c, \cdot)$ is holomorphic in $\mathbb{D}(R_0) \setminus (M_q)_{(c,\cdot)}$.

Consequently, applying Lemma 10.4.5 to the cross

$$\boldsymbol{Z}_q := \mathbb{X}(C, \mathbb{D}(R'); \mathbb{P}_p(r) \times \mathbb{P}_{q-1}(R'), \mathbb{D}(R_0)),$$

we conclude that there exists a relatively closed pluripolar set $S_q \subset \widehat{\boldsymbol{Z}}_q$ such that

- $S_q \cap \boldsymbol{Z}_q^0 \subset M_q$, where $\boldsymbol{Z}_q^0$ is constructed according to Lemma 10.4.5,
- any function $h \in \mathcal{O}_s(\boldsymbol{Y} \setminus M)$ extends holomorphically to a $\tilde{h}_q \in \mathcal{O}(\widehat{\boldsymbol{Z}}_q \setminus S_q)$ with $\tilde{h}_q = h$ on $\boldsymbol{Z}_q^0 \setminus M_q$,
- S_q is singular with respect to the family $\{\tilde{h}_q : h \in \mathcal{O}_s(\boldsymbol{Y} \setminus M)\}$,
- if all the fibers $M_{(z,\cdot)}$, $z \in A^0$, are thin, then S_q is analytic.

Using the product property of the relative extremal function (Theorem 3.2.17), we get

$$\begin{aligned}\widehat{\boldsymbol{Z}}_q &= \{(z, w', w_q) \in \mathbb{P}_p(r) \times \mathbb{P}_{q-1}(R') \times \mathbb{D}(R_0) : \\ &\qquad \boldsymbol{h}^*_{C,\mathbb{P}_p(r)\times\mathbb{P}_{q-1}(R')}(z, w') + \boldsymbol{h}^*_{\mathbb{D}(R'),\mathbb{D}(R_0)}(w_q) < 1\} \\ &= \{(z, w', w_q) \in \mathbb{P}_p(r) \times \mathbb{P}_{q-1}(R') \times \mathbb{D}(R_0) : \\ &\qquad \boldsymbol{h}^*_{A^0\times\mathbb{P}_{q-1}(R'),\mathbb{P}_p(r)\times\mathbb{P}_{q-1}(R')}(z, w') + \boldsymbol{h}^*_{\mathbb{D}(R'),\mathbb{D}(R_0)}(w_q) < 1\} \\ &= \{(z, w', w_q) \in \mathbb{P}_p(r) \times \mathbb{P}_{q-1}(R') \times \mathbb{D}(R_0) : \\ &\qquad \boldsymbol{h}^*_{A,\mathbb{P}_p(r)}(z) + \boldsymbol{h}^*_{\mathbb{D}(R'),\mathbb{D}(R_0)}(w_q) < 1\}.\end{aligned}$$

Consequently, since $R'' < R_0$, we find an $r_q \in (0, r)$ such that

$$\mathbb{P}_p(r_q) \times \mathbb{P}_{q-1}(R') \times \mathbb{D}(R'') \subset \widehat{\boldsymbol{Z}}_q.$$

Thus any function $h \in \mathcal{O}_s(\boldsymbol{Y} \setminus M)$ extends holomorphically to a function $\tilde{h}_q$ on $(\mathbb{P}_p(r_q) \times \mathbb{P}_{q-1}(R') \times \mathbb{D}(R'')) \setminus S_q$ and S_q is singular with respect to the family $\{\tilde{h}_q : h \in \mathcal{O}_s(\boldsymbol{Y} \setminus M)\}$.

Repeating the above argument for the coordinates w_ν, $\nu = 1, \dots, q-1$, and gluing the obtained sets, we find an $r_* \in (0, r)$ and a relatively closed pluripolar set $S_0 := \bigcup_{j=1}^q S_j$ such that any function $h \in \mathcal{O}_s(\boldsymbol{Y} \setminus M)$ extends holomorphically to a function $\tilde{h}_0 := \bigcup_{j=1}^q \tilde{h}_j$ holomorphic in $\mathbb{P}_p(r_*) \times W \setminus S_0$, where

$$W := \bigcup_{j=1}^{q} \mathbb{P}_{j-1}(R') \times \mathbb{P}(R'') \times \mathbb{P}_{q-j}(R').$$

Observe that W is a complete Reinhardt domain in $\mathbb{C}^q$. Let $\widehat{W}$ denote the envelope of holomorphy of W (it is known that $\widehat{W}$ is a complete logarithmically convex Reinhardt domain in $\mathbb{C}^q$ – cf. § 2.9). Applying Theorem 9.4.2, we find a relatively closed pluripolar subset $\widehat{S}_0$ of $\mathbb{P}_p(r_*) \times \widehat{W}$ such that

- $\widehat{S}_0 \cap (\mathbb{P}_p(r_*) \times \widehat{W}) \subset S_0$,
- any function $h \in \mathcal{O}_s(\boldsymbol{Y} \setminus M)$ extends to an $\tilde{h} \in \mathcal{O}(\mathbb{P}_p(r_*) \times \widehat{W} \setminus \widehat{S}_0)$,
- $\widehat{S}_0$ is singular with respect to the family $\{\tilde{h} : h \in \mathcal{O}_s(\boldsymbol{Y} \setminus M)\}$,
- if all the fibers $M_{(a,\cdot)}$, $a \in A^0$, are thin, then $\widehat{S}_0$ is analytic.

Since $\widehat{W}$ is logarithmically convex, we must have $\mathbb{P}_q(\sqrt[q]{R'^{q-1}R'_0}) \subset \widehat{W}$. Consequently, $\mathbb{P}_q(R''') \subset \widehat{W}$. Recall that $R''' > R'_0$. Let $0 < \rho < r_*$ be such that $\mathbb{P}_p(\rho) \times \mathbb{P}_q(R''') \subset \mathbb{P}_p(r_*) \times \widehat{W}$. Put $r_{R'''} := \rho$, $S' = S_{R'''} := \widehat{S}_0 \cap (\mathbb{P}_p(\rho) \times \mathbb{P}_q(R'''))$. Then any function $h \in \mathcal{O}_s(\boldsymbol{Y} \setminus M)$ extends holomorphically to $(\mathbb{P}_p(\rho) \times \mathbb{P}_q(R''')) \setminus S'$.

To get a contradiction it remains to show that $S' \cap (A^0[0,\rho] \times \mathbb{P}_q(R''')) \subset M$. Take $(z,w) \in (A^0[0,\rho] \times \mathbb{P}_q(R''')) \setminus M$. Since $M_{(z,\cdot)}$ is pluripolar, there exists a curve $\gamma\colon [0,1] \to \mathbb{P}_q(R''') \setminus M_{(z,\cdot)}$ such that $\gamma(0) = 0$, $\gamma(1) = w$. We may assume that for small $\varepsilon > 0$ we have

$$\mathbb{P}_p(z,\varepsilon) \times (\gamma([0,1]) + \mathbb{P}_q(\varepsilon)) \subset\subset (\mathbb{P}_p(\rho) \times \mathbb{P}_q(R''')) \setminus M.$$

Put $V_w := \gamma([0,1]) + \mathbb{P}_q(\varepsilon)$. Consider the cross

$$\boldsymbol{W} := \mathbb{X}(A[z,\varepsilon], \mathbb{P}_q(\varepsilon); \mathbb{P}_p(z,\varepsilon), V_w).$$

Then $h \in \mathcal{O}_s(\boldsymbol{W})$ for any $h \in \mathcal{O}_s(\boldsymbol{Y} \setminus M)$. Consequently, by Theorem 5.4.1, $(z,w) \in \widehat{\boldsymbol{W}} \subset \mathbb{P}_p(r) \times \mathbb{P}_q(R''') \setminus S'$. □

Thus, it remains to prove Lemma 10.4.5.

Proof of Lemma 10.4.5. We are going to apply Theorem 10.4.2 (with $D := \mathbb{P}(r_0)$, $G := \mathbb{D}(R_0)$, $B := \mathbb{D}(r_0)$, $B^0 := \{b \in B : M_{(\cdot,b)} \in \mathcal{PLP}\}$). Keep all the notation from Theorem 10.4.2. Assume additionally that $B = \mathbb{D}(r_0) \subset\subset G_k$ for every k. Take $(a,b) \in \Xi_k = A^0_k \times B^0_k \setminus M$.

The "horizontal" direction is simple: we take $s = s_{k,b} > 0$ such that $\mathbb{D}(b,s) \subset \mathbb{D}(r_0)$ and let $\widetilde{S}^{k,b} := M \cap (D_k \times \mathbb{D}(b,s)) =: V^{k,b}$; $\widetilde{S}^{k,b}$ is relatively closed pluripolar. Let $S^{k,b}$ be the singular part of $\widetilde{S}^{k,b}$ with respect to the family $\{f|_{V^{k,b}\setminus M} : f \in \mathcal{F}\}$ and let $\tilde{f}^{k,b}$ denote the extension of $f|_{V^{k,b}\setminus M}$ to $V^{k,b} \setminus S^{k,b}$.

The "vertical" direction is more complicated: we have to show that there exist an $r = r_{k,a} > 0$ and a relatively closed pluripolar set $S \subset \widehat{\mathbb{P}}(a,r) \times G_k =: V_{k,a}$ such that

- $\widehat{\mathbb{P}}(a,r) \subset D_k$,
- $S \cap (A^0[a,r] \times G_k) \subset M$,
- any function $f \in \mathcal{F}$ extends to an $\tilde{f} \in \mathcal{O}(V_{k,a} \setminus S)$ with $\tilde{f} = f$ on $(A^0[a,r] \times G_k) \setminus M$.

For $c \in \mathbb{D}(R_0)$, let $\rho = \rho_c > 0$ be such that $\mathbb{D}(c,\rho) \subset\subset \mathbb{D}(R_0)$ and $M_{(a,\cdot)} \cap \partial\mathbb{D}(c,\rho) = \varnothing$ (cf. Proposition 2.3.21). Take $\rho^- = \rho^-_c > 0$, $\rho^+ = \rho^+_c > 0$ such that $\rho^- < \rho < \rho^+$, $\mathbb{D}(c, \rho^+) \subset\subset \mathbb{D}(R_0)$, and $M_{(a,\cdot)} \cap \overline{P} = \varnothing$, where $P = P_c := \mathbb{A}(c, \rho^-, \rho^+)$. Let $\gamma\colon [0, 1] \to G \setminus M_{(a,\cdot)}$ be a curve such that $\gamma(0) = 0$ and $\gamma(1) \in \partial\mathbb{D}(c, \rho)$. There exists an $\varepsilon = \varepsilon_c > 0$ such that

$$\big(\mathbb{P}(a, \varepsilon) \times ((\gamma([0, 1]) + \mathbb{D}(\varepsilon)) \cup P)\big) \cap M = \varnothing.$$

Put $V = V_c := \mathbb{D}(r_0) \cup (\gamma([0, 1]) + \mathbb{D}(\varepsilon)) \cup P$ and consider the cross

$$\boldsymbol{Y} = \boldsymbol{Y}_c := \mathbb{X}(A[a, \varepsilon], \mathbb{D}(r_0); \mathbb{P}(a, \varepsilon), V).$$

Then $f \in \mathcal{O}_s(\boldsymbol{Y})$ for any $f \in \mathcal{F}$. Consequently, by Theorem 5.4.1, any function from $\mathcal{F}$ extends holomorphically to $\widehat{\boldsymbol{Y}} \supset \{a\} \times V$. Shrinking P, ε and V, we may assume that any function $f \in \mathcal{F}$ extends to a function $\tilde{f} = \tilde{f}_c \in \mathcal{O}(\mathbb{P}(a, \varepsilon) \times W)$, where

$$W = W_c := \mathbb{D}(r_0 - \varepsilon) \cup (\gamma([0, 1]) + \mathbb{D}(\varepsilon)) \cup P.$$

In particular, $\tilde{f}$ is holomorphic in $\mathbb{P}(a, \varepsilon) \times P$, and therefore may be represented by the Hartogs–Laurent series

$$\tilde{f}(z, w) = \sum_{\nu=0}^{\infty} \tilde{f}_\nu(z)(w - c)^\nu + \sum_{\nu=1}^{\infty} \tilde{f}_{-\nu}(z)(w - c)^{-\nu} =: \tilde{f}^+(z, w) + \tilde{f}^-(z, w),$$
$$(z, w) \in \mathbb{P}(a, \varepsilon) \times P,$$

where $\tilde{f}^+ \in \mathcal{O}(\mathbb{P}(a, \varepsilon) \times \mathbb{P}(c, \rho^+))$ and $\tilde{f}^- \in \mathcal{O}(\mathbb{P}(a, \varepsilon) \times (\mathbb{C} \setminus \overline{\mathbb{P}}(c, \rho^-)))$. Recall that for any $z \in A^0[a, \varepsilon]$ the function $\tilde{f}(z, \cdot)$ extends holomorphically to $\mathbb{D}(R_0) \setminus M_{(z,\cdot)}$. Consequently, for any $z \in A^0[a, \varepsilon]$ the function $\tilde{f}^-(z, \cdot)$ extends holomorphically to $\mathbb{C} \setminus (M_{(z,\cdot)} \cap \overline{\mathbb{D}}(c, \rho^-))$. Now, by Theorem 9.2.24, there exists a relatively closed pluripolar set $S = S_c \subset \mathbb{P}(a, \varepsilon) \times \overline{\mathbb{D}}(c, \rho^-)$ such that

- $S \cap (A^0[a, \varepsilon] \times \overline{\mathbb{D}}(c, \rho^-)) \subset M$,
- any function $\tilde{f}^-$ extends holomorphically to an $\approx\!\!f^- \in \mathcal{O}(\mathbb{P}(a, \varepsilon) \times \mathbb{C} \setminus S)$.

Since $\tilde{f} = \tilde{f}^+ + \tilde{f}^-$, the function $\tilde{f}$ extends holomorphically to a function $\hat{f} = \hat{f}_c \in \mathcal{O}(\mathbb{P}(a, \varepsilon) \times \mathbb{D}(c, \rho^+) \setminus S)$. We may assume that the set S is singular with respect to the family $\{\hat{f} : f \in \mathcal{F}\}$. In particular, if $c', c'' \in \mathbb{D}(R_0)$ and $C := \mathbb{D}(c', \rho^+_{c'}) \cap \mathbb{D}(c'', \rho^+_{c''}) \neq \varnothing$, then

$$S_{c'} \cap (\mathbb{P}(a, \eta) \times C) = S_{c''} \cap (\mathbb{P}(a, \eta) \times C), \quad \hat{f}_{c'} = \hat{f}_{c''} \text{ on } \mathbb{P}(a, \eta) \times C,$$

where $\eta := \min\{\varepsilon_{c'}, \varepsilon_{c''}\}$. Thus the functions $\hat{f}_{c'}$, $\hat{f}_{c''}$ and sets $S_{c'}$, $S_{c''}$ may be glued together.

Now select $c_1, \dots, c_s \in \mathbb{D}(R_0)$ so that $\overline{G}_k \subset \bigcup_{j=1}^s \mathbb{D}(c_j, \rho^+_{c_j})$. Put

$$r = r_{k,a} := \min\{\varepsilon_{c_j} : j = 1, \dots, s\}, \quad V_{k,a} := \mathbb{P}(a, r) \times G_k.$$

Then $S := V_{k,a} \cap \bigcup_{j=1}^{s} S_{c_j}$ gives the required relatively closed pluripolar subset of $V_{k,a}$ such that $S \cap (A^0[a,r] \times G_k) \subset M$ and for any $f \in \mathcal{F}$, the function $\tilde{f} := \bigcup_{j=1}^{s} \hat{f}_{c_j}$ extends holomorphically f to $V_{k,a} \setminus S$ with $\tilde{f} = f$ on $(A^0[a,r] \times G_k) \setminus M$. □

The proof of Theorem 10.4.1 is completed.

Theorem 10.4.1 gives the following generalization of Theorem 9.2.24.

Theorem 10.4.7. *Let $D \subset \mathbb{C}^p$ be a domain of holomorphy, let $A \subset D$ be locally pluriregular, and let $\varnothing \neq \Delta_0 \subset \mathbb{C}^q$ be a domain. Assume that for each $a \in A$ we are given a closed pluripolar set $M(a) \subset \mathbb{C}^q$ with $M(a) \cap \Delta_0 = \varnothing$. Define*

$$\mathcal{S} := \{f \in \mathcal{O}(D \times \Delta_0) : \forall_{a \in A}\ \exists_{\tilde{f}_a \in \mathcal{O}(\mathbb{C}^q \setminus M(a))} : \ \tilde{f}_a(w) = f(a,w),\ w \in \Delta_0\}.$$

Then the $\mathcal{S}$-envelope of holomorphy of $D \times \Delta_0$ is of the form $(D \times \mathbb{C}^q) \setminus \widehat{M}$, where $\widehat{M}$ is a pluripolar set such that

- $\widehat{M} \cap (D \times \Delta_0) = \varnothing$,
- $\widehat{M}_{(a,\cdot)} \subset M(a)$, *$a \in A \setminus P$, where $P \subset A$ is pluripolar,*
- *if all the sets $M(a)$, $a \in A$, are thin, then $\widehat{M}$ is analytic.*

Proof. By Lemma 9.1.5 there exists a pluripolar set $P \subset A$ such that the set $M := \bigcup_{a \in A \setminus P} \{a\} \times M(a)$ is relatively closed in $(A \setminus P) \times \mathbb{C}^q$. Now we apply Theorem 10.4.1 to $(D, G, A, A_0, B, B_0, M) := (D, \mathbb{C}^q, A \setminus P, A \setminus P, \Delta_0, \Delta_0, M)$; notice that $\mathcal{S} \subset \mathcal{O}_s(X \setminus M)$. □

10.5 Separately pluriharmonic functions III

Having Theorem 10.4.7 we may partially generalize Proposition 9.3.1.

Proposition 10.5.1. *Let $D \subset \mathbb{C}^p$ be a domain of holomorphy, let $A \subset D$ be locally pluriregular, and let $\varnothing \neq \Delta_0 \subset \mathbb{C}^q$ be a domain. Assume that for each $a \in A$ we are given a closed thin set $M(a) \subset \mathbb{C}^q$ with $M(a) \cap \Delta_0 = \varnothing$. Define*

$$\mathcal{F} := \{u \in \mathcal{PH}(D \times \Delta_0) : \forall_{a \in A}\ \exists_{\tilde{u}_a \in \mathcal{PH}(\mathbb{C}^q \setminus M(a))} : \ \tilde{u}_a(w) = u(a,w),\ w \in \Delta_0\}.$$

Then there exists a pseudoconcave analytic set $\widehat{M} \subset D \times \mathbb{C}^q$ such that

- $\widehat{M}_{(a,\cdot)} \subset M(a)$, *$a \in A \setminus P$, where $P \subset A$ is pluripolar,*
- *every function $u \in \mathcal{F}$ extends to a multivalued pluriharmonic function $\hat{u}$ on $(D \times \mathbb{C}^q) \setminus \widehat{M}$.*

Moreover, if A is dense in D, then $\hat{u}$ is univalent (*cf.* [Sad 2005], [Sad-Imo 2006b] *for the case $q = 1$*).

[?] It is an open question what are "optimal" conditions on A, under which the extension $\hat{u}$ is univalent for each $u \in \mathcal{F}$ [?] (cf. Proposition 9.3.1).

Proof. We may assume that $0 \in \Delta_0$. Observe that for each $z \in D$ the fiber $\widehat{M}_{(z,\cdot)}$ is pluripolar. Let $f_j := \frac{\partial u}{\partial w_j}$, $j = 1, \dots, q$. Then $f_j \in \mathcal{O}(D \times \Delta_0)$ and for each $a \in A$ the function $\frac{\partial \tilde{u}_a}{\partial w_j}$ gives a holomorphic extension of $f_j(a, \cdot)$ to $\mathbb{C}^q \setminus M(a)$, $j = 1, \dots, q$. Thus, we may apply Theorem 10.4.7 and we get a pseudoconcave analytic set $\widehat{M} \subset D \times \mathbb{C}^q$ such that

- $\widehat{M}_{(a,\cdot)} \subset M(a)$, $a \in A \setminus P$, where $P \subset A$ is pluripolar,
- f_j extends to an $\hat{f}_j \in \mathcal{O}((D \times \mathbb{C}^q) \setminus \widehat{M})$, $j = 1, \dots, q$.

Define

$$F(z, w) := \int_0^w \sum_{j=1}^q \hat{f}_j(z, \zeta) d\zeta_j, \quad (z, w) \in (D \times \mathbb{C}^q) \setminus \widehat{M},$$

where the integral is taken over an arbitrary piecewise $\mathcal{C}^1$ curve $\gamma\colon [0, 1] \to \mathbb{C}^q \setminus \widehat{M}_{(z,\cdot)}$ with $\gamma(0) = 0$, $\gamma(1) = w$. Then F defines a multivalued holomorphic function in $(D \times \mathbb{C}^q) \setminus \widehat{M}$. Thus $\hat{u}(z, w) := 2\operatorname{Re}(F(z, w)) + u(z, 0)$ defines a multivalued pluriharmonic function on $(D \times \mathbb{C}^q) \setminus \widehat{M}$. Observe that if $a \in A \setminus P$, then

$$\begin{aligned}\hat{u}(a, w) &= 2\operatorname{Re}\Big(\int_0^w \sum_{j=1}^q \frac{\partial \tilde{u}_a}{\partial w_j} d\zeta_j\Big) + u(a, 0)\\ &\overset{\zeta_j = \xi_j + i\eta_j}{=} \operatorname{Re}\Big(\int_0^w \sum_{j=1}^q \Big(\frac{\partial \tilde{u}_a}{\partial \xi_j} d\xi_j + \frac{\partial \tilde{u}_a}{\partial \eta_j} d\eta_j\Big) - i\int_0^w \sum_{j=1}^q \Big(\frac{\partial \tilde{u}_a}{\partial \eta_j} d\xi_j - \frac{\partial \tilde{u}_a}{\partial \xi_j} d\eta_j\Big)\Big)\\ &\qquad + u(a, 0) = \tilde{u}_a(w).\end{aligned}$$

Consequently, $\hat{u}$ is an extension of u.

Now assume that A is dense in D. For any non-empty domain $U \subset D$ and a piecewise $\mathcal{C}^1$-curve $\gamma\colon [0, 1] \to \mathbb{C}^q$ with $\gamma(0) = \gamma(1)$ and $U \times \gamma([0, 1]) \subset (D \times \mathbb{C}^q) \setminus \widehat{M}$, define

$$\varphi_{U,\gamma}(z) := \operatorname{Re}\Big(\int_\gamma \sum_{j=1}^q \hat{f}_j(z, \zeta) d\zeta_j\Big), \quad z \in U.$$

Then

- $\varphi_{U,\gamma} \in \mathcal{PH}(U)$;
- $\varphi_{U,\gamma} = 0$ on $U \cap (A \setminus P)$ and so $\varphi_{U,\gamma} \equiv 0$.

So it remains to note that $\hat{u}$ is univalent iff for all (U, γ) (as above) we have $\varphi_{U,\gamma} \equiv 0$. $\square$

10.6 Proof of Theorem 10.2.6 in the general case

$\boxed{>}$ §§ 1.4, 2.3, 2.4, 5.1, 7.1, 9.1, 9.4, 10.2, 10.4.

As we already mentioned, the geometric structure of generalized N-fold crosses with $N \geq 3$ is much more complicated than in the case $N = 2$. This will be reflected through the proof. The main tool is again a gluing result (cf. Theorem 10.4.2).

Theorem 10.6.1 (Gluing theorem II). *Assume that Theorem 10.2.6 was already proved for all $(N-2)$-fold crosses if $N \geq 4$.*

Let $D_j, A_j, \Sigma_j, \Sigma_j^0$, $j = 1, \dots, N$, $\boldsymbol{X}$, $\boldsymbol{T}$, $\boldsymbol{T}^0$, M, $\mathcal{F}$ be as in Theorem 10.2.6 (with (C1)–(C8) from § 10.2).

Let $(D_{j,k})_{k=1}^{\infty}$ be an exhaustion sequence for D_j (cf. Definition 1.4.5) such that each $D_{j,k}$ is a domain of holomorphy and $A_{j,k} := A_j \cap D_{j,k} \neq \varnothing$, $k \in \mathbb{N}$, $j = 1, \dots, N$. Put

$$A'_{j,k} := A_{1,k} \times \dots \times A_{j-1,k}, \quad A''_{j,k} := A_{j+1,k} \times \dots \times A_{N,k},$$

$$\Sigma_{j,k} := \Sigma_j \cap (A'_{j,k} \times A''_{j,k}), \quad \Sigma^0_{j,k} := \Sigma^0_j \cap (A'_{j,k} \times A''_{j,k}),$$

$$\boldsymbol{X}_k := \boldsymbol{X} \cap (D_{1,k} \times \dots \times D_{N,k}) = \mathbb{X}((A_{j,k}, D_{j,k})_{j=1}^{N}),$$

$$\boldsymbol{T}_k := \boldsymbol{T} \cap (D_{1,k} \times \dots \times D_{N,k}) = \mathbb{T}((A_{j,k}, D_{j,k}, \Sigma_{j,k})_{j=1}^{N}),$$

$$\boldsymbol{T}^0_k := \boldsymbol{T}^0 \cap (D_{1,k} \times \dots \times D_{N,k}) = \mathbb{T}((A_{j,k}, D_{j,k}, \Sigma^0_{j,k})_{j=1}^{N}).$$

Assume that for any $j \in \{1, \dots, N\}$, $k \in \mathbb{N}$, and $a \in \Xi_k := \boldsymbol{c}(\boldsymbol{T}^0_k) \setminus M$, there exist

- $r = r_{k,a} \in (0, \rho_a)$,
- *a relatively closed pluripolar set* $S_{j,k,a} \subset \widehat{\mathbb{P}}(a'_j, r) \times D_{j,k} \times \widehat{\mathbb{P}}(a''_j, r) =: V_{j,k,a}$,
- *a pluripolar set* $P_{j,k,a} \subset (A'_{j,k} \times A''_{j,k}) \setminus \Sigma^0_j$,

such that

- $\widehat{\mathbb{P}}(a, r) \subset D_{1,k} \times \dots \times D_{N,k}$,
- $S_{j,k,a} \cap \boldsymbol{T}^0_{j,k,a} \subset M$, *where*

$$\boldsymbol{T}^0_{j,k,a} := \{(z'_j, z_j, z''_j) \in A'_{j,k}[a'_j, r] \times A_{j,k} \times A''_{j,k}[a''_j, r] : (z'_j, z''_j) \notin \Sigma^0_{j,k} \cup P_{j,k,a}\},$$

- *for any $f \in \mathcal{F}$ there exists an $\tilde{f}_{j,k,a} \in \mathcal{O}(V_{j,k,a} \setminus S_{j,k,a})$ with $\tilde{f}_{j,k,a} = f$ on $\boldsymbol{T}^0_{j,k,a} \setminus M$.*

Then there exists a relatively closed pluripolar set $\widehat{M} \subset \widehat{\boldsymbol{X}}$ such that

- $\widehat{M} \cap \boldsymbol{c}(\boldsymbol{T}^0) \subset M$,
- *for any $f \in \mathcal{F}$ there exists an $\hat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus \widehat{M})$ with $\hat{f} = f$ on $\boldsymbol{c}(\boldsymbol{T}^0) \setminus M$,*

- *$\widehat{M}$ is singular with respect to the family $\{\hat{f} : f \in \mathcal{F}\}$,*
- *if each set $S_{j,k,a}$ is thin in $V_{j,k,a}$, then $\widehat{M}$ is analytic.*

Proof. Step 1^0. If $N \geq 4$, then for any $1 \leq \mu < \nu \leq N$, define an auxiliary $(N-2)$-fold cross

$$\boldsymbol{Y}_{\mu,\nu} := \mathbb{X}((A_j, D_j)_{j\in\{1,\dots,\mu-1,\mu+1,\dots,\nu-1,\nu+1,\dots,N\}}).$$

We may assume that for each pair (k,a) the number $r_{k,a}$ is so small that

$$\widehat{\mathbb{P}}((a_1,\dots,a_{\mu-1},a_{\mu+1},\dots,a_{\nu-1},a_{\nu+1},\dots,a_N),r_{k,a}) \subset \widehat{\boldsymbol{Y}}_{\mu,\nu}, \quad 1 \leq \mu < \nu \leq N.$$

Step 2^0. We may assume that each set $S_{j,k,a}$ is singular with respect to the family $\{\tilde{f}_{j,k,a} : f \in \mathcal{F}\}$. In particular, $S_{j,k,a}$ is thin in $V_{j,k,a}$ iff $S_{j,k,a}$ is analytic in $V_{j,k,a}$.

Step 3^0. Observe that

$$S_{j,k,a} \cap \boldsymbol{c}(\boldsymbol{T}_k^0) \subset M, \quad \tilde{f}_{j,k,a} = f \text{ on } (V_{j,k,a} \cap \boldsymbol{c}(\boldsymbol{T}_k^0)) \setminus M. \tag{10.6.1}$$

Indeed, fix a point $b \in (V_{j,k,a} \cap \boldsymbol{c}(\boldsymbol{T}_k^0)) \setminus M$. Let $\rho_b > 0$, $\tilde{f}_b \in \mathcal{O}(\widehat{\mathbb{P}}(b,\rho_b))$ be as in (C8). So $\tilde{f}_b = f = \tilde{f}_{j,k,a}$ on $(\widehat{\mathbb{P}}(b,\rho_b) \cap \boldsymbol{T}_{j,k,a}^0) \setminus M$. The set $\boldsymbol{T}_{j,k,a}^0 \setminus M$ is locally pluriregular and $b \in \overline{\boldsymbol{T}_{j,k,a}^0 \setminus M}$ (Remark 10.2.1 (f)). In particular, $(\widehat{\mathbb{P}}(b,\rho_b) \cap \boldsymbol{T}_{j,k,a}^0) \setminus M$ is not pluripolar and so, $\tilde{f}_b = \tilde{f}_{j,k,a}$ on $\widehat{\mathbb{P}}(b,\rho_b) \setminus S_{j,k,a}$. Hence $\widehat{\mathbb{P}}(b,\rho_b) \cap S_{j,k,a} = \varnothing$ (because $S_{j,k,a}$ is singular) and $\tilde{f}_{j,k,a} = f$ on $\widehat{\mathbb{P}}(b,\rho_b) \cap \boldsymbol{c}(\boldsymbol{T}_k^0) \setminus M$.

Step 4^0. Fix a $k \in \mathbb{N}$. Put

$$V_k := \bigcup_{a\in\Xi_k} V_{j,k,a}, \quad S_k := \bigcup_{a\in\Xi_k} S_{j,k,a}.$$

Then $\boldsymbol{T}_k^0 \subset V_k$.

Indeed, let $c \in \boldsymbol{T}_k^0$, e.g. $c = (c', c_N) \in (A'_{N-1,k} \setminus \Sigma_{N,k}^0) \times D_{N,k}$. Since $M_{(c',\cdot)}$ is pluripolar, there exists an $a_N \in A_{N,k} \setminus M_{(c',\cdot)}$. Then $a := (c', a_N) \in \boldsymbol{c}(\boldsymbol{T}_k^0) \setminus M$ and $c \in \widehat{\mathbb{P}}(c', r_{k,a}) \times D_{N,k} = V_{N,k,a}$.

Step 5^0. The main problem is to show that

$$\begin{gathered}\text{for arbitrary } i,j \in \{1,\dots,N\},\ a,b \in \Xi_k \text{ with } W_{i,j,k,a,b} := V_{i,k,a} \cap V_{j,k,b} \neq \varnothing \\ \text{we have } \tilde{f}_{i,k,a} = \tilde{f}_{j,k,b} \text{ on } W_{i,j,k,a,b} \setminus (S_{i,k,a} \cup S_{j,k,b}) \text{ for all } f \in \mathcal{F}.\end{gathered} \tag{10.6.2}$$

Suppose for a moment that (10.6.2) is proved. We finish the main proof.

Since the sets $S_{i,k,a}$ and $S_{j,k,b}$ are singular, we conclude that

$$S_{i,k,a} \cap W_{i,j,k,a,b} = S_{j,k,b} \cap W_{i,j,k,a,b},$$

which implies that the function $\tilde{f}_k := \bigcup_{a\in\Xi_k} \tilde{f}_{j,k,a}$ is well defined on $V_k \setminus S_k$. Moreover,

- S_k is a relatively closed pluripolar subset of V_k,
- $S_k \cap \boldsymbol{c}(\boldsymbol{T}_k^0) \subset M$,
- $\tilde{f}_k \in \mathcal{O}(V_k \setminus S_k)$,
- $\tilde{f}_k = f$ on $\boldsymbol{c}(\boldsymbol{T}_k^0) \setminus M$,
- S_k is singular with respect to the family $\{\tilde{f}_k : f \in \mathcal{F}\}$,
- S_k is analytic provided that each set $S_{j,k,a}$ is analytic.

Let U_k denote the connected component of $V_k \cap \widehat{\boldsymbol{X}}_k$ that intersect $\boldsymbol{T}_k^0$. Then $\widehat{\boldsymbol{X}}_k$ is the envelope of holomorphy of U_k.

Indeed, since $\widehat{\boldsymbol{X}}_k$ is a domain of holomorphy (Remark 5.1.8 (e)), we only need to show that any function from $\mathcal{O}(U_k)$ extends holomorphically to $\widehat{\boldsymbol{X}}_k$. Take a $g \in \mathcal{O}(U_k)$. Then $g|_{\boldsymbol{T}_k^0} \in \mathcal{O}_s(\boldsymbol{T}_k^0) \cap \mathcal{C}(\boldsymbol{T}_k^0)$. By Theorem 7.1.4, g extends to a $\widehat{g} \in \mathcal{O}(\widehat{\boldsymbol{X}}_k)$ with $\widehat{g} = g$ on $\boldsymbol{T}_k$. Observe that $\boldsymbol{T}_k^0$ is locally pluriregular. In particular, $U_k \cap \boldsymbol{T}_k^0$ is not pluripolar. Hence, by the identity principle, $\widehat{g} = g$ on U_k.

By virtue of the Chirka theorem (Theorem 9.4.2), there exists a relatively closed pluripolar set $\widehat{M}_k$ of $\widehat{\boldsymbol{X}}_k$, $\widehat{M}_k \cap U_k \subset S_k$, such that $\widehat{\boldsymbol{X}}_k \setminus \widehat{M}_k$ is the envelope of holomorphy of $U_k \setminus S_k$. Moreover, if S_k is analytic, then so is $\widehat{M}_k$. In particular, for each $f \in \mathcal{F}$ there exists an $\hat{f}_k \in \mathcal{O}(\widehat{\boldsymbol{X}}_k \setminus \widehat{M}_k)$ with $\hat{f}_k = \tilde{f}_k$ on $U_k \setminus S_k$.

We may assume that $\widehat{M}_k$ is singular with respect to the family $\{\hat{f}_k : f \in \mathcal{F}\}$. Hence, $\widehat{M}_{k+1} \cap \widehat{\boldsymbol{X}}_k = \widehat{M}_k$. Recall that $\widehat{\boldsymbol{X}}_k \nearrow \widehat{\boldsymbol{X}}$. Consequently:

- $\widehat{M} := \bigcup_{k=1}^{\infty} \widehat{M}_k$ is a relatively closed pluripolar subset of $\widehat{\boldsymbol{X}}$ with $\widehat{M} \cap \boldsymbol{c}(\boldsymbol{T}^0) \subset M$,
- for each $f \in \mathcal{F}$, the function $\hat{f} := \bigcup_{k=1}^{\infty} \hat{f}_k$ is holomorphic on $\widehat{\boldsymbol{X}} \setminus \widehat{M}$ with $\hat{f} = f$ on $\boldsymbol{c}(\boldsymbol{T}^0) \setminus M$,
- $\widehat{M}$ is singular with respect to the family $\{\hat{f} : f \in \mathcal{F}\}$,
- if each set $S_{j,k,a}$ is analytic in $V_{j,k,a}$, then $\widehat{M}$ is analytic.

This completes the proof of Theorem 10.6.1 (modulo (10.6.2)).

Step 6^0. We move to (10.6.2). Fix $i, j \in \{1, \dots, N\}$ and $a, b \in \Xi_k$ with $W_{i,j,k,a,b} \neq \varnothing$ and take an $f \in \mathcal{F}$. We have the following two cases:

(a) $i \neq j$: We may assume that $i = N-1$, $j = N$. Write

$$w = (w', w'') \in (D_1 \times \dots \times D_{N-2}) \times (D_{N-1} \times D_N).$$

Observe that

$$W_{N-1,N,k,a,b} = (\widehat{\mathbb{P}}(a', r_{k,a}) \cap \widehat{\mathbb{P}}(b', r_{k,b})) \times \widehat{\mathbb{P}}(b_{N-1}, r_{k,b}) \times \widehat{\mathbb{P}}(a_N, r_{k,a}).$$

Consider the following three subcases:

Subcase $N = 2$: Then we have $W_{1,2,k,a,b} = \widehat{\mathbb{P}}(b_1, r_{k,b}) \times \widehat{\mathbb{P}}(a_2, r_{k,a})$. We know (cf. (10.6.1)) that $\tilde{f}_{1,k,a} = f = \tilde{f}_{2,k,b}$ on the set $B \setminus M$, $B := W_{1,2,k,a,b} \cap \boldsymbol{c}(\boldsymbol{T}_k^0)$.

Observe that

$$(A_{1,k}[b_1, r_{k,b}] \setminus \Sigma_2^0) \times (A_{2,k}[a_2, r_{k,a}] \setminus \Sigma_1^0) \subset B.$$

Thus $B \setminus M$ is not pluripolar. Hence, by the identity principle, $\tilde{f}_{1,k,a} = \tilde{f}_{2,k,b}$ on $W_{1,2,k,a,b} \setminus (S_{1,k,a} \cup S_{2,k,b})$.

Subcase $N = 3$: Then $W_{2,3,k,a,b} = (\widehat{\mathbb{P}}(a_1, r_{k,a}) \cap \widehat{\mathbb{P}}(b_1, r_{k,b})) \times \widehat{\mathbb{P}}(b_2, r_{k,b}) \times \widehat{\mathbb{P}}(a_3, r_{k,a})$. We are going to show that

$$\tilde{f}_{N-1,k,a} = \tilde{f}_{N,k,b} \text{ on } ((\widehat{\mathbb{P}}(a_1, r_{k,a}) \cap \widehat{\mathbb{P}}(b_1, r_{k,b})) \times C) \setminus (S_{2,k,a} \cup S_{3,k,b}),$$

where $C \subset \widehat{\mathbb{P}}(b_2, r_{k,b}) \times \widehat{\mathbb{P}}(a_3, r_{k,a})$ is a non-pluripolar set; then, by the identity principle, we obtain $\tilde{f}_{2,k,a} = \tilde{f}_{3,k,b}$ on $W_{2,3,k,a,b} \setminus (S_{2,k,a} \cup S_{3,k,b})$.

Let

$$C := \{c \in (A_{2,k}[b_2, r_{k,b}] \times A_{3,k}[a_3, r_{k,a}]) \setminus \Sigma_{1,k}^0 : (S_{2,k,a})_{(\cdot,c)} \in \mathcal{PLP}, \\ (S_{3,k,b})_{(\cdot,c)} \in \mathcal{PLP}\}.$$

The set C is locally pluriregular (Remark 10.2.1 (f)). Fix a $c = (c_2, c_3) \in C$. Recall that $\widehat{\mathbb{P}}(a_1, r_{k,a}) \cup \widehat{\mathbb{P}}(b_1, r_{k,b}) \subset D_{1,k}$. Thus, the functions $\tilde{f}_{3,k,b}(\cdot, c)$ and $f(\cdot, c)$ are holomorphic on

$$\widehat{\mathbb{P}}(b_1, r_{k,b}) \setminus ((S_{3,k,b})_{(\cdot,c)} \cup M_{(\cdot,c)}).$$

Moreover, they are equal on the non-pluripolar set $A_{1,k}[b_1, r_{k,b}] \setminus M_{(\cdot,c)}$. Hence, since the set $(S_{3,k,b})_{(\cdot,c)} \cup M_{(\cdot,c)}$ is pluripolar, we get

$$\tilde{f}_{3,k,b}(\cdot, c) = f(\cdot, c) \text{ on } \widehat{\mathbb{P}}(b_1, r_{k,b}) \setminus ((S_{3,k,b})_{(\cdot,c)} \cup M_{(\cdot,c)}).$$

An analogous argument shows that

$$\tilde{f}_{2,k,a}(\cdot, c) = f(\cdot, c) \text{ on } \widehat{\mathbb{P}}(a_1, r_{k,a}) \setminus ((S_{2,k,a})_{(\cdot,c)} \cup M_{(\cdot,c)}).$$

Hence, by the identity principle,

$$\tilde{f}_{2,k,a}(\cdot, c) = \tilde{f}_{3,k,b}(\cdot, c) \text{ on } (\widehat{\mathbb{P}}(a_1, r_{k,a}) \cap \widehat{\mathbb{P}}(b_1, r_{k,b})) \setminus ((S_{2,k,a})_{(\cdot,c)} \cup (S_{3,k,b})_{(\cdot,c)}).$$

Consequently,

$$\tilde{f}_{2,k,a} = \tilde{f}_{3,k,b} \text{ on } ((\widehat{\mathbb{P}}(a_1, r_{k,a}) \cap \widehat{\mathbb{P}}(b_1, r_{k,b})) \times C) \setminus (S_{2,k,a} \cup S_{3,k,b}).$$

Subcase $N \geq 4$: We are going to show that

$$\tilde{f}_{N-1,k,a} = \tilde{f}_{N,k,b} \text{ on } ((\widehat{\mathbb{P}}(a', r_{k,a}) \cap \widehat{\mathbb{P}}(b', r_{k,b})) \times C) \setminus (S_{N-1,k,a} \cup S_{N,k,b}),$$

where $C \subset \widehat{\mathbb{P}}(b_{N-1}, r_{k,b}) \times \widehat{\mathbb{P}}(a_N, r_{k,a})$ is a non-pluripolar set; then, by the identity principle, we obtain

$$\tilde{f}_{N-1,k,a} = \tilde{f}_{N,k,b} \text{ on } W_{N-1,N,k,a,b} \setminus (S_{N-1,k,a} \cup S_{N,k,b}).$$

Let

$$B_{N-1} := \{c_{N-1} \in A_{N-1,k}[b_{N-1}, r_{k,b}] : (\Sigma_N^0)_{(\cdot,c_{N-1})} \in \mathcal{PLP}\}.$$

By Proposition 2.3.31 the set B_{N-1} is locally pluriregular. Analogously, the set

$$B_N := \{c_N \in A_{N,k}[a_N, r_{k,a}] : (\Sigma_{N-1}^0)_{(\cdot,c_N)} \in \mathcal{PLP}\}$$

is locally pluriregular. Let

$$C := \{c \in B_{N-1} \times B_N : (S_{N-1,k,a})_{(\cdot,c)} \in \mathcal{PLP},\ (S_{N,k,b})_{(\cdot,c)} \in \mathcal{PLP},\\ (\Sigma_\nu^0)_{(\cdot,c)} \in \mathcal{PLP},\ \nu = 1, \dots, N-2\}.$$

The set C is also locally pluriregular. Fix a $c = (c_{N-1}, c_N) \in C$. Note that $\boldsymbol{T}^0_{(\cdot,c)} \supset \boldsymbol{T}^0_c$, where

$$\boldsymbol{T}^0_c := \mathbb{T}((A_\nu, D_\nu, (\Sigma_\nu^0)_{(\cdot,c)})_{\nu=1}^{N-2}).$$

Observe that the $(N-2)$-fold cross $\boldsymbol{T}^0_c$, the sets $(\Sigma_\nu^0)_{(\cdot,c)}$, $\nu = 1, \dots, N-2$, the set $M_{(\cdot,c)}$, and the family $\mathcal{F}_c := \{f(\cdot,c)|_{\boldsymbol{T}^0_c \setminus M_{(\cdot,c)}} : f \in \mathcal{F}\}$ satisfy all the assumptions of Theorem 10.2.6 (with $N-2$).

Indeed, put $A_\nu''' := A_{\nu+1} \times \cdots \times A_{N-2}$, $\nu = 1, \dots, N-2$. Then:

- The sets $(\Sigma_\nu^0)_{(\cdot,c)}$, $\nu = 1, \dots, N-2$, are pluripolar.
- For any $\nu \in \{1, \dots, N-2\}$ and $(\zeta_\nu', \zeta_\nu''') \in A_\nu' \times A_\nu'''$ we have

  $$(M_{(\cdot,c)})_{(\zeta_\nu',\cdot,\zeta_\nu''')} = M_{(\zeta_\nu',\cdot,\zeta_\nu''',c)}.$$

  Consequently, if $(\zeta_\nu', \zeta_\nu''') \notin (\Sigma_\nu^0)_{(\cdot,c)}$, then $(M_{(\cdot,c)})_{(\zeta_\nu',\cdot,\zeta_\nu''')}$ is closed and pluripolar.
- $f(\cdot,c) \in \mathcal{O}_s(\boldsymbol{T}^0_c \setminus M_{(\cdot,c)})$, $f \in \mathcal{F}$.
- For every $\zeta \in \boldsymbol{c}(\boldsymbol{T}^0_c) \setminus M_{(\cdot,c)} \subset (\boldsymbol{c}(\boldsymbol{T}^0) \setminus M)_{(\cdot,c)}$, the function $\tilde{f}_{(\zeta,c)}(\cdot,c) \in \mathcal{O}(\widehat{\mathbb{P}}(\zeta, \rho_{(\zeta,c)}))$ gives the required holomorphic extension of $f(\cdot,c)$ with

  $$\tilde{f}_{(\zeta,c)}(\cdot,c) = f(\cdot,c)\\ \text{on } ((\widehat{\mathbb{P}}((\zeta,c), \rho_{(\zeta,c)}) \cap \boldsymbol{c}(\boldsymbol{T}^0)) \setminus M)_{(\cdot,c)} \supset \widehat{\mathbb{P}}(\zeta, \rho_{(\zeta,c)}) \cap \boldsymbol{c}(\boldsymbol{T}^0_c) \setminus M_{(\cdot,c)}.$$

Since we have assumed that Theorem 10.2.6 is true for $(N-2)$, we conclude that there exists a relatively closed pluripolar set $\widehat{M}(c) \subset \widehat{\boldsymbol{Y}}_{N-1,N}$ such that

- $\widehat{M}(c) \cap \boldsymbol{c}(\boldsymbol{T}^0_c) \subset M_{(\cdot,c)}$,
- for any $f \in \mathcal{F}_c$ there exists an $\hat{f}_c \in \mathcal{O}(\widehat{\boldsymbol{Y}}_{N-1,N} \setminus \widehat{M}(c))$ with $\hat{f}_c = f(\cdot,c)$ on $\boldsymbol{c}(\boldsymbol{T}^0_c) \setminus M_{(\cdot,c)}$.

Recall that $\widehat{\mathbb{P}}(a', r_{k,a}) \cup \widehat{\mathbb{P}}(b', r_{k,b}) \subset \widehat{\boldsymbol{Y}}_{N-1,N}$. Thus, the functions $\tilde{f}_{N,k,b}(\cdot,c)$ and $\hat{f}_c$ are holomorphic on

$$\widehat{\mathbb{P}}(b', r_{k,b}) \setminus ((S_{N,k,b})_{(\cdot,c)} \cup \widehat{M}(c)).$$

Moreover, they are equal to $f(\cdot, c)$ on the set $(\boldsymbol{c}(\boldsymbol{T}_k^0))_{(\cdot,c)} \cap \boldsymbol{c}(\boldsymbol{T}_c^0) \setminus M_{(\cdot,c)} =: L$. Observe that L is not pluripolar.

Indeed, put $\widetilde{A}_\nu := A_{\nu,k}[b_\nu, r_{k,b}]$, $\nu = 1, \dots, N-2$. First observe that

$$(\widetilde{A}_1 \times \dots \times \widetilde{A}_{N-2}) \setminus (\Sigma_N^0)_{(\cdot,c_{N-1})} \subset (\boldsymbol{c}(\boldsymbol{T}_k^0))_{(\cdot,c)}.$$

On the other hand, $(\widetilde{A}_1 \times \dots \times \widetilde{A}_{N-2}) \setminus P \subset \boldsymbol{c}(\boldsymbol{T}_c^0)$, where P is pluripolar. Hence, in view of the definition of the set B_{N-1}, we conclude that

$$(\widetilde{A}_1 \times \dots \times \widetilde{A}_{N-2}) \setminus Q \subset (\boldsymbol{c}(\boldsymbol{T}_k^0))_{(\cdot,c)} \cap \boldsymbol{c}(\boldsymbol{T}_c^0),$$

where Q is pluripolar. In particular, the set

$$R := \{\xi \in \widetilde{A}_1 \times \dots \times \widetilde{A}_{N-3} : Q_{(\xi,\cdot)} \notin \mathcal{PLP}\}$$

is pluripolar. Moreover, for any $\xi \in (\widetilde{A}_1 \times \dots \times \widetilde{A}_{N-3}) \setminus (\Sigma_{N-2}^0)_{(\cdot,c)}$, the fiber $(M_{(\cdot,c)})_{(\xi,\cdot)} = M_{(\xi,\cdot,c)}$ is pluripolar. Thus, for any

$$\xi \in (\widetilde{A}_1 \times \dots \times \widetilde{A}_{N-3}) \setminus (R \cup (\Sigma_{N-2}^0)_{(\cdot,c)})$$

the set $(Q \cup M_{(\cdot,c)})_{(\xi,\cdot)}$ is pluripolar. Now we are in a position to apply Proposition 2.3.31 and to conclude that L is not pluripolar.

Hence, since the set $(S_{N,k,b})_{(\cdot,c)} \cup \widehat{M}(c)$ is pluripolar, we get

$$\tilde{f}_{N,k,b} = \hat{f}_c \text{ on } \widehat{\mathbb{P}}(b', r_{k,b}) \setminus ((S_{N,k,b})_{(\cdot,c)} \cup \widehat{M}(c)).$$

An analogous argument shows that

$$\tilde{f}_{N-1,k,a} = \hat{f}_c \text{ on } \widehat{\mathbb{P}}(a', r_{k,a}) \setminus ((S_{N-1,k,a})_{(\cdot,c)} \cup \widehat{M}(c)).$$

Hence,

$$\begin{aligned}\tilde{f}_{N-1,k,a}(\cdot, c) &= \tilde{f}_{N,k,b}(\cdot, c) \\ &\text{on } (\widehat{\mathbb{P}}(a', r_{k,a}) \cap \widehat{\mathbb{P}}(b', r_{k,b})) \setminus ((S_{N-1,k,a})_{(\cdot,c)} \cup (S_{N,k,b})_{(\cdot,c)}).\end{aligned}$$

Consequently,

$$\tilde{f}_{N-1,k,a} = \tilde{f}_{N,k,b} \text{ on } ((\widehat{\mathbb{P}}(a', r_{k,a}) \cap \widehat{\mathbb{P}}(b', r_{k,b})) \times C) \setminus (S_{N-1,k,a} \cup S_{N,k,b}).$$

(b) $i = j$: We may assume that $i = j = N$. Write

$$w = (w', w_N) \in (D_1 \times \dots \times D_{N-1}) \times D_N.$$

Observe that $\varnothing \neq W_{k,N,N,a,b} = (\widehat{\mathbb{P}}(a', r_{k,a}) \cap \widehat{\mathbb{P}}(b', r_{k,b})) \times D_{N,k}$. By (a) we know that

$$\begin{aligned}\tilde{f}_{N,k,a} &= \tilde{f}_{N-1,k,a} \text{ on } (V_{N,k,a} \cap V_{N-1,k,a}) \setminus (S_{N,k,a} \cup S_{N-1,k,a}),\\ \tilde{f}_{N-1,k,a} &= \tilde{f}_{N,k,b} \text{ on } (V_{N-1,k,a} \cap V_{N,k,b}) \setminus (S_{N-1,k,a} \cup S_{N,k,b}).\end{aligned}$$

Hence $\tilde{f}_{N,k,a} = \tilde{f}_{N,k,b}$ on

$$\begin{aligned}(V_{N,k,a} \cap V_{N-1,k,a} \cap V_{N,k,b}) \setminus (S_{N-1,k,a} \cup S_{N,k,a} \cup S_{N,k,b})\\ = ((\widehat{\mathbb{P}}(a', r_{k,a}) \cap \widehat{\mathbb{P}}(b', r_{k,b})) \times \widehat{\mathbb{P}}(a_N, r_{k,a})) \setminus (S_{N-1,k,a} \cup S_{N,k,a} \cup S_{N,k,b}),\end{aligned}$$

and finally, by the identity principle,

$$\tilde{f}_{N,k,a} = \tilde{f}_{N,k,b} \text{ on } W_{N,N,k,a,b} \setminus (S_{N,k,a} \cup S_{N,k,b}).$$

The proof of (10.6.2) is completed. □

Proof of Theorem 10.2.6. Observe that we only need to apply inductively Theorem 10.6.1 in such a way that if for all $j \in \{1, \dots, N\}$ and $(a'_j, a''_j) \in (A'_j \times A''_j) \setminus \Sigma^0_j$, the fiber $M_{(a'_j,\cdot,a''_j)}$ is thin in D_j, then each set $S_{j,k,a}$ is thin in $V_{j,k,a}$.

We keep notation from Theorem 10.6.1. Fix $j \in \{1, \dots, N\}$, $k \in \mathbb{N}$, and $a \in \boldsymbol{c}(\boldsymbol{T}^0_k) \setminus M$. Let $\widehat{\mathbb{P}}(a, \rho_a)$ and $\tilde{f}_a \in \mathcal{O}(\widehat{\mathbb{P}}(a, \rho_a))$ be as in (C8). We may assume that $\widehat{\mathbb{P}}(a, \rho_a) \subset D_{k,1} \times \cdots \times D_{N,k}$. For simplicity, assume that $j = N$.

For each $b'_N \in A'_N \setminus \Sigma^0_N$, let $\widetilde{M}_{N,b'_N}$ be the singular part of $M_{(b'_N,\cdot)}$ with respect to the family $\{f(b'_N, \cdot) : f \in \mathcal{F}\}$ (taken in the sense of § 2.4) and let $\tilde{f}_{N,b'_N}$ stand for the holomorphic extension of $f(b'_N, \cdot)$ to $D_N \setminus \widetilde{M}_{N,b'_N}$. Since $\tilde{f}_{N,b'_N} = f(b'_N, \cdot) = \tilde{f}_a(b'_N, \cdot)$ on $A_N(a_N, \rho_a) \setminus M_{(b'_N,\cdot)}$, we conclude that $\tilde{f}_{N,b'_N} = \tilde{f}_a(b'_N, \cdot)$ on $\widehat{\mathbb{P}}(a_N, \rho_a) \setminus \widetilde{M}_{N,b'_N}$. In particular, $\widetilde{M}_{N,b'_N} \cap \widehat{\mathbb{P}}(a_N, \rho_a) = \varnothing$.

We are going to apply Lemma 9.1.5 with

- $k := n_1 + \cdots + n_{N-1}$, $\ell := n_N$,
- $D := \widehat{\mathbb{P}}(a'_N, \rho_a)$, $G := D_N$, $G_0 := \widehat{\mathbb{P}}(a_N, \rho_a)$,
- $A := A'_N(a'_N, \rho_a) \setminus \Sigma^0_N$,
- $M(b'_N) := \widetilde{M}_{N,b'_N}$, $b'_N \in A$.

Notice that $\{\tilde{f}_a : f \in \mathcal{F}\} \subset \mathcal{S}$, where $\mathcal{S}$ denotes the set of all functions $g \in \mathcal{O}(D \times G_0)$ such that for every $b'_N \in A$, the function $g(b'_N, \cdot)$ extends holomorphically to $G \setminus M(b'_N)$. Since the set $M(b'_N)$ is singular with respect to the family $\{\tilde{f}_{N,b'_N} : f \in \mathcal{F}\}$ for every $b'_N \in A$, it is obviously singular with respect to the family $\{\widehat{g}(b'_N, \cdot) : g \in \mathcal{S}\}$. Consequently, Lemma 9.1.5 applies and we get a pluripolar set $P = P_{N,a} \subset A$ such that the set

$$M_{N,a} := \bigcup_{b'_N \in A \setminus P} \{b'_N\} \times M(b'_N)$$

is relatively closed in $(A \setminus P) \times G$. Consider the 2-fold cross

$$\boldsymbol{Y}_{N,a} := \mathbb{X}(A \setminus P, G_0; D, G) = (D \times G_0) \cup ((A \setminus P) \times G) \subset (D \times G_0) \cup \boldsymbol{T}.$$

Since $M_{N,a}$ is relatively closed in $\boldsymbol{Y}_{N,a}$, we may apply Theorem 10.4.1 and we get a relatively closed pluripolar set $S_{N,a} \subset \widehat{\boldsymbol{Y}}_{N,a}$ such that

- $S_{N,a} \cap \boldsymbol{Z}_{N,a} \subset M_{N,a} \subset M$, where $\boldsymbol{Z}_{N,a} := \mathbb{X}(A \setminus (P \cup Q), G_0 \setminus R; D, G)$ for certain pluripolar sets $Q = Q_{N,a} \subset A \setminus P$, $R = R_{N,a} \subset G_0$,
- for any $f \in \mathcal{O}_s(\boldsymbol{Y}_{N,a} \setminus M_{N,a})$ there exists an $\hat{f}_{N,a} \in \mathcal{O}(\widehat{\boldsymbol{Y}}_{N,a} \setminus S_{N,a})$ such that $\hat{f}_{N,a} = f$ on $\boldsymbol{Z}_{N,a} \setminus M_{N,a}$,
- $S_{N,a}$ is singular with respect to the family $\{\hat{f}_{N,a} : f \in \mathcal{F}_{N,a}\}$, where

$$\mathcal{F}_{N,a} := \{g \in \mathcal{O}_s(\boldsymbol{Y}_{N,a} \setminus M_{N,a}) : \exists_{f \in \mathcal{F}} : \ g = \tilde{f}_a \text{ on } D \times G_0\};$$

  in particular, $S_{N,a} \cap \widehat{\mathbb{P}}(a, \rho_a) = \varnothing$,
- if all the fibers $M(b'_N)$, $b'_N \in A \setminus P$, are thin (in fact, analytic), then $S_{N,a}$ is analytic.

Since $\{a'_N\} \times G \subset \widehat{\boldsymbol{Y}}_{N,a}$, there exists an $\tau = r_{N,k,a} \in (0, \rho_a)$ so small that $V_{N,k,a} := \widehat{\mathbb{P}}(a'_N, \tau) \times D_{N,k} \subset \widehat{\boldsymbol{Y}}_{N,a}$. Define $S_{N,k,a} := S_{N,a} \cap V_{N,k,a}$, $\tilde{f}_{N,k,a} := \hat{f}_{N,a}|_{V_{N,k,a} \setminus S_{N,k,a}}$, and $P_{N,k,a} := Q_{N,a} \cap \widehat{\mathbb{P}}(a'_N, \tau)$ (observe that $\boldsymbol{T}^0_{N,k,a} \subset \boldsymbol{Z}_{N,a}$).

Obviously, an analogous construction may be done for all $j \in \{1, \dots, N-1\}$. We set $r_{k,a} := \min\{r_{k,1,a}, \dots, r_{N,k,a}\}$. This shows that all the assumptions of Theorem 10.6.1 are satisfied.

The proof of Theorem 10.2.6 is completed. □

10.7 Example and application

It is natural to ask how large is the class of sets $M \subset \boldsymbol{T}$ with pluripolar fibers (cf. (C7)), that are not pluripolar.

Proposition 10.7.1. *Let S be a d-dimensional $\mathcal{C}^1$-submanifold of an open set $\Omega \subset \mathbb{C}^n$ with $1 \le d \le 2n-2$. Then for every point $z_0 \in S$ there exist an open neighborhood U and a $\mathbb{C}$-affine isomorphism $L : \mathbb{C}^n \to \mathbb{C}^n$ such that $L(U) = \mathbb{D}^n$ and if $M := L(S \cap U)$, then for any $a = (a_1, \dots, a_n) \in \mathbb{D}^n$ and $j \in \{1, \dots, n\}$, the fiber $M_{(a'_j, \cdot, a''_j)}$ is finite.*

Remark 10.7.2. (a) M satisfies (C1)–(C7) for an arbitrary cross with $D_j := \mathbb{D}$, $j = 1, \dots, N = n$.

(b) Notice that there are real analytic 2-dimensional submanifolds of $\mathbb{C}^2$ that are not locally pluripolar. For example (cf. [Sad 2005]):

$$S := \{(x_1 + i(x_1 + x_2^2), x_2 + i(x_1^2 + x_2)) : x_1, x_2 \in \mathbb{R}\} \subset \mathbb{C}^2.$$

Consequently, one may easily produce examples of sets M satisfying (C1)–(C7) that are not pluripolar.

Proof of Proposition 10.7.1. The idea of the proof is due to W. Jarnicki.

Without loss of generality, we may assume that $z_0 = 0$ and $d = 2n-2$ (EXERCISE). We shall prove that there exists an $X_0 \in \mathbb{C}^n$ such that $(\mathbb{C}X_0) \cap T_0 S = \{0\}$. Let $\alpha, \beta \in \mathbb{C}^n \simeq \mathbb{R}^{2n}$ be two $\mathbb{R}$-linearly independent vectors such that

$$T_0 S = \{x \in \mathbb{R}^{2n} : \operatorname{Re}\langle \alpha, x\rangle = \operatorname{Re}\langle \beta, x\rangle = 0\},$$

where $\langle\ ,\ \rangle$ denotes the standard Hermitian scalar product in $\mathbb{C}^n$.

Assume that there exists a $j \in \{1, \dots, n\}$ such that α_j and β_j are $\mathbb{R}$-linearly independent. Then, for any $\xi \in \mathbb{C}_* \simeq (\mathbb{R}^2)_*$ we have $\operatorname{Re}(\alpha_j \bar{\xi}) \neq 0$ or $\operatorname{Re}(\beta_j \bar{\xi}) \neq 0$. Hence, we get $(\mathbb{C}X_0) \cap T_0 S = \{0\}$ for $X_0 := e_j$.

If such a j does not exist, then for any $j \in \{1, \dots, n\}$ we have $\alpha_j = \beta_j = 0$ or there exists a $\lambda_j \in \mathbb{C}_*$ such that for any $\xi \in \mathbb{C} \simeq \mathbb{R}^2$ we have $\operatorname{Re}(\alpha_j \bar{\xi}) = \operatorname{Re}(\beta_j \bar{\xi}) = 0 \iff \xi \in \mathbb{R}\lambda_j$ (EXERCISE). Fix $j, k \in \{1, \dots, n\}$ with $j \neq k$ such that (α_j, α_k), $(\beta_j, \beta_k) \in \mathbb{C}^2$ are $\mathbb{R}$-linearly independent. Obviously, $(\alpha_j, \beta_j) \neq 0 \neq (\alpha_k, \beta_k)$. Take $\mu_j, \mu_k \in \mathbb{C}$ such that $\mu_j \lambda_k, \mu_k \lambda_j$ are $\mathbb{R}$-linearly independent (EXERCISE). Put $X_0 := \mu_j e_j + \mu_k e_k$. Take a $\lambda \in \mathbb{C}$ and assume that $\lambda X_0 \in T_0 S$. We have $\lambda \mu_j \in \mathbb{R}\lambda_j$ and $\lambda \mu_k \in \mathbb{R}\lambda_k$. It follows that $\lambda \mu_j \mu_k \in (\mathbb{R}\lambda_j \mu_k) \cap (\mathbb{R}\lambda_k \mu_j) = \{0\}$, hence $\lambda = 0$.

Let $\varepsilon_0 > 0$ be such that $U_0 := \mathbb{P}_n(\varepsilon_0) \subset\subset \Omega$ and $(\mathbb{C}X) \cap T_z S = \{0\}$ whenever $X \in X_0 + U_0$ and $z \in \bar{U}_0$ (EXERCISE). Take $(X_1, \dots, X_n) \in (X_0 + U_0)^n$ with $\det(X_1, \dots, X_n) \neq 0$ and let L_0 be a $\mathbb{C}$-linear isomorphism with $L_0(X_j) = e_j$, $j = 1, \dots, n$. Choose an $\varepsilon > 0$ such that $L_0^{-1}(\mathbb{P}_n(\varepsilon)) \subset U_0$ and put $L := (1/\varepsilon)L_0$, $U := L^{-1}(\mathbb{D}^n)$. Take $a \in \mathbb{D}^n$ and $j \in \{1, \dots, n\}$ and consider the set

$$A := \{\lambda \in \mathbb{C} : L^{-1}(a_1, \dots, a_{j-1}, \lambda, a_{j+1}, \dots, a_n) \in S\}.$$

Observe that $M_{(a_j', \cdot, a_j'')} = A \cap \mathbb{D}$ (EXERCISE). Hence, it suffices to show that $A \cap \mathbb{D}$ is finite. In order to do that we will show that each point of $A \cap \bar{\mathbb{D}}$ is an isolated point of A. Take a $\mu \in A \cap \bar{\mathbb{D}}$. Let $z := L^{-1}(a_1, \dots, a_{j-1}, \mu, a_{j+1}, \dots, a_n)$. Since $L^{-1}(e_j) = \varepsilon X_j$ and $(\mathbb{C}X_j) \cap T_z S = \{0\}$, there exists a $\delta > 0$ with $(z + \mathbb{D}(\varepsilon\delta)X_j) \cap S = \{z\}$ (EXERCISE) and hence $A \cap (\mu + \mathbb{D}(\delta)) = \{\mu\}$ (EXERCISE). ☐

As an application of Theorem 10.2.9 and Proposition 10.7.1 we get a simple proof of the following extension theorem (cf. [Shi 1968], [Sto 1991]).

Theorem 10.7.3. *Let S be a connected d-dimensional $\mathcal{C}^1$-submanifold of a domain $D \subset \mathbb{C}^n$ with $1 \le d \le 2n-2$. Then every function $f \in \mathcal{O}(D \setminus S)$ extends holomorphically to D unless S is a complex submanifold of codimension* 1.

Notice that an analogous result is also true for plurisubharmonic functions – compare [Pfl 1980], [Kar 1991].

Proof. Take a point $a \in S$. Using Proposition 10.7.1 we find a neighborhood $U_a \subset D$ of a and a $\mathbb{C}$-affine isomorphism $L_a : \mathbb{C}^n \to \mathbb{C}^n$ such that

- $G_a = L_a(U_a) = (I_1 + iI_2) \times \cdots \times (I_{2n-1} + iI_{2n})$, where $I_1, \dots, I_{2n} \subset \mathbb{R}$ are open intervals,

- for any $b = (b_1, \dots, b_n) \in (I_1 + iI_2) \times \cdots \times (I_{2n-1} + iI_{2n})$ and $j \in \{1, \dots, n\}$, the fiber $(M_a)_{(b'_j, \cdot, b''_j)}$ is finite, where $M_a := L_a(S \cap U_a)$,
- M_a is the graph of a $\mathcal{C}^1$ mapping with respect to real variables $x_{i_1}, \dots, x_{i_d}$, $1 \le i_1 < \cdots < i_{i_d} \le 2n$ (in particular, M_a is connected).

  Now we apply Theorem 10.2.9, with $N := n$, $D_j = A_j := I_{2j-1} + iI_{2j}$, $\Sigma_j = \Sigma_j^0 := \varnothing$, $j = 1, \dots, n$ (in particular, $\boldsymbol{T}^0 = \boldsymbol{c}(\boldsymbol{T}^0) = \boldsymbol{X}$), and we get an analytic subset $\widehat{M}_a$ of G_a with the following properties:
- $\widehat{M}_a \subset M_a$,
- every function $g \in \mathcal{O}(G_a \setminus M_a)$ extends to a $\widehat{g} \in \mathcal{O}(G_a \setminus \widehat{M}_a)$ with $\widehat{g} = g$ on $G_a \setminus M_a$,
- $\widehat{M}_a$ is either empty or of codimension 1.

It is clear that if $d \le 2n - 3$, then each set $\widehat{M}_a$ must be empty and consequently, S is removable. Thus assume that $d = 2n - 2$. We have to prove that if there exists an $a \in S$ such that $\widehat{M}_a \neq \varnothing$, then S is a complex manifold (and, consequently, $\widehat{M}_a \neq \varnothing$ for every $a \in S$). Since S is connected, we only need show that for a fixed $a \in S$, either $\widehat{M}_a = \varnothing$ or $\widehat{M}_a = M_a$. Fix an $a \in S$ and suppose that $\widehat{M}_a \neq \varnothing$. Let $\Sigma_a := \operatorname{Sing}(\widehat{M}_a)$. Recall that Σ_a is an analytic subset of G_a of codimension ≥ 2. In particular, the manifold $M_a \setminus \Sigma_a$ is connected. Since $\operatorname{Reg}(\widehat{M}_a) \subset M_a \setminus \Sigma_a$, we conclude that $\operatorname{Reg}(\widehat{M}_a) = M_a \setminus \Sigma_a$ (Exercise). The set $M_a \setminus \Sigma_a$ is dense in M_a. Consequently, $T_x M_a = T_x^{\mathbb{C}} M_a$ for every $x \in M_a$ (Exercise). Thus, by the Levi–Civita theorem (cf. [Chi 1993], Appendix § 2.3), M_a is a complex manifold. □

Chapter 11

Separately meromorphic functions

Summary. As an application of previous results we discuss various problems related to separately meromorphic functions. We begin with Rothstein's theorems (Theorems 11.1.1, 11.1.2) which extend Proposition 1.1.10. The main result of the chapter is a cross theorem with singularities for separately meromorphic functions (Theorem 11.2.1). In particular, we get a generalization of Hartogs' theorem for separately holomorphic functions (Theorem 11.2.4). Some special cases are discussed in §§ 11.3, 11.4.

11.1 Rothstein theorem

$\boxed{>}$ §§ 1.1.3, 2.4, 2.8, 5.1, 5.4, 9.1, 10.2.

Recall (cf. § 2.8) that a function $f : \Omega \setminus S \to \mathbb{C}$, where Ω is a Riemann region and $S = \mathcal{S}(f) \subsetneq \Omega$ is an analytic set, is said to be meromorphic on Ω ($f \in \mathcal{M}(\Omega)$) if

- $f \in \mathcal{O}(\Omega \setminus S)$ and S is singular for f (consequently, either $S = \varnothing$ or S is of pure codimension 1),
- for any point $a \in S$ there exist an open connected neighborhood U of a and functions $\varphi, \psi \in \mathcal{O}(U)$, $\psi \not\equiv 0$, such that $\psi f = \varphi$ on $U \setminus S$.

Recall also (Theorem 2.8.4) that for any Riemann domain Ω its envelope of holomorphy $\widetilde{\Omega}$ coincides with the envelope of meromorphy.

We begin with the following Rothstein theorem which may be considered as a generalization of Proposition 1.1.10 for meromorphic functions.

Theorem 11.1.1 (Cf. [Rot 1950]). *Let D, G be Riemann domains over $\mathbb{C}^p$ and $\mathbb{C}^q$, respectively. Assume that D is a Riemann domain of holomorphy and $\mathcal{O}(G)$ separates points in G. Let $A \subset D$ be locally pluriregular and let $\varnothing \neq G_0 \subset G$ be a domain. Define $\boldsymbol{X} := \mathbb{X}(A, G_0; D, G) = (A \times G) \cup (D \times G_0)$. Assume that $f \in \mathcal{M}(D \times G_0)$ is such that for each $a \in A$ the fiber $\mathcal{S}(f)_{(a,\cdot)}$ is thin (equivalently, $\mathcal{S}(f)_{(a,\cdot)} \neq G_0$) and the function $f(a,\cdot)$ extends meromorphically to G, i.e. there exists an $\tilde{f}_a \in \mathcal{M}(G)$ with $\mathcal{S}(\tilde{f}_a) \cap G_0 \subset \mathcal{S}(f)_{(a,\cdot)}$ and $\tilde{f}_a = f(a,\cdot)$ on $G_0 \setminus \mathcal{S}(f)_{(a,\cdot)}$. Then there exists an $\tilde{f} \in \mathcal{M}(\widehat{\boldsymbol{X}})$ such that $\tilde{f} = f$ on $D \times G_0$, i.e. $\mathcal{S}(\tilde{f}) \cap (D \cap G_0) = \mathcal{S}(f)$ and $\tilde{f} = f$ on $(D \times G_0) \setminus \mathcal{S}(f)$.*

Proof. Our proof is based on Theorem 10.2.6 (with $N = 2$) and is essentially different than standard proofs.

First, observe that we may assume that G is also a Riemann domain of holomorphy. Indeed, let $\widetilde{G}$ be the envelope of holomorphy of G such that G is an open set in $\widetilde{G}$. Then every function $\tilde{f}_a$ extends to an $\tilde{\tilde{f}}_a \in \mathcal{M}(\widetilde{G})$. Consequently, if the theorem is true in the case of domains of holomorphy, then f extends to an $\tilde{f} \in \mathcal{M}(\widehat{\boldsymbol{Y}})$ with $\boldsymbol{Y} := \mathbb{X}(A, G_0; D, \widetilde{G})$. Clearly, $\widehat{\boldsymbol{X}} \subset \widehat{\boldsymbol{Y}}$.

Notice that the set A may be always substituted by a set $A \setminus P$, where P is pluripolar. In particular, using Proposition 9.1.4, we may always assume that for each $a \in A$ the set $\mathcal{S}(f)_{(a,\cdot)}$ is singular for $f(a,\cdot)$ and, consequently, $\mathcal{S}(\tilde{f}_a) \cap G_0 = \mathcal{S}(f)_{(a,\cdot)}$.

Take $N := 2$, $D_1 := D$, $D_2 := G$, $A_1 := A$,

$$A_2 = B := \{b \in G_0 : \mathcal{S}(f)_{(\cdot,b)} \neq D\},$$

$\Sigma_1 = \Sigma_1^0 = \Sigma_2 = \Sigma_2^0 := \varnothing$, $\boldsymbol{T} = \boldsymbol{T}^0 := \mathbb{X}(A, B; D, G)$,

$$M := \mathcal{S}(f) \cup \bigcup_{a \in A} \{a\} \times \mathcal{S}(\tilde{f}_a),$$

$\mathcal{F} := \{f_0\}$, where

$$f_0(z,w) := \begin{cases} f(z,w) & \text{if } (z,w) \in (D \times B) \setminus M, \\ \tilde{f}_z(w) & \text{if } (z,w) \in (A \times G) \setminus M, \end{cases} \qquad (z,w) \in \boldsymbol{T} \setminus M.$$

Observe that $G_0 \setminus B$ is pluripolar. In particular, $\widehat{\boldsymbol{T}} = \widehat{\boldsymbol{X}}$. One can easily check (Exercise) that (C1)–(C8) are satisfied. Moreover, for any $(a,b) \in A \times B$, the fibers $M_{(a,\cdot)} = \mathcal{S}(\tilde{f}_a)$, $M_{(\cdot,b)} = \mathcal{S}(f)_{(\cdot,b)}$ are thin. By Theorem 10.2.6 there exist an analytic set $\widehat{M} \subset \widehat{\boldsymbol{T}}$ and an $\hat{f}_0 \in \mathcal{O}(\widehat{\boldsymbol{T}} \setminus \widehat{M})$ such that $\widehat{M} \cap (A \times B) \subset M$, $\hat{f}_0 = f_0 = f$ on $(A \times B) \setminus M$, and $\widehat{M}$ is singular for $\hat{f}_0$.

Since the set $(A \times B) \setminus M$ is not pluripolar (cf. Remark 10.2.1 (f)), we get $\hat{f}_0 = f$ on $(D \times G_0) \setminus (\widehat{M} \cup \mathcal{S}(f))$. Consequently, since $\mathcal{S}(f)$ is singular for f, we conclude that $\mathcal{S}(f) = \widehat{M} \cap (D \times G_0)$ and $\hat{f}_0 = f$ on $(D \times G_0) \setminus \mathcal{S}(f)$. Moreover, $\mathcal{S}(\tilde{f}_a) \subset \widehat{M}_{(a,\cdot)}$ for $a \in A$. Using once again Proposition 9.1.4 (and substituting A by $A \setminus P$, where P is pluripolar), we may assume that $\widehat{M}_{(a,\cdot)}$ is singular for $\hat{f}_0(a,\cdot)$ for every $a \in A$. Thus $\mathcal{S}(\tilde{f}_a) = \widehat{M}_{(a,\cdot)}$ and $\tilde{f}_a = \hat{f}_0(a,\cdot)$ on $G \setminus \widehat{M}_{(a,\cdot)}$ for $a \in A$.

We only need to show that $\hat{f}_0 \in \mathcal{M}(\widehat{\boldsymbol{T}})$. The case $\widehat{M} = \varnothing$ is trivial. Thus, assume that $\widehat{M}$ is of pure codimension 1.

Observe that the main problem is to show that $\hat{f}_0 \in \mathcal{M}(\Omega)$, where $\Omega \subset \widehat{\boldsymbol{T}}$ is an open connected neighborhood of $\boldsymbol{T}$ (recall that $\boldsymbol{T}$ as a 2-fold cross is connected – cf. Remark 5.1.8 (a)).

Indeed, by Theorem 2.8.4, it is enough to show that the envelope of holomorphy of Ω coincides with $\widehat{\boldsymbol{T}}$. Take a $g \in \mathcal{O}(\Omega)$. Then $g|_{\boldsymbol{T}} \in \mathcal{O}_s(\boldsymbol{T})$. Thus, by the main cross theorem (Theorem 5.4.1), there exists a $\hat{g} \in \mathcal{O}(\widehat{\boldsymbol{T}})$ with $\hat{g} = g$ on $\boldsymbol{T}$. Hence, by the identity principle, $\hat{g} = g$ on Ω.

Let us start with the following "generic" situation, when $\widehat{M}$ is locally a special graph of a holomorphic function. Suppose that $(a_0, b_0) \in (A \times G) \cap \widehat{M}$ is such that there exist $0 < r, r_q < \min\{d_D(a_0), d_G(b_0)\}$ and a holomorphic function

$$\varphi\colon \mathbb{P}((a_0, b_0'), r) \to \mathbb{D}(b_{0,q}, r_q), \quad \varphi(a_0, b_0') = b_{0,q},$$

such that

$$\widehat{M} \cap (\mathbb{P}((a_0, b_0'), r) \times \mathbb{D}(b_{0,q}, r_q)) = \{(z, w', \varphi(z, w')) : (z, w') \in \mathbb{P}((a_0, b_0'), r)\},$$

where (using the standard identifications) we assume that $\widehat{\mathbb{P}}_D(a_0) \subset \mathbb{C}^p$, $\widehat{\mathbb{P}}_G(b_0) \subset \mathbb{C}^q$ and $b_0 = (b_0', b_{0,q})$ in local coordinates. Since $\hat{f}_0 \in \mathcal{O}(\widehat{\boldsymbol{T}} \setminus \widehat{M})$, we have

$$\hat{f}_0(z, w', w_q) = \sum_{k=-\infty}^{\infty} f_k(z, w')(w_q - \varphi(z, w'))^k,$$
$$(z, w') \in \mathbb{P}((a_0, b_0'), r), \ |w_q - \varphi(z, w')| < \varepsilon(z, w') := r_q - |\varphi(z, w') - b_{0,q}|,$$

where $f_k \in \mathcal{O}(\mathbb{P}((a_0, b_0'), r))$, $k \in \mathbb{Z}$. Observe that for each

$$(z, w') \in C := (A \cap \mathbb{P}(a_0, r)) \times \mathbb{P}(b_0', r),$$

the function $\hat{f}_0(z, w', \cdot)$ extends meromorphically to $\mathbb{D}(\varphi(z, w'), \varepsilon(z, w'))$.

Indeed, fix $(z, w') \in C$. Recall that $\mathcal{S}(\tilde{f}_z) = \widehat{M}_{(z,\cdot)}$ and $\tilde{f}_z = f(z, \cdot) = \hat{f}_0(z, \cdot)$ on $B \setminus \widehat{M}_{(z,\cdot)}$. Since $B \setminus \widehat{M}_{(z,\cdot)}$ is not pluripolar, we get $\tilde{f}_z = \hat{f}_0(z, \cdot)$ on $G \setminus \widehat{M}_{(z,\cdot)}$. Observe that the function $\tilde{f}_z(w', \cdot)$ is meromorphic on $\mathbb{D}(\varphi(z, w'), \varepsilon(z, w'))$ and $\mathcal{S}(\tilde{f}_z(w', \cdot) \subset \{\varphi(z, w')\}$. Moreover, $\tilde{f}_z(w', \cdot) = \hat{f}_0(z, w', \cdot)$ on $\mathbb{D}(\varphi(z, w'), \varepsilon(z, w')) \setminus \{\varphi(z, w')\}$.

Thus for each $(z, w') \in C$ there exists an $s(z, w') \in \mathbb{N}$ such that $f_{-k}(z, w') = 0$ for $k \geq s(z, w')$. Let $C_s := \{(z, w') \in C : s(z, w') \leq s\}$. It is clear that there exists an s_0 such that C_{s_0} is not pluripolar, which implies (via the identity principle) that $f_{-k} \equiv 0$ for $k \geq s_0$. Thus $\hat{f}_0$ is meromorphic in a neighborhood of (a_0, b_0).

In the general case we proceed as in the proof of Theorem 10.2.12. Fix a point $(a_0, b_0) \in (A \times G) \cap \widehat{M}$ and let $g \in \mathcal{O}(\widehat{\mathbb{P}}((a_0, b_0), \rho))$ be a defining function for $\widehat{M} \cap \widehat{\mathbb{P}}((a_0, b_0), \rho)$. Since the fiber $\widehat{M}_{(a_0,\cdot)} = \mathcal{S}(\tilde{f}_{a_0})$ is thin, we conclude that $g(a_0, \cdot) \not\equiv 0$ on $\widehat{\mathbb{P}}(b_0, \rho)$. Choose a unitary operator $\boldsymbol{U} : \mathbb{C}^q \to \mathbb{C}^q$ such that $\tilde{g}(a_0, b_0', \cdot) \not\equiv 0$ near $b_{0,q}$, where $\tilde{g} := g \circ \Phi$, $\Phi(z, w) := (z, b_0 + \boldsymbol{U}(w - b_0))$, $(z, w) \in \widehat{\mathbb{P}}(a_0, \rho) \times \widehat{\mathbb{P}}(b_0, \rho')$ and $\rho' > 0$ is so small that $b_0 + \boldsymbol{U}(w - b_0) \in \widehat{\mathbb{P}}(b_0, \rho)$ for all $w \in \widehat{\mathbb{P}}(b_0, \rho')$. Put $H := \Phi^{-1}(\widehat{M})$, $h := \hat{f}_0 \circ \Phi$. Then $h \in \mathcal{O}((\widehat{\mathbb{P}}(a_0, \rho) \times \widehat{\mathbb{P}}(b_0, \rho')) \setminus H)$ and for each $a \in A \cap \widehat{\mathbb{P}}(a_0, \rho))$ the function $h(a, \cdot)$ extends meromorphically to $\widehat{\mathbb{P}}(b_0, \rho')$. Using the same argument as in the proof of Theorem 10.2.12, we find $r, r_q > 0$ such that $\mathbb{P}((a_0, b_0'), r) \times \mathbb{D}(b_{0,q}, r_q) \subset \widehat{\mathbb{P}}(a_0, \rho) \times \widehat{\mathbb{P}}(b_0, \rho')$ and the projection

$$\mathrm{pr}_{\mathbb{C}^p \times \mathbb{C}^{q-1}} : H \cap (\mathbb{P}((a_0, b_0'), r) \times \mathbb{D}(b_{0,q}, r_q)) \xrightarrow{\pi} \mathbb{P}((a_0, b_0'), r)$$

is a ramified holomorphic covering. We may also assume that $(\mathbb{P}((a_0, b_0'), r) \times \mathbb{D}(c, \tau)) \cap H = \varnothing$ for certain $\mathbb{D}(c, \tau) \subset \mathbb{D}(b_{0,q}, r_q)$. It is known that there exists an analytic set $\Sigma \subsetneq \mathbb{P}((a_0, b_0'), r)$ such that

$$\pi|_{\pi^{-1}(\mathbb{P}((a_0,b_0'),r)\setminus\Sigma)} : \pi^{-1}(\mathbb{P}((a_0, b_0'), r) \setminus \Sigma) \to \mathbb{P}((a_0, b_0'), r) \setminus \Sigma$$

is an unramified holomorphic covering. Put $C := (A \cap \widehat{\mathbb{P}}(a_0, r)) \times \mathbb{P}(b_0', r)$. Applying the "generic" case shows that there exists an open neighborhood U of $C \setminus \Sigma$ such that $h \in \mathcal{M}(U \times \mathbb{D}(b_{0,q}, r_q))$. It remains to show that the envelope of holomorphy of the domain $\Omega := (U \times \mathbb{D}(b_{0,q}, r_q)) \cup (\mathbb{P}((a_0, b_0'), r) \times \mathbb{D}(c, \tau))$ contains a neighborhood of (a_0, b_0) (and to use Theorem 2.8.4). Observe that $\Omega \supset \boldsymbol{Y} := \mathbb{X}(C \setminus \Sigma, \mathbb{D}(c, \tau); \mathbb{P}((a_0, b_0'), r), \mathbb{D}(b_{0,q}, r_q))$. Consequently, every function holomorphic on Ω extends holomorphically to $\widehat{\boldsymbol{Y}}$ and

$$\begin{aligned}\boldsymbol{h}^*_{C\setminus\Sigma,\mathbb{P}((a_0,b_0'),r)}(z, w') &= \boldsymbol{h}^*_{C,\mathbb{P}((a_0,b_0'),r)}(z, w') \\ &= \max\{\boldsymbol{h}^*_{A\cap\widehat{\mathbb{P}}(a_0,r),\widehat{\mathbb{P}}(a_0,r)}(z),\ \boldsymbol{h}^*_{\mathbb{P}(b_0',r),\mathbb{P}(b_0',r)}(w')\} = \boldsymbol{h}^*_{A\cap\widehat{\mathbb{P}}(a_0,r),\widehat{\mathbb{P}}(a_0,r)}(z),\end{aligned}$$

which implies that $(a_0, b_0) \in \widehat{\boldsymbol{Y}}$. □

In the case where $A = D$ the result may be strengthened as follows.

Theorem 11.1.2 (Cf. [Rot 1950]). *Let D, G, G_0 be as in Theorem 11.1.1. Let $f \in \mathcal{M}(D \times G_0)$ be such that for every $a \in D$ with $\mathcal{S}(f)_{(a,\cdot)} \neq G_0$, the function $f(a, \cdot)$ extends meromorphically to G, i.e. there exists an $\tilde{f}_a \in \mathcal{M}(G)$ with $\mathcal{S}(\tilde{f}_a) \cap G_0 \subset \mathcal{S}(f)_{(a,\cdot)}$ and $\tilde{f}_a = f$ on $G_0 \setminus \mathcal{S}(f)_{(a,\cdot)}$. Then there exists an $\tilde{f} \in \mathcal{M}(D \times G)$ such that $\tilde{f} = f$ on $D \times G_0$, i.e. $\mathcal{S}(\tilde{f}) \cap (D \times G_0) = \mathcal{S}(f)$ and $\tilde{f} = f$ on $(D \times G_0) \setminus \mathcal{S}(f)$.*

Proof. Let $\Sigma := \{z \in D : \{z\} \times G_0 \subset \mathcal{S}(f)\}$. We know that Σ is a proper analytic set. Using locally Theorem 11.1.1 on $(D \setminus \Sigma) \times G$, we easily prove that f extends meromorphically to an $\tilde{f} \in \mathcal{M}((D \times G_0) \cup ((D \setminus \Sigma) \times G))$. Using Proposition 2.4.4, we conclude that the envelope of holomorphy of the domain $(D \times G_0) \cup ((D \setminus \Sigma) \times G)$ contains $D \times G$. Finally, Theorem 2.8.4 implies that $\tilde{f}$ extends meromorphically on $D \times G$. □

11.2 Cross theorem with singularities for meromorphic functions

> §§ 1.1.3, 2.8, 5.4, 10.2.

After the Rothstein theorem (which may be considered as a generalization of Proposition 1.1.10) it is natural to look for an analogue of the main cross theorem (Theorem 5.4.1) and an extension theorem with pluripolar singularities (Theorem 10.2.9) for meromorphic functions.

Theorem 11.2.1 (Cross theorem with singularities for meromorphic functions). *Assume that* (*cf.* (C1)–(C7))

- *D_j is a Riemann domain of holomorphy over $\mathbb{C}^{n_j}$,*
- *$A_j \subset D_j$, A_j is locally pluriregular,*
- *$\Sigma_j \subset \Sigma_j^0 \subset A_j' \times A_j''$, Σ_j^0 is pluripolar, $j = 1, \dots, N$,*
- *$\boldsymbol{X} := \mathbb{X}((A_j, D_j)_{j=1}^N)$, $\boldsymbol{T} := \mathbb{T}((A_j, D_j, \Sigma_j)_{j=1}^N)$,*
- *$M \subset S \subset \boldsymbol{T}$, M, S are relatively closed,*
- *for any $j \in \{1, \dots, N\}$ and $(a_j', a_j'') \in (A_j' \times A_j'') \setminus \Sigma_j^0$, the fiber $S_{(a_j', \cdot, a_j'')}$ is thin.*

Let

$$\mathcal{G} \subset \begin{cases} \mathcal{O}_s(\boldsymbol{X} \setminus S) & \text{if } \Sigma_1 = \dots = \Sigma_N = \varnothing, \\ \mathcal{O}_s^c(\boldsymbol{T} \setminus S) & \text{otherwise,} \end{cases}$$

and assume that for all $f \in \mathcal{G}$, $j \in \{1, \dots, N\}$, and $(a_j', a_j'') \in (A_j' \times A_j'') \setminus \Sigma_j^0$, the function $f(a_j', \cdot, a_j'')$ extends meromorphically to $D_j \setminus M_{(a_j', \cdot, a_j'')}$, i.e. there exists a function $\tilde{f}_{j,a_j',a_j''} \in \mathcal{M}(D_j \setminus M_{(a_j', \cdot, a_j'')})$ such that $\mathcal{S}(\tilde{f}_{j,a_j',a_j''}) \subset S_{(a_j', \cdot, a_j'')} \setminus M_{(a_j', \cdot, a_j'')}$ and $\tilde{f}_{j,a_j',a_j''} = f(a_j', \cdot, a_j'')$ on $D_j \setminus S_{(a_j', \cdot, a_j'')}$. Let $\widehat{M} \subset \widehat{\boldsymbol{X}}$ be constructed via Theorem 10.2.9 with respect to the family

$$\mathcal{F}(\boldsymbol{T} \setminus M) := \begin{cases} \mathcal{O}_s(\boldsymbol{X} \setminus M) & \text{if } \Sigma_1 = \dots = \Sigma_N = \varnothing, \\ \mathcal{O}_s^c(\boldsymbol{T} \setminus M) & \text{otherwise.} \end{cases}$$

Then there exists a generalized cross $\boldsymbol{T}' = \mathbb{T}((A_j, D_j, \Sigma_j')_{j=1}^N)$ with $\Sigma_j^0 \subset \Sigma_j'$, Σ_j' pluripolar, $j = 1, \dots, N$, such that for each $f \in \mathcal{G}$ there exists an $\tilde{f} \in \mathcal{M}(\widehat{\boldsymbol{X}} \setminus \widehat{M})$ with

- *$\widehat{M} \cap \boldsymbol{T}' \subset M$,*
- *$\mathcal{S}(\tilde{f}) \cap \boldsymbol{T}' \subset S$,*
- *$\tilde{f} = f$ on $\boldsymbol{T}' \setminus S$.*

Definition 11.2.2. In the above situation we say that f is *separately meromorphic on* $\boldsymbol{T} \setminus M$ ($f \in \mathcal{M}_s(\boldsymbol{T} \setminus M)$).

In the case where $M = \varnothing$ the theorem is a simultaneous generalization of some results presented in [Sak 1957], [Kaz 1976], [Kaz 1978], [Kaz 1984], [Shi 1986], [Kaz 1988], and [Shi 1989], namely:

Theorem 11.2.3 (Cross theorem for meromorphic functions). *Assume that $\boldsymbol{X}$, $\boldsymbol{T}$, and Σ_j^0, $j = 1, \dots, N$, are as in Theorem 11.2.1. Let $S \subset \boldsymbol{T}$ be relatively closed such that for any $j \in \{1, \dots, N\}$ and $(a_j', a_j'') \in (A_j' \times A_j'') \setminus \Sigma_j^0$, the fiber $S_{(a_j', \cdot, a_j'')}$ is thin. Let*

$$\mathcal{G} \subset \begin{cases} \mathcal{O}_s(\boldsymbol{X} \setminus S) & \text{if } \Sigma_1 = \dots = \Sigma_N = \varnothing, \\ \mathcal{O}_s^c(\boldsymbol{T} \setminus S) & \text{otherwise,} \end{cases}$$

and assume that for all $f \in \mathcal{G}$, $j \in \{1, \dots, N\}$, *and* $(a'_j, a''_j) \in (A'_j \times A''_j) \setminus \Sigma^0_j$, *the function* $f(a'_j, \cdot, a''_j)$ *extends meromorphically to* D_j, *i.e. there exists a function* $\tilde{f}_{j,a'_j,a''_j} \in \mathcal{M}(D_j)$ *such that* $\mathcal{S}(\tilde{f}_{j,a'_j,a''_j}) \subset S_{(a'_j,\cdot,a''_j)}$ *and* $\tilde{f}_{j,a'_j,a''_j} = f(a'_j, \cdot, a''_j)$ *on* $D_j \setminus S_{(a'_j,\cdot,a''_j)}$. *Then there exists a generalized cross* $\boldsymbol{T}' = \mathbb{T}((A_j, D_j, \Sigma'_j)_{j=1}^N)$ *with* $\Sigma^0_j \subset \Sigma'_j$, Σ'_j *pluripolar,* $j = 1, \dots, N$, *such that for each* $f \in \mathcal{G}$ *there exists an* $\tilde{f} \in \mathcal{M}(\widehat{\boldsymbol{X}})$ *with*

- $\widehat{M} \cap \boldsymbol{T}' \subset M$,
- $\mathcal{S}(\tilde{f}) \cap \boldsymbol{T}' \subset S$,
- $\tilde{f} = f$ *on* $\boldsymbol{T}' \setminus S$.

If $A_j = D_j$, $\Sigma_j = \varnothing$, $j = 1, \dots, N$, and $M = \varnothing$, then Theorem 11.2.3 gives the following analogue of the Hartogs theorem for meromorphic functions.

Theorem 11.2.4 (Hartogs theorem for meromorphic functions). *Let* D_j *be a Riemann domain of holomorphy over* $\mathbb{C}^{n_j}$ *and let* $\Sigma^0_j \subset D'_j \times D''_j$ *be a pluripolar set,* $j = 1, \dots, N$. *Assume that* $S \subset D_1 \times \cdots \times D_N$ *is a closed set such that for all* $j \in \{1, \dots, N\}$ *and* $(a'_j, a''_j) \in (D'_j \times D''_j) \setminus \Sigma^0_j$, *the fiber* $S_{(a'_j,\cdot,a''_j)}$ *is thin. Then there exist pluripolar sets* $\Sigma'_j \subset D'_j \times D''_j$ *with* $\Sigma^0_j \subset \Sigma'_j$, $j = 1, \dots, N$, *satisfying the following property.*

If $f \in \mathcal{O}((D_1 \times \cdots \times D_N) \setminus S)$ *is such that for all* $j \in \{1, \dots, N\}$ *and* $(a'_j, a''_j) \in (D'_j \times D''_j) \setminus \Sigma^0_j$, *the function* $f(a'_j, \cdot, a''_j)$ *extends meromorphically to* D_j, *then there exists an* $\tilde{f} \in \mathcal{M}(D_1 \times \cdots \times D_N)$ *such that for all* $j \in \{1, \dots, N\}$ *and* $(a'_j, a''_j) \in (D'_j \times D''_j) \setminus \Sigma'_j$, *we have* $\tilde{f}(a'_j, \cdot, a''_j) = f(a'_j, \cdot, a''_j)$ *on* $D_j \setminus S_{(a'_j,\cdot,a''_j)}$.

Proof of Theorem 11.2.1. Our proof is based on Theorem 10.2.9. Using this theorem we get an analytic set $\widehat{S} \subset \widehat{\boldsymbol{X}}$ and a generalized cross $\boldsymbol{T}' \subset \boldsymbol{T}^0$, $\boldsymbol{T}' = \mathbb{T}((A_j, D_j, \Sigma'_j)_{j=1}^N)$, $\boldsymbol{T}^0 := \mathbb{T}((A_j, D_j, \Sigma^0_j)_{j=1}^N)$, with $\Sigma^0_j \subset \Sigma'_j$, Σ'_j pluripolar, $j = 1, \dots, N$) such that for each $f \in \mathcal{G}$ there exists an $\hat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}} \setminus \widehat{S})$ with

- $\widehat{S} \cap (\boldsymbol{c}(\boldsymbol{T}^0) \cup \boldsymbol{T}') \subset S$,
- $\hat{f} = f$ on $(\boldsymbol{c}(\boldsymbol{T}^0) \cup \boldsymbol{T}') \setminus S$,
- $\widehat{S}$ is singular with respect to $\{\hat{f} : f \in \mathcal{G}\}$,
- $\widehat{M} \cap (\boldsymbol{c}(\boldsymbol{T}^0) \cup \boldsymbol{T}') \subset M$,
- for any $j \in \{1, \dots, N\}$ and $(a'_j, a''_j) \in (A'_j \times A''_j) \setminus \Sigma'_j$, the fiber $\widehat{S}_{(a'_j,\cdot,a''_j)}$ is singular with respect to the family $\{\hat{f}(a'_j, \cdot, a''_j) : f \in \mathcal{G}\}$ (cf. the proof of Lemma 10.2.8).

Similarly as in the proof of the Rothstein theorem (Theorem 11.1.1), we only need to show that there exists an open neighborhood $\Omega \subset \widehat{\boldsymbol{X}} \setminus \widehat{M}$ of $\boldsymbol{T}' \setminus \widehat{M}$ such that

- every connected component of Ω intersects $\boldsymbol{T}' \setminus \widehat{M}$,

- for every $f \in \mathcal{G}$ there exists an $\tilde{\tilde{f}} \in \mathcal{M}(\Omega)$ such that $\mathcal{S}(\tilde{\tilde{f}}) \subset \hat{S}$ and $\tilde{\tilde{f}} = \hat{f}$ on $\Omega \setminus \hat{S}$ (equivalently, $\tilde{\tilde{f}} = f$ on $\boldsymbol{T}' \setminus S$).

In fact, if $g \in \mathcal{O}(\Omega)$, then $g|_{\boldsymbol{T}'\setminus\hat{M}} \in \mathcal{O}_s^c(\boldsymbol{T}'\setminus\hat{M})$. Hence, by Theorem 10.2.12 there exists a $\hat{g} \in \mathcal{O}(\hat{\boldsymbol{X}} \setminus \hat{M})$ with $\hat{g} = g$ on $\boldsymbol{T}' \setminus \hat{M}$. Consequently, $\hat{g} = g$ on $\Omega \setminus \hat{M}$. This shows that the envelope of holomorphy of Ω coincides with $\hat{\boldsymbol{X}} \setminus \hat{M}$ (Theorem 2.8.4) and, therefore, for each $f \in \mathcal{G}$ the function $\tilde{\tilde{f}}$ extends to an $\tilde{f} \in \mathcal{M}(\hat{\boldsymbol{X}} \setminus \hat{M})$ with $\mathcal{S}(\tilde{f}) \cap \boldsymbol{T}' \subset S$ and $\tilde{f} = f$ on $\boldsymbol{T}' \setminus S$.

Take an $a \in \boldsymbol{T}' \setminus \hat{M}$. We may assume that $a = (a'_N, a_N) \in (A'_N \setminus \Sigma'_N) \times D_N$. Fix a $b_N \in A_N \setminus \hat{S}_{(a'_N,\cdot)}$ (recall that $\hat{S}_{(a'_N,\cdot)}$ is pluripolar). Put $b := (a'_N, b_N) \in \boldsymbol{T}' \setminus \hat{S}$. Let $r > 0$ be such that $\hat{\mathbb{P}}(b,r) \subset\subset \hat{\boldsymbol{X}} \setminus \hat{S}$. In particular, $\hat{f} \in \mathcal{O}(\hat{\mathbb{P}}(b,r))$, $f \in \mathcal{G}$.

Since $\hat{M}_{(a'_N,\cdot)}$ is pluripolar, there exists a domain $G_N \subset\subset D_N \setminus \hat{M}_{(a'_N,\cdot)}$ with $\{a_N\} \cup \hat{\mathbb{P}}(b_N, r) \subset G_N$. Take $r' \in (0, r)$ so small that $\hat{\mathbb{P}}(a'_N, r') \times G_N \subset \hat{\boldsymbol{X}} \setminus \hat{M}$. We going to apply the Rothstein theorem (Theorem 11.1.1) with $D := \hat{\mathbb{P}}(a'_N, r')$, $G := G_N$, $G_0 := \hat{\mathbb{P}}(b_N, r)$, $A := (A'_N \setminus \Sigma'_N) \cap \hat{\mathbb{P}}(a'_N, r')$. Observe that for every $f \in \mathcal{G}$ we have $\tilde{f}_{N,a'_N} = f(a'_N, \cdot) = \hat{f}(a'_N, \cdot)$ on $D_N \setminus S_{(a'_N,\cdot)}$, which implies that $\mathcal{S}(\tilde{f}_{N,a'_N}) \cap \hat{\mathbb{P}}(b_N, r) = \varnothing$ and $\tilde{f}_{N,a'_N} = \hat{f}(a'_N, \cdot)$ on $\hat{\mathbb{P}}(b_N, r)$. Put $\boldsymbol{Y}_a := \mathbb{X}(A, \hat{\mathbb{P}}(b_N, r); \hat{\mathbb{P}}(a'_N, r'), G_N)$. By the Rothstein theorem, for every $f \in \mathcal{G}$ there exists an $\tilde{\tilde{f}} \in \mathcal{M}(\hat{\boldsymbol{Y}}_a)$ such that $\tilde{\tilde{f}} = \hat{f}$ on $\hat{\mathbb{P}}(a'_N, r')$. Hence, by the identity principle, $\tilde{\tilde{f}} = \hat{f}$ in $\hat{\boldsymbol{Y}}_a \setminus \hat{S}$.

We put $\Omega := \bigcup_{a \in \boldsymbol{T}'\setminus\hat{M}} \hat{\boldsymbol{Y}}_a$. □

Remark 11.2.5. [?] It is an open problem to characterize those closed sets $S \subset D_1 \times \dots \times D_N$ without the assumption that the fiber $S_{(a'_j,\cdot,a''_j)}$ is thin for all $j \in \{1, \dots, N\}$ and $(a'_j, a''_j) \in (D'_j \times D''_j) \setminus \Sigma^0_j$, for which Theorem 11.2.4 remains true [?] Some partial answers will be given in the next sections.

11.3 The case $N = 2$

[>] §§ 2.8, 3.6, 5.4, 10.4, 11.1, 11.2.

In the case where $N = 2$, Theorem 11.2.3 may be strengthened as follows.

Theorem 11.3.1 (Extension theorem for meromorphic functions). *Let D, G be Riemann domains over $\mathbb{C}^p$ and $\mathbb{C}^q$, respectively, let $\varnothing \neq A \subset D$, $\varnothing \neq B \subset G$ be locally pluriregular sets, and let*

$$\boldsymbol{X} := \mathbb{X}(A, B; D, G) = (A \times G) \cup (D \times B).$$

Let $S \subset \boldsymbol{X}$ be a relatively closed set. Assume that there exist exhaustions $(D_k)_{k=1}^{\infty}$ and $(G_k)_{k=1}^{\infty}$ of D and G, respectively, by domains such that

(A) $A_k := A \cap D_k \neq \varnothing$, $B_k := B \cap G_k \neq \varnothing$,

(B) *for every $(a,b) \in A_k \times B_k$ we have $B_k \setminus S_{(a,\cdot)} \neq \varnothing$, $A_k \setminus S_{(\cdot,b)} \neq \varnothing$, $k \in \mathbb{N}$,*

(C) $A \times B \subset \overline{(A \times B) \setminus S}$.

Then for every function $f \in \mathcal{O}_s(\boldsymbol{X} \setminus S) \cap \mathcal{M}_s(\boldsymbol{X})$ there exists a function $\hat{f} \in \mathcal{M}(\widehat{\boldsymbol{X}})$ such that $\mathcal{S}(\hat{f}) \cap \boldsymbol{X} \subset S$ and $\hat{f} = f$ on $\boldsymbol{X} \setminus S$.

Remark 11.3.2. (a) Recall that if $f \in \mathcal{O}_s(\boldsymbol{X} \setminus S) \cap \mathcal{M}_s(\boldsymbol{X})$, then for all $(a,b) \in A \times B$ there exist functions $\tilde{f}_a \in \mathcal{M}(G)$, $\tilde{f}^b \in \mathcal{M}(D)$ such that

- $\mathcal{S}(\tilde{f}_a) \subset S_{(a,\cdot)}$ and $\tilde{f} = f(a,\cdot)$ in $G \setminus S_{(a,\cdot)}$,
- $\mathcal{S}(\tilde{f}^b) \subset S_{(\cdot,b)}$ and $\tilde{f}^b = f(\cdot,b)$ in $D \setminus S_{(\cdot,b)}$.

(b) Note that if $\boldsymbol{h}^*_{A,D} \equiv 0$ (e.g. $D \setminus A$ is of zero Lebesgue measure), then $\widehat{\boldsymbol{X}} = D \times G$.

Exercise 11.3.3. Check that for $N = 2$ and $M = \varnothing$, Theorem 11.3.1 generalizes Theorem 11.2.1.

Proof of Theorem 11.3.1. It suffices to prove that for each k there exists an open connected neighborhood $\Omega_k \subset \widehat{\boldsymbol{X}}_k$ of the cross $\boldsymbol{X}_k := \mathbb{X}(A_k, B_k; D_k, G_k) = (A_k \times G_k) \cup (D_k \times B_k)$ such that for each $f \in \mathcal{O}_s(\boldsymbol{X} \setminus S) \cap \mathcal{M}_s(\boldsymbol{X})$ there exists an $\tilde{f}_k \in \mathcal{M}(\Omega_k)$ with $\mathcal{S}(\tilde{f}_k) \cap \boldsymbol{X}_k \subset S$ and $\tilde{f}_k = f$ on $\boldsymbol{X}_k \setminus S$.

Indeed, the envelope of holomorphy of Ω_k coincides with $\widehat{\boldsymbol{X}}_k$ (cf. the proof of Theorem 10.4.2). Hence, by Theorem 2.8.4, the function $\tilde{f}_k$ extends to a function $\hat{f}_k \in \mathcal{M}(\widehat{\boldsymbol{X}}_k)$ i.e. $\mathcal{S}(\hat{f}_k) \cap \Omega_k = \mathcal{S}(\tilde{f}_k)$ and $\hat{f}_k = \tilde{f}_k$ on $\Omega_k \setminus \mathcal{S}(\tilde{f}_k)$. Observe that $\hat{f}_{k+1} = \hat{f}_k$ on $\widehat{\boldsymbol{X}}_k$ (equivalently, $\hat{f}_{k+1} = \hat{f}_k$ on $\widehat{\boldsymbol{X}}_k \setminus (\mathcal{S}(\hat{f}_{k+1}) \cup \mathcal{S}(\hat{f}_k))$).

Indeed, we have $\hat{f}_{k+1} = f = \hat{f}_k$ on $(A_k \times B_k) \setminus S$. Hence, in view of (C), we get $\hat{f}_{k+1} = \hat{f}_k$ on $(A_k \times B_k) \setminus (\mathcal{S}(\hat{f}_{k+1}) \cup \mathcal{S}(\hat{f}_k))$. Thus, by the identity principle, $\hat{f}_{k+1} = \hat{f}_k$ on the non-pluripolar set $\widehat{\boldsymbol{X}}_k \setminus (\mathcal{S}(\hat{f}_{k+1}) \cup \mathcal{S}(\hat{f}_k))$.

Now we glue the functions $(\hat{f}_k)_{k=1}^{\infty}$ and we get the required extension $\hat{f}$.

Fix a $k \in \mathbb{N}$. We are going to show that for any $(a,b) \in \Xi_k := (A_k \times B_k) \setminus S$ there exists an $r_{a,b} > 0$ with $\widehat{\mathbb{P}}((a,b), r_{a,b}) \subset D_k \times G_k$ such that every function $f \in \mathcal{O}_s(\boldsymbol{X} \setminus S) \cap \mathcal{M}_s(\boldsymbol{X})$ extends to an $\hat{f}_{a,b} \in \mathcal{M}(\Omega_{a,b})$, where

$$\Omega_{a,b} := (\widehat{\mathbb{P}}(a, r_{a,b}) \times G_k) \cup (D_k \times \widehat{\mathbb{P}}(b, r_{a,b})) \subset \widehat{\boldsymbol{X}},$$

i.e. $\mathcal{S}(\hat{f}_{a,b}) \cap \boldsymbol{X}_k \subset S$ and $\hat{f}_{a,b} = f$ on $\boldsymbol{X}_k \cap \Omega_{a,b} \setminus S$. Moreover, $\hat{f}_{a,b} = \hat{f}_{c,d}$ on $\Omega_{a,b} \cap \Omega_{c,d}$.

Suppose for a moment that $\Omega_{a,b}$, $(a,b) \in \Xi_k$, are already constructed. Put $\widetilde{\Omega}_k := \bigcup_{(a,b) \in \Xi_k} \Omega_{a,b}$. Observe that $\boldsymbol{X}_k \subset \widetilde{\Omega}_k$.

Indeed, let $(z,w) \in \boldsymbol{X}_k$, e.g. $(z,w) \in A_k \times G_k$. By (B) there exists a $b \in B_k \setminus S_{(z,\cdot)}$. Then $(z,b) \in \Xi_k$ and $(z,w) \in \Omega_{z,b}$.

For every $f \in \mathcal{O}_s(\boldsymbol{X} \setminus S) \cap \mathcal{M}_s(\boldsymbol{X})$ the functions $\hat{f}_{a,b}$, $(a, b) \in \Xi_k$, may be glued to an $\hat{f}_k \in \mathcal{M}(\widetilde{\Omega}_k)$ with $\mathcal{S}(\hat{f}_k) \cap \boldsymbol{X}_k \subset S$ and $\hat{f}_k = f$ on $\boldsymbol{X}_k \setminus S$. Thus, we may define Ω_k as the connected component of $\widetilde{\Omega}_k$ that contains $\boldsymbol{X}_k$. This will finish the proof of the theorem

We move to the construction of $\Omega_{a,b}$ and $\hat{f}_{a,b}$. Fix $(a, b) \in \Xi_k$ and $f \in \mathcal{O}_s(\boldsymbol{X} \setminus S) \cap \mathcal{M}_s(\boldsymbol{X})$. Let $\tau > 0$ be such that $\widehat{\mathbb{P}}((a, b), \tau) \subset (D_k \times G_k) \setminus S$. Define

$$\boldsymbol{Y}_{a,b} := \mathbb{X}(A[a, \tau], B[b, \tau]; \widehat{\mathbb{P}}(a, \tau), \widehat{\mathbb{P}}(b, \tau)) \subset \boldsymbol{X}_k.$$

Then $f \in \mathcal{O}_s(\boldsymbol{Y}_{a,b})$ and hence, by Theorem 5.4.1, $f|_{\boldsymbol{Y}_{a,b}}$ extends holomorphically on $\widehat{\boldsymbol{Y}}_{a,b}$. In particular, f extends holomorphically to an $\tilde{f}_{a,b} \in \mathcal{O}(\widehat{\mathbb{P}}((a, b), \rho))$ with $\rho \in (0, \tau)$, $\widehat{\mathbb{P}}((a, b), \rho) \subset \widehat{\boldsymbol{Y}}_{a,b}$.

Observe that for each $\alpha \in A[a, \rho]$, $\tilde{f}_\alpha = \tilde{f}_{a,b}(\alpha, \cdot)$ on $\widehat{\mathbb{P}}(b, \rho)$. Thus all the assumptions of the Rothstein theorem (Theorem 11.1.1) are satisfied, and consequently, there exists an $\tilde{\tilde{f}}_{a,b} \in \mathcal{M}(\widehat{\boldsymbol{Z}}_{a,b})$ with $\boldsymbol{Z}_{a,b} := \mathbb{X}(A[a, \rho], \widehat{\mathbb{P}}(b, \rho); \widehat{\mathbb{P}}(a, \rho), G)$ such that $\tilde{\tilde{f}}_{a,b} = \tilde{f}_{a,b}$ on $\widehat{\mathbb{P}}((a, b), \rho)$. Notice that $\tilde{\tilde{f}}_{a,b}(\alpha, \cdot) = \tilde{f}_\alpha$ for every $\alpha \in A[a, \rho]$. We take an $r \in (0, \rho)$ so small that $\widehat{\mathbb{P}}(a, r) \times G_k \subset \widehat{\boldsymbol{Z}}_{a,b}$. Repeating the same construction in the "horizontal" direction gives $\Omega_{a,b}$ and $\hat{f}_{a,b}$. It remains to show that $\hat{f}_{a,b} = \hat{f}_{c,d}$ on $\Omega_{a,b} \cap \Omega_{c,d}$. Observe that

$$\begin{aligned}\Omega_{a,b} \cap \Omega_{c,d} = \big((\widehat{\mathbb{P}}(a, r_{a,b}) \cap \widehat{\mathbb{P}}(c, r_{c,d})) \times G_k\big) \cup \big(\widehat{\mathbb{P}}(a, r_{a,b}) \times \widehat{\mathbb{P}}(d, r_{c,d})\big) \\ \cup \big(\widehat{\mathbb{P}}(c, r_{c,d}) \times \widehat{\mathbb{P}}(b, r_{a,b})\big) \cup \big(D_k \times (\widehat{\mathbb{P}}(b, r_{a,b}) \cap \widehat{\mathbb{P}}(d, r_{c,d}))\big).\end{aligned}$$

Note that the main problem is to show that $\hat{f}_{a,b} = \hat{f}_{c,d}$ on $\widehat{\mathbb{P}}(a, r_{a,b}) \times \widehat{\mathbb{P}}(d, r_{c,d})$ (resp. $\widehat{\mathbb{P}}(c, r_{c,d}) \times \widehat{\mathbb{P}}(b, r_{a,b})$). Then the identity principle for meromorphic functions immediately gives the equality on

$$\big(\widehat{\mathbb{P}}(a, r_{a,b}) \cap \widehat{\mathbb{P}}(c, r_{c,d})\big) \times G_k \quad (\text{resp. } D_k \times \big(\widehat{\mathbb{P}}(b, r_{a,b}) \cap \widehat{\mathbb{P}}(d, r_{c,d})\big)),$$

provided that this set is not empty.

To prove that $\hat{f}_{a,b} = \hat{f}_{c,d}$ on $\widehat{\mathbb{P}}(a, r_{a,b}) \times \widehat{\mathbb{P}}(d, r_{c,d})$ we may argue as follows. For $(\alpha, \beta) \in (A[a, r_{a,b}] \times B[d, r_{c,d}]) \setminus S$ we have

$$\hat{f}_{a,b}(\alpha, \beta) = \tilde{f}_\alpha(\beta) = f(\alpha, \beta) = \tilde{f}^\beta(\alpha) = \hat{f}_{c,d}(\alpha, \beta).$$

Hence, by (C), we get $\hat{f}_{a,b} = \hat{f}_{c,d}$ on the non-pluripolar set

$$(A[a, r_{a,b}] \times B[d, r_{c,d}]) \setminus (\mathcal{S}(\hat{f}_{a,b}) \cup \mathcal{S}(\hat{f}_{c,d})).$$

Consequently, by the identity principle,

$$\hat{f}_{a,b} = \hat{f}_{c,d} \quad \text{on } (\widehat{\mathbb{P}}(a, r_{a,b}) \times \widehat{\mathbb{P}}(d, r_{c,d})) \setminus (\mathcal{S}(\hat{f}_{a,b}) \cup \mathcal{S}(\hat{f}_{c,d})). \qquad \square$$

Corollary 11.3.4 (Cf. [Sak 1957]). *Let $S \subset \mathbb{D} \times \mathbb{D}$ be a relatively closed set such that*

- $\operatorname{int} S = \varnothing$,
- *for every domain $U \subset \mathbb{D} \times \mathbb{D}$ the set $U \setminus S$ is connected (we say in brief that S **does not separate domains**).*

Let A (resp. B) denote the set of all $a \in \mathbb{D}$ (resp. $b \in \mathbb{D}$) such that $\operatorname{int}_{\mathbb{C}} S_{(a,\cdot)} = \varnothing$ (resp. $\operatorname{int}_{\mathbb{C}} S_{(\cdot,b)} = \varnothing$). Put $\boldsymbol{X} := \mathbb{X}(A, B; \mathbb{D}, \mathbb{D}) = (A \times \mathbb{D}) \cup (\mathbb{D} \times B)$.

Then for every function $f \in \mathcal{O}_s(\boldsymbol{X} \setminus S) \cap \mathcal{M}_s(\boldsymbol{X})$ there exists an $\hat{f} \in \mathcal{M}(\mathbb{D} \times \mathbb{D})$ such that $\hat{f} = f$ on $\boldsymbol{X} \setminus S$.

Notice that the original proof of the above result is not correct – details may be found in [Jar-Pfl 2003c], see also § 11.4.

Proof. We are going to apply Theorem 11.3.1. First we check that the sets A and B are not plurithin at any point of $\mathbb{D}$ (in particular, they are dense in $\mathbb{D}$).

Indeed, suppose that A is plurithin at a point $a \in \mathbb{D}$. By Remark 3.6.2 (e), there exists a circle $C \subset \mathbb{D}$ such that $C \cap A = \varnothing$. Let $(G_j)_{j=1}^{\infty}$ be a countable basis of the topology of $\mathbb{D}$ consisting of relatively compact discs. Put $C_j := \{z \in C : G_j \subset S_{(z,\cdot)}\}$. Observe that C_j is closed and $C = \bigcup_{j=1}^{\infty} C_j$. Using a Baire category argument, we find a non-empty open arc $\Gamma \subset C$ and a disc $\mathbb{D}(b, r)$ such that $\Sigma := \Gamma \times \mathbb{D}(b, r) \subset S$. Since $\dim \Sigma = 3$, the surface Σ separates domains. Hence, since S is nowhere dense, we easily conclude that S separates domains; a contradiction.

Consequently, by Remark 3.6.2 (d), the sets A and B are locally regular and $\boldsymbol{h}^*_{A,\mathbb{D}} = \boldsymbol{h}^*_{B,\mathbb{D}} = 0$. In particular, $\widehat{\boldsymbol{X}} = \mathbb{D} \times \mathbb{D}$.

It remains to check (A)–(C) with arbitrary exhaustions $D_k := \mathbb{D}(r_k)$, $G_k := \mathbb{D}(r_k)$, $0 < r_k \nearrow 1$ (Exercise). □

Corollary 11.3.5 (Cf. [Shi 1989]). *Let $D, G, A, B, \boldsymbol{X}$ be as in Theorem 11.3.1. Assume that $S \subset \boldsymbol{X}$ is a relatively closed set such that*

- *the set $D \setminus A$ is of zero Lebesgue measure,*
- *for every $a \in A$ the fiber $S_{(a,\cdot)}$ is pluripolar,*
- *for every $b \in B$ the fiber $S_{(\cdot,b)}$ is of zero Lebesgue measure.*

Then for every function $f \in \mathcal{O}_s(\boldsymbol{X} \setminus S) \cap \mathcal{M}_s(\boldsymbol{X})$ there exists an $\hat{f} \in \mathcal{M}(D \times G)$ such that $\hat{f} = f$ on $\boldsymbol{X} \setminus S$.

Proof. One can easily check that all the assumptions of Theorem 11.3.1 are satisfied with arbitrary exhaustions and $k \gg 1$ (Exercise). □

11.4 Counterexamples

$\boxed{>}$ §§ 2.1.7, 2.3, 2.5.3, 3.2, 7.2, 11.3.

This section is based on [Pfl-NVA 2003].

In [Sak 1957], E. Sakai claimed (without proof) that the n-dimensional analogue of Corollary 11.3.4 is also true. Unfortunately, this is false as Propositions 11.4.4, 11.4.5 will show. We need the following preparations.

Let $v \in \mathcal{SH}(\mathbb{D}(2))$ be such that $v(0) = 0$ and the set $v^{-1}(-\infty)$ is dense in $\mathbb{D}(2)$. For example,

$$v(z) := \sum_{k=1}^{\infty} \frac{1}{d_k} \log \frac{|z - q_k|}{4} - \sum_{k=1}^{\infty} \frac{1}{d_k} \log \frac{|q_k|}{4},$$

where $(\mathbb{Q} + i\mathbb{Q}) \cap \mathbb{D}(2) = \{q_1, q_2, \dots\}$ and $(d_k)_{k=1}^{\infty} \subset \mathbb{R}_{>0}$ is such that

$$\sum_{k=1}^{\infty} \frac{1}{d_k} \log \frac{|q_k|}{4} > -\infty.$$

Put

$$u_n(z) := \sum_{k=1}^{n} v(z_k), \qquad z = (z_1, \dots, z_n) \in \mathbb{P}_n(2),$$
$$\boldsymbol{A}_n := \{z \in \mathbb{D}^n : u_n(z) < -1\}.$$

Observe that $\boldsymbol{A}_n$ is an open set dense in $\mathbb{D}^n$.

Lemma 11.4.1. *If $S_1, \dots, S_N \subset \mathbb{D}^n$ are relatively closed sets which do not separate domains, then the set $\bigcup_{j=1}^{N} S_j$ does not separate domains.*

Proof. Use finite induction on N and the equality

$$U \setminus \bigcup_{j=1}^{N} S_j = \Big(U \setminus \bigcup_{j=1}^{N-1} S_j\Big) \setminus S_N. \qquad \square$$

Lemma 11.4.2. *Let $S \subset \mathbb{D}^n \setminus \boldsymbol{A}_n$ be relatively closed, $n \geq 2$. Then:*

(a) *S does not separate domains,*

(b) *for any closed sets $F_p' \subset \mathbb{C}^p$, $F_q'' \subset \mathbb{C}^q$ $(p, q \geq 0)$, the closed set $F_p' \times S \times F_q''$ does not separate domains in $\mathbb{C}^p \times \mathbb{D}^n \times \mathbb{C}^q$.*

Proof. (a) It suffices to show that for every open convex domain $P = P_1 \times \dots \times P_n \subset\subset \mathbb{D}^n$, the set $P \setminus S$ is connected.

Indeed, let $U \subset \mathbb{D}^n$ be a domain and let $a, b \in U \setminus S$. Let $\gamma : [0, 1] \to U$ be a curve such that $\gamma(0) = a$, $\gamma(1) = b$. Choose polydiscs $Q_1, \dots, Q_N \subset\subset U$ such that

$\gamma([0,1]) \subset \bigcup_{j=1}^{N} P_j, a \in Q_1, Q_j \cap Q_{j+1} \neq \varnothing, j = 1, \dots, N-1, b \in Q_N$. If $Q_j \setminus S$ is connected, $j = 1, \dots, N$, and $(Q_j \cap Q_{j+1}) \setminus S$ is connected, $j = 1, \dots, N-1$, then $(Q_j \cup Q_{j+1}) \setminus S$ is also connected, $j = 1, \dots, N-1$, and finally, $(Q_1 \cup \dots \cup Q_N) \setminus S$ is connected.

Take $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n) \in P \setminus S$. We are going to connect the points a and b with a curve $\gamma \colon [0,1] \to P \setminus S$. We may assume that $v(a_j) = v(b_j) = -\infty$, $j = 1, \dots, n$.

Indeed, as the set $v^{-1}(-\infty)$ is dense, we find $a' = (a'_1, \dots, a'_n)$, $b' = (b'_1, \dots, b'_n) \in P$ with $v(a'_j) = v(b'_j) = -\infty$, $j = 1, \dots, n$, such that the segments $[a, a']$, $[b, b']$ are contained in $P \setminus S$.

Define

$$\gamma(t) := (b_1, \dots, b_j, (j+1-nt)a_{j+1} + (nt-j)b_{j+1}, a_{j+2}, \dots, a_n),$$
$$t \in \left[\frac{j}{n}, \frac{j+1}{n}\right], \quad j = 0, \dots, n-1.$$

Observe that $\gamma \colon [0,1] \to P$ is well defined, continuous, $\gamma(0) = a$, $\gamma(1) = b$, and $u_n(\gamma(t)) = -\infty$ for every $t \in [0,1]$ (because $n > 1$). Since

$$S \subset \mathbb{D}^n \setminus \boldsymbol{A}_n = \{z \in \mathbb{D}^n : u(z) \geq -1\},$$

we conclude that $\gamma \colon [0,1] \to P \setminus S$.

(b) As in (a), it suffices to show that $P \setminus (F'_p \times S \times F''_q)$ is connected for every open convex domain $P = P'_p \times P_n \times P''_q \subset\subset \mathbb{C}^p \times \mathbb{D}^n \times \mathbb{C}^q$. Fix (a'_p, a, a''_q), $(b'_p, b, b''_q) \in P \setminus (F'_p \times S \times F''_q)$. Since $v^{-1}(-\infty)$ is dense in $\mathbb{D}$, we may assume that $a, b \notin S$. By (a) there exists a curve $\gamma \colon [0,1] \to P_n \setminus S$ such that $\gamma(0) = a$, $\gamma(1) = b$. Consequently, the curve

$$[0,1] \ni t \mapsto ((1-t)a'_p + tb'_p, \gamma(t), (1-t)a''_q + tb''_q)$$

connects the points (a'_p, a, a''_q), (b'_p, b, b''_q) in $P \setminus (F'_p \times S \times F''_q)$. □

Lemma 11.4.3. *Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex domain, $\varphi \in \mathcal{PSH}(\Omega)$, $\Omega' := \{z \in \Omega : \varphi(z) < 0\}$. Assume that $\Omega' \subsetneq \Omega$. Then there exists an $f \in \mathcal{O}(\Omega')$ that cannot be extended meromorphically to Ω.*

Proof. By Theorem 2.5.5 (e) Ω' is pseudoconvex. Take an arbitrary $f \in \mathcal{O}(\Omega')$ that cannot be extended holomorphically beyond Ω' (cf. Proposition 2.1.27) and suppose that $\hat{f} \in \mathcal{M}(\Omega)$ is such that $\hat{f} = f$ on Ω'. Then $\Omega \setminus \mathcal{S}(\hat{f}) \subset \Omega'$. Since $\mathcal{S}(\hat{f})$ is a proper analytic set, we easily conclude (cf. Proposition 2.3.29) that $\varphi \leq 0$ on $\mathcal{S}(\hat{f})$; a contradiction. □

For a set $S \subset \mathbb{D}^n$ put

$$S_j := \{(a', a'') \in \mathbb{D}^{j-1} \times \mathbb{D}^{n-j} : \operatorname{int}_{\mathbb{C}} S_{(a',\cdot,a'')} \neq \varnothing\}, \quad j = 1, \dots, n.$$

Proposition 11.4.4. *For all $n \geq 3$ there exist*

- *a relatively closed nowhere dense set $S \subset \mathbb{D}^n$ such that S does not separate domains and $S_j \subsetneq \mathbb{D}^{n-1}$, $j = 1, \dots, n$,*
- *a domain $\Omega' \subsetneq \mathbb{D}^n$ with $\boldsymbol{T} := \mathbb{T}((\mathbb{D}, \mathbb{D}, S_j)_{j=1}^n) \subset \Omega'$, and*
- *a function $f \in \mathcal{O}(\Omega')$ that cannot be extended meromorphically to $\mathbb{D}^n$.*

In particular, $f|_{\boldsymbol{T} \setminus S} \in \mathcal{O}_s^c(\boldsymbol{T} \setminus S) \cap \mathcal{M}_s(\boldsymbol{T})$ cannot be extended meromorphically to $\mathbb{D}^n$.

Proof. Let

$$B := \{z \in \mathbb{D} : v(z) < -1/(n-1)\}.$$

Note that B is dense in $\mathbb{D}$ and

$$(\mathbb{D} \setminus B)^{n-1} = \underbrace{(\mathbb{D} \setminus B) \times \cdots \times (\mathbb{D} \setminus B)}_{(n-1)\text{-times}} \subset \mathbb{D}^{n-1} \setminus \boldsymbol{A}_{n-1}.$$

Fix an arbitrary closed disc $F \subset B$. Then, by Lemmas 11.4.1 and 11.4.2, the closed set

$$S := (F \times (\mathbb{D} \setminus B)^{n-1}) \cup \cdots \cup ((\mathbb{D} \setminus B)^{n-1} \times F)$$

does not separate domains in $\mathbb{D}^n$. Since B is dense, we easily conclude that $\operatorname{int} S = \varnothing$. Observe that $S_j = (\mathbb{D} \setminus B)^{n-1}$, $j = 1, \dots, n$. Consequently,

$$\boldsymbol{T} = (B \times \mathbb{D}^{n-1}) \cup \cdots \cup (\mathbb{D}^{n-1} \times B) = \mathbb{X}_{n,n-1}((B, \mathbb{D})_{j=1}^n) =: \boldsymbol{X}_{n,n-1}$$

(cf. § 7.2); note that $\boldsymbol{T}$ is open. Define

$$\Omega' := \Big\{(z_1, \dots, z_n) \in \mathbb{D}^n : \sum_{j=1}^n \boldsymbol{h}^*_{B,\mathbb{D}}(z_j) < n-1\Big\} = \widehat{\boldsymbol{X}}_{n,n-1}.$$

Observe that:

- $\Omega' \subsetneq \mathbb{D}^n$.
  Indeed, let $C := \sup_{\mathbb{D}} v$. Note that $C > 0$ (because $v(0) = 0$). Put

  $$w := \frac{v - C}{C + 1/2} + 1.$$

  Then $\boldsymbol{h}^*_{B,\mathbb{D}} \geq w$. In particular,

  $$\boldsymbol{h}^*_{B,\mathbb{D}}(0) \geq w(0) = \frac{-C}{C + 1/2} + 1 > 0.$$

  Now it remains to apply Remark 5.1.6 (b).
- Ω' is connected – cf. Remark 7.2.3 (f).

Finally, by Lemma 11.4.3, there exists an $f \in \mathcal{O}(\Omega')$ that cannot be continued meromorphically to $\mathbb{D}^n$. □

For domains $D \subset \mathbb{C}^p$, $G \subset \mathbb{C}^q$ and $S \subset D \times G$ put

$$A_S := \{a \in D : \operatorname{int}_G S_{(a,\cdot)} = \varnothing\}, \quad B^S := \{b \in G : \operatorname{int}_D S_{(\cdot,b)} = \varnothing\}.$$

Proposition 11.4.5. *Let $D \subset \mathbb{C}^p$, $G \subset \mathbb{C}^q$ be bounded pseudoconvex domains ($p, q \geq 2$). Then there exist*

- *a relatively closed nowhere dense set $S \subset D \times G$ such that S does not separate domains and A_S, B^S are locally pluriregular,*
- *a domain $\Omega' \subsetneq D \times G$ with $\boldsymbol{X} := \mathbb{X}(A, B; D, G) \subset \Omega'$, and*
- *a function $f \in \mathcal{O}(\Omega')$ that cannot be extended meromorphically to $D \times G$.*

In particular, $f|_{\boldsymbol{X} \setminus S} \in \mathcal{O}_s(\boldsymbol{X} \setminus S) \cap \mathcal{M}_s(\boldsymbol{X})$ cannot be extended meromorphically to $D \times G$.

Proof. We may assume that $0 \in D \subset \mathbb{D}^p$, $0 \in G \subset \mathbb{D}^q$. Fix closed polydiscs $F_p' := \overline{\mathbb{P}}(r_p) \subset\subset D$ and $F_q'' := \overline{\mathbb{P}}(r_q) \subset\subset G$. Recall that $\boldsymbol{A}_p = \{z \in \mathbb{D}^p : u_p(z) < -1\}$ – cf. the proof of Proposition 11.4.4. Put

$$S := ((D \setminus \boldsymbol{A}_p) \times F_q'') \cup (F_p' \times (G \setminus \boldsymbol{A}_q)).$$

It is clear that S is nowhere dense. Lemmas 11.4.1, 11.4.2 imply that S does not separate domains. Moreover, $A_S = \boldsymbol{A}_p \cap D$ and $B^S = \boldsymbol{A}_q \cap G$; $A = A_S$, $B = B^S$ are open, in particular, locally pluriregular. Let

$$\Omega' := \widehat{\mathbb{X}}(A, B; D, G) = \{(z, w) \in D \times G : \boldsymbol{h}^*_{\boldsymbol{A}_p, D}(z) + \boldsymbol{h}^*_{\boldsymbol{A}_q, G}(w) < 1\}.$$

Observe that $\Omega' \subsetneq D \times G$. Indeed, let $C := \sup_D u_p$. Observe that $C > 0$ (because $u_p(0) = pv(0) = 0$). Put

$$w := \frac{u_p - C}{C + 1} + 1.$$

Then $\boldsymbol{h}^*_{\boldsymbol{A}_p, D} \geq w$. In particular,

$$\boldsymbol{h}^*_{\boldsymbol{A}_p, D}(0) \geq w(0) = \frac{-C}{C + 1} + 1 > 0.$$

Hence, by Remark 5.1.6 (b), $\Omega' \subsetneq D \times G$.

We conclude the proof as in Proposition 11.4.4. □

Bibliography

The numbers at the end of each item refer to the pages on which the respective work is cited.

[Agr 2010] M. L. Agranovsky, Boundary Forelli theorem for the sphere in $\mathbb{C}^n$ and $n+1$ bundles of complex lines; preprint 2010 arXiv:1003.6125v1 [math.CV]. 92

[Agr-Sem 1991] M. L. Agranovsky and A. M. Semenov, Boundary analogues of Hartogs' theorem. *Siberian Math. J.* **32** (1991), 137–139. 92

[AEK 1996] V. Aguilar, L. Ehrenpreis, and P. Kuchment, Range conditions for the exponential Radon transform. *J. Anal. Math.* **68** (1996), 1–13. 222

[Aiz 1990] L. A. Aizenberg, *Carleman formulae in complex analysis*. Nauka Sibirsk. Otdel., Novosibirsk 1990 (in Russian). 170

[Akh-Ron 1973] N. I. Akhiezer and L. I. Ronkin, On separately analytic functions of several variables and theorems on "the thin end of the wedge". *Uspehi Mat. Nauk.* **28** (171) (1973), 27–42; English. transl. *Russian Math. Surveys* **28** (1973), 27–44. 4, 104, 108, 160, 167

[Akh-Ron 1976] N. I. Akhiezer and L. I. Ronkin, Separately analytic functions of several variables. In *Questions on mathematical physics and functional analysis* (Proceedings of research reminars of the Institute for Low Temperature Physics and Engineering, Kharkov), "Naukova Dumka", Kiev 1976, 3–10, (in Russian). 167

[Ale-Ama 2003] O. Alehyane and H. Amal, Separately holomorphic functions with pluripolar singularities. *Vietnam J. Math.* **31** (2003), 333–340. 221

[Ale-Hec 2004] O. Alehyane and J.-M. Hecart, Propriété de stabilité de la fonction extrémale relative. *Potential Anal.* **21** (2004), 363–373. 64, 68

[Ale-Zer 2001] O. Alehyane and A. Zeriahi, Une nouvelle version du théorème d'extension de Hartogs pour les applications séparément holomorphes entre espaces analytiques. *Ann. Polon. Math.* **76** (2001), 245–278. 4, 71, 104, 117

[Arm-Gar 2001] D. H. Armitage and S. J. Gardiner, *Classical potential theory*. Springer Monogr. Math., Springer-Verlag, London 2001. 43, 81, 84

[Ars 1966] M. G. Arsove, On the subharmonicity of doubly subharmonic functions. *Proc. Amer. Math. Soc.* **17** (1966), 622–626. 129

[Ava 1961] V. Avanissian, Fonctions plurisousharmoniques et fonctions doublement sousharmoniques. *Ann. Sci. École Norm. Sup.* **78** (1961), 101–161. 129

[Bai 1899] R. Baire, Sur les fonctions des variables réelles. *Ann. Mat. Pura Appl.* **3** (1899), 1–122. 1, 3

[Bar-Zam 2009] L. Baracco and G. Zampieri, The edge of the wedge theorem for separately holomorphic functions with singularities. *Commun. Anal. Geom.* **17** (2009), 327–342. 222

[Bar 1975] T. Barth, Families of holomorphic maps into Riemann surfaces. *Trans. Amer. Math. Soc.* **207** (1975), 175–187. 5

[Bed 1981] E. Bedford, The operator $(dd^c)^n$ on complex spaces. In *Séminaire d'analyse Lelong–Skoda*, Lecture Notes in Math. 919, Springer-Verlag, Berlin 1981, 294–324. 71

[Bed-Tay 1976] E. Bedford and B. A. Taylor, The Dirichlet problem for a complex Monge–Ampère equation. *Invent. Math.* **37** (1976), 1–44. 71

[Bed-Tay 1982] E. Bedford and B. A. Taylor, A new capacity for plurisubharmonic functions. *Acta Math.* **149** (1982), 1–40. 71

[Ber-Gay 1991] C. A. Berenstein and R. Gay, *Complex variables*. Grad. Texts in Math. 125, Springer-Verlag, New York 1991. 190

[Ber 1912] S. N. Bernstein, *Sur l'ordre de la meilleure approximation des fonctions continues par des polynomes de degré donné*. Mémoires de l'Académie Royale de Belgique, Bruxelles 1912. 4, 104, 108

[Bło 1992] Z. Błocki, Singular sets of separately analytic functions. *Ann. Polon. Math.* **56** (1992), 219–225. 3, 129

[Bło 2000] Z. Błocki, Equilibrium measure of a product subset of $\mathbb{C}^n$. *Proc. Amer. Math. Soc.* **128** (2000), 3595–3599. 66

[Boc 1938] S. Bochner, A theorem on analytic continuation of functions in several variables. *Ann. of Math.* **39** (1938), 14–19. 124

[Boc-Mar 1948] S. Bochner and W. T. Martin, *Several complex variables*. Princeton Math. Ser. 10, Princeton University Press, Princeton, N. J., 1948. 124

[Bro 1961] F. Browder, Real analytic functions on product spaces and separate analyticity. *Canad. J. Math.* **13** (1961), 650–656. 126

[Cam-Sto 1966] R. H. Cameron and D. A. Storvick, Analytic continuation for functions of several complex variables. *Trans. Amer. Math. Soc.* **125** (1966), 7–12. 4, 93, 104, 108, 109

[Cau 1821] A. Cauchy, *Cours d'analyse de l'École royale polytechnique*. Imprimerie royale Debure frères, Paris 1821; translated into English by R. E. Bradley and C. E. Sandifer, *Cauchy's cours d'analyse*, Sources Stud. Hist. Math. Phys. Sci., Springer-Verlag, New York 2009. 1

[Chi 1989] E. M. Chirka, *Complex analytic sets*. Math. Appl. (Soviet Ser.) 46, Kluwer Acadademic Publishers, Dordrecht 1989. 46, 47, 230, 231

[Chi 1993] E. M. Chirka, The extension of pluripolar singularity sets. *Proc. Steklov Inst. Math.* **200** (1993), 369–373. 214, 254

[Chi 2006] E. M. Chirka, Variations of Hartogs' theorem. *Proc. Steklov Inst. Math.* **253** (2006), 212–220. 91

[Chi-Sad 1987] E. M. Chirka and A. Sadullaev, On continuation of functions with polar singularities. *Mat. Sb.* (*N.S.*) **132** (174) (1987), 383–390; English. transl. *Math. USSR-Sb.* **60** (1988), 377–384. 4, 209

[Dlo 1977] G. Dloussky, Envelopes d'holomorphie et prolongement d'hypersurfaces. In *Séminaire Pierre Lelong* (*Analyse*), Lecture Notes in Math. 578, Springer-Verlag, Berlin 1977, 217–235. 213

[Doc-Gra 1960] F. Docquier, H. Grauert, Levisches Problem und Rungescher Satz für Teilgebiete Steinscher Mannigfaltigkeiten. *Math. Ann.* **140** (1960), 94–123. 51

[Dru 1977] L. M. Drużkowski, Extension of separately analytic functions defined on a cross in the space $\mathbb{C}^n$. *Zeszyty Nauk. Uniw. Jagiell.* **18** (1977), 35–44. 108

[Dru 1980] L. M. Drużkowski, A generalization of the Malgrange-Zerner theorem. *Ann. Polon. Math.* **38** (1980), 181–186. 4, 158, 160, 169

[Edi 2002] A. Edigarian, Analytic discs method in complex analysis. *Diss. Math.* **402** (2002), 1–56. 66

[Edi 2003] A. Edigarian, A note on Rosay's paper. *Ann. Polon. Math.* **80** (2003), 125–132. 132

[Edi-Pol 1997] A. Edigarian and E. A. Poletsky, Product property of the relative extremal function. *Bull. Sci. Acad. Polon.* **45** (1997), 331–335. 66

[El 1980] H. El Mir, Fonctions plurisousharmoniques et ensembles polaires. In *Séminaire Pierre Lelong-Henri Skoda* (*Analyse*), Lecture Notes in Math. 822, Springer-Verlag, Berlin 1980, 61–76. 44

[Eps 1966] H. Epstein, Some analytic properties of the scattering amplitudes in quantum field theory. In *Particle symmetries and axiomatic field theory*, Vol. 1, Axiomatic field theory, Brandeis University Summer Institute in Theoretical Physics, 1965, Gordon and Breach Science Publishers, Inc., New York, London, Paris 1966. 160

[Fil 1973] A. F. Filippov, The analytic continuation of functions of several variables. *Mat. Zametki* **14** (1973), 49–54; English transl. *Math. Notes* **14** (1973), 582–585. 88

[For 1977] F. Forelli, Pluriharmonicity in terms of harmonic slices. *Math. Scand.* **41** (1977), 358–364. 92

[Fuk 1983] J. Fuka, A remark on Hartogs' double series. In *Complex analysis* (Warsaw, 1979), Banach Center Publ. 11, PWN, Warsaw 1983, 77–78. 21

[Gau-Zer 2009] P. M. Gauthier and E. S. Zeron, Hartogs' theorem on separate holomorphicity for projective spaces. *Canad. Math. Bull.* **52** (2009), 84–86. 6

[Gen 1884] A. Genocchi and G. Peano, *Calcolo differenziale e principi di calcolo integrale*. Bocca, Torino 1884. 1

[Glo 2009] J. Globevnik, Small families of complex lines for testing holomorphic extendibility. *Amer. J. Math.*, to appear; preprint 2009 arXiv:0911.5088v2 [math.CV]. 92

[Gol 1983] G. M. Goluzin, *Geometric theory of functions of a complex variable*. Transl. Math. Monogr. 26, Amer. Math. Soc., Prividence, RI, 1983. 109, 170, 173, 175, 176, 177

[Gol-Kry 1933] G. M. Goluzin and W. I. Krylow, General formulae of Carleman and applications in analytic continuation. *Mat. Sb.* 40(1933), 144–149 (in Russian). 170

[Gon 1967] G. A. Gonchar, Estimates of the growth of rational functions and some of their applications. *Mat. Sb.* (*N.S.*) **72** (114) (1967), 489–503; English transl. *Math. USSR-Sb.* **1** (1967), 445–456. 191

[Gon 1968] G. A. Gonchar, Generalized analytic continuation. *Mat. Sb.* (*N.S.*) **76** (118) (1968), 135–146; English transl. *Math. USSR-Sb.* **5** (1968), 129–140. 191

[Gon 1969] G. A. Gonchar, Zolotarev problems connected with rational functions. *Mat. Sb.* (*N.S.*) **78** (120) (1969), 148–164; English transl. *Math. USSR-Sb.* **7** (1969), 640–654. 191

[Gon 1972] G. A. Gonchar, A local condition of single-valuedness of analytic functions. *Mat. Sb.* (*N.S.*) **89** (131) (1972), 148–164; English transl. *Math. USSR-Sb.* **18** (1972), 151–167. 188, 192, 193

[Gon 1974] G. A. Gonchar, A local condition for the single-valuedness of analytic functions of several variables. *Mat. Sb.* (*N.S.*) **93** (135) (1974), 296–313; English transl. *Math. USSR-Sb.* **22** (1974), 305–322. 188, 193

[Gon 1985] G. A. Gonchar, On analytic continuation from the "edge of the wedge". *Ann. Acad. Sci. Fenn. Ser. A I Math.* **10** (1985), 221–225. 4, 158, 170

[Gon 2000] G. A. Gonchar, On Bogulyubov's "edge-of-the-wedge" theorem. *Proc. Steklov Inst. Math.* **223** (2000), 18–24. 158, 170

[Gór 1975] J. Górski, Holomorphic continuation of separately holomorphic functions on a 3-dimensional set in $\mathbb{C}^2$. *Univ. Śląski w Katowicach Prace Naukowe Prace Mat.* **59** (1975), 33–40. 109

[Gra 1956] H. Grauert, Charakterisierung der Holomorphiegebiete durch die vollständige Kählersche Metrik. *Math. Ann.* **131** (1956), 38–75. 51

[Gra-Rem 1956] H. Grauert and R. Remmert, Konvexität in der komplexen Analysis. *Comment. Math. Helv.* **31** (1956–57), 152–183. 51, 213

[Gra-Rem 1957] H. Grauert and R. Remmert, Singularitäten komplexer Mannigfaltigkeiten und Riemannsche Gebiete. *Math. Z.* **67** (1957), 103–128. 51

[Har 1906] F. Hartogs, Zur Theorie der analytischen Funktionen mehrerer unabhängiger Veränderlichen, insbesondere über die Darstellung derselben durch Reihen, welche nach Potenzen einer Veränderlichen fortschreiten. *Math. Ann.* **62** (1906), 1–88. 3, 12, 21

[Héc 2000] J. Hécart, Ouverts d'harmonicité pour les fonctions séparément harmoniques. *Potential Anal.* **13** (2000), 115–126. 127

[Hör 1973] L. Hörmander, *An introduction to complex analysis in several variables*. North-Holland Math. Library 7, North Holland Publishing Company, Amsterdam 1990. 124

[Hou 2009] J. Hounie, A proof of Bochner's tube theorem. *Proc. Amer. Math. Soc.* **137** (2009), 4203–4207. 124

[Huk 1942] M. Hukuhara, Extension of a theorem of Osgood and Hartogs. *Kansu-hoteisiki ogobi Oyo-Kaiseki* (1942), 48–49 (in Japanese). 3, 25, 281

[Imo-Khu 2000] S. A. Imomkulov and Z. U. Khuzhamov, On separately analytic functions of several variables. *Uzbek. Mat. Zh.* **3** (2000), 3–7 (in Russian). 121

[Jar-Pfl 1993] M. Jarnicki and P. Pflug, *Invariant distances and metrics in complex analysis*. De Gruyter Exp. Math. 9, Walter de Gruyter, Berlin 1993. 74

[Jar-Pfl 2000] M. Jarnicki and P. Pflug, *Extension of holomorphic functions*. De Gruyter Exp. Math. 34, Walter de Gruyter, Berlin 2000. 28, 30, 32, 33, 35, 36, 38, 39, 46, 47, 50, 51, 53, 54, 117, 118, 149, 188, 209, 213, 214, 218, 219, 220

[Jar-Pfl 2001] M. Jarnicki and P. Pflug, Cross theorem. *Ann. Polon. Math.* **77** (2001), 295–302. 5, 221, 223

[Jar-Pfl 2003a] M. Jarnicki and P. Pflug, An extension theorem for separately holomorphic functions with analytic singularities. *Ann. Polon. Math.* **80** (2003), 143–161. 5, 221, 223, 239

[Jar-Pfl 2003b] M. Jarnicki and P. Pflug, An extension theorem for separately holomorphic functions with pluripolar singularities. *Trans. Amer. Math. Soc.* **355** (2003), 1251–1267. 5, 221, 223, 239

[Jar-Pfl 2003c] M. Jarnicki and P. Pflug, An extension theorem for separately meromorphic functions with pluripolar singularities. *Kyushu J. Math.* **57** (2003), 291–302. 5, 221, 264

[Jar-Pfl 2005] M. Jarnicki and P. Pflug, A remark on separate holomorphy. *Studia Math.* **174** (2006), 309–317. 183

[Jar-Pfl 2007] M. Jarnicki and P. Pflug, A general cross theorem with singularities. *Analysis (Munich)* **27** (2007), 181–212. 5, 221

[Jar-Pfl 2008] M. Jarnicki and P. Pflug, *First steps in several complex variables: Reinhardt domains*. EMS Textbk. Math., European Math. Soc. Publishing House, Zürich 2008. 24, 38, 56, 213, 223

[Jar-Pfl 2010a] M. Jarnicki and P. Pflug, An elementary proof of the cross theorem in the Reinhardt case. *Michigan Math. J.* **59** (2010), 411–417. 5, 58

[Jar-Pfl 2010b] M. Jarnicki and P. Pflug, A new cross theorem for separately holomorphic functions. *Proc. Amer. Math. Soc.* **138** (2010), 3923–3932. 5

[Jar-Pfl 2011] M. Jarnicki and P. Pflug, A remark on the identity principle for analytic sets. *Colloq. Math.* **123** (2011), 21–26. 5, 221, 230

[Jos 1978] B. Josefson, On the equivalence between polar and globally polar sets for plurisubharmonic functions on $\mathbb{C}^n$. *Ark. Mat.* **16** (1978), 109–115. 43

[Kar 1976] J. Karlsson, Rational interpolation and best rational approximation. *J. Math. Anal. Appl.* **53** (1976), 38–52. 193, 198

[Kar 1991] N. G. Karpova, On removal of singularities of plurisubharmonic functions. *Mat. Zametki* **49** (1991), 35–40 (in Russian). 253

[Kaz 1976] M. V. Kazarjan, On functions of several complex variables that are separately meromorphic. *Mat. Sb. (N.S.)* **99** (141) (1976), 538–547; English transl. *Math. USSR-Sb.* **28** (1976), 481–489. 5, 259

[Kaz 1978] M. V. Kazarjan, Separately meromorphic functions. *Akad. Nauk Armyan. SSR Dokl.* **67** (1978), 69–73 (in Russian). 5, 259

[Kaz 1984] M. V. Kazaryan, Meromorphic extension with respect to groups of variables. *Mat. Sb.* (*N.S.*) **125** (167) (1984), 384–397; English transl. *Math. USSR-Sb.* **53** (1986), 385–398. 5, 259

[Kaz 1988] M. V. Kazaryan, On holomorphic continuation of functions with pluripolar singularities. *Dokl. Akad. Nauk Arm. SSR* **87** (1988), 106–110 (in Russian). 259

[Ker 1943] R. Kershner, The continuity of functions of many variables. *Trans. Amer. Math. Soc.* **53** (1943), 83–100. 2

[Kli 1991] M. Klimek, *Pluripotential theory*. London Math. Soc. Monogr. (N.S.) 6, Oxford University Press, New York 1991. 46, 69, 71, 81

[Kno 1969] N. Knoche, Der Satz von Osgood-Hartogs in Polynomringen. *Schriftenr. Math. Inst. Univ. Münster* **41** (1969), 1–50. 103

[Kno 1971] N. Knoche, Der Satz von Osgood und Hartogs für reelle Funktionen. *Jahresber. Dtsch. Math.-Ver.* **73** (1971), 138–148. 103

[Kno 1974] N. Knoche, Ergänzung zu: Der Satz von Osgood und Hartogs für reelle Funktionen. *Jahresber. Dtsch. Math.-Ver.* **75** (1974), 144–149. 103

[Koł-Tho 1996] S. Kołodziej and J. Thorbiörnson, Separately harmonic and subharmonic functions. *Potential Anal.* **5** (1996), 463–466. 130

[Kom 1972] H. Komatsu, A local version of Bochner's theorem. *J. Fac. Sci. Univ. Tokyo Sect. I A* **19** (1972), 201–214. 160

[Kos 1966] K. Koseki, Neuer Beweis des Hartogsschen Satzes. *Math. J. Okayama Univ.* **12** (1966), 63–70. 12, 18

[Leb 1905] H. Lebesque, Sur les fonctions représentables analytiquement. *J. Math. Pures Appl.* **1** (1905), 139–215. 2

[Lej 1933a] F. Leja, Sur les suites de polynômes bornées presque partout sur une courbe. *Math. Ann.* **107** (1933), 68–82. 12

[Lej 1933b] F. Leja, Sur les suites de polynômes bornées presque partout sur la frontière d'une domaine. *Math. Ann.* **108** (1933), 517–524. 12

[Lej 1950] F. Leja, Une nouvelle démonstration d'un théorème sur les séries de fonctions analytiques. *Actas Acad. Ci. Lima* **13** (1950), 3–7. 12, 16, 21

[Lel 1945] P. Lelong, Les fonctions plurisousharmoniques. *Ann. Sci. École Norm. Sup.* **62** (1945), 301–338. 129

[Lel 1961] P. Lelong, Fonctions plurisousharmoniques et fonctions analytiques de variables réelles. *Ann. Inst. Fourier* **11** (1961), 515–562. 3, 126

[LMH-NVK 2005] Le Mau Hai and Nguyen Van Khue, Hartogs spaces, spaces having the Forelli property and Hartogs holomorphic extension spaces. *Vietnam J. Math.* **33** (2005), 43–53. 6

[Mas-Mik 2000] V. K. Maslyuchenko and V. V. Mikhailyuk, Characterization of the sets of discontinuity points of separately continuous functions of many variables on the products of metrizable spaces. *Ukrainian Math. J.* **52** (2000), 847–855. 2

[Mit 1961] B. S. Mitiagin, Approximative dimension and bases in nuclear spaces. *Russian Math. Surveys* **16** (1961), 59–127. 117

[NTV 1997] Nguyen Thanh Van, Separate analyticity and related subjects. *Vietnam J. Math.* **25** (1997), 81–90. 4, 104, 127

[NTV 2000] Nguyen Thanh Van, Fonctions séparément harmoniques, un théorème de type Terada. *Potential Anal.* **12** (2000), 73–80. 127

[NTV-Sic 1991] Nguyen Thanh Van and J. Siciak, Fonctions plurisousharmoniques extrémales et systèmes doublement orthogonaux de fonctions analytiques. *Bull. Sci. Math.* **115** (1991), 235–244. 4, 66, 104

[NTV-Zer 1991] Nguyen Thanh Van and A. Zeriahi, Une extension du théorème de Hartogs sur les fonctions séparément analytiques. In *Analyse complexe multivariables, récents dévelopements*, EditEl, Rende 1991, 183–194. 4, 104, 117

[NTV-Zer 1995] Nguyen Thanh Van and A. Zeriahi, Systèmes doublement othogonaux de fonctions holomorphes et applications. In *Topics in complex analysis*, Banach Center Publ. 31, Polish Academy of Sciences, Institute of Mathematics, Warsaw 1995, 281–297. 4, 104

[NVA 2005] Nguyên Viêt-Ahn, A general version of the Hartogs extension theorem for separately holomorphic mappings between complex analytic spaces. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **4** (2005), 219–254. 81, 131, 136

[NVA 2008] Nguyên Viêt-Ahn, A unified approach to the theory of separately holomorphic mappings. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **7** (2008), 181–240. 81, 131, 136, 180

[NVA 2009] Nguyên Viêt-Ahn, Recent developments in the theory of separately holomorphic mappings. *Colloq. Math.* **117** (2009), 175–206. 81, 131, 136, 180

[NVA 2010] Nguyên Viêt-Ahn, Conical plurisubharmonic measure and new cross theorems. *J. Math. Anal. Appl.* **365** (2010), 429-434. 180

[NVA-Pfl 2009] Nguyên Viêt-Ahn and P. Pflug, Boundary cross theorem in dimension 1 with singularities. *Indiana Math. J.* **58** (2009), 393–414. 229

[NVA-Pfl 2010] Nguyên Viêt-Ahn and P. Pflug, Cross theorems with singularities. *J. Geom. Anal.* **20** (2010), 193–218. 229

[Nis 2001] T. Nishino, *Function theory in several complex variables*. Transl. Math. Monogr. 193, Amer. Math. Soc., Providence, RI, 2001. 207, 208

[Non 1977a] K. Nono, Separately holomorphic functions. *Mem. Fac. Sci. Kyushu Univ. Ser. A* **31** (1977), 87–93. 109

[Non 1977b] K. Nono, Holomorphy of separately holomorphic functions. *Mem. Fac. Sci. Kyushu Univ. Ser. A* **31** (1977), 285–288. 109

[Non 1979] K. Nono, Holomorphy and boundedness of separately holomorphic functions. *Bull. Fukuoka Univ. Ed.* III **29** (1979), 33–41. 109

[Ökt 1998] O. Öktem, Extension of separately analytic functions and applications to range characterization of exponential Radon transform. *Ann. Polon. Math.* **70** (1998), 195–213. 5, 221, 222

[Ökt 1999] O. Öktem, Extending separately analytic functions in $\mathbb{C}^{n+m}$ with singularities. In *Extension of separately analytic functions and applications to mathematical tomography* (Thesis), Department of Mathematics, Stockholm University 1999. 5, 221, 222

[Osg 1899] W. F. Osgood, Note über analytische Functionen mehrerer Veränderlichen. *Math. Annalen* **52** (1899), 462–464. 3, 10

[Osg 1900] W. F. Osgood, Zweite Note über analytische Functionen mehrerer Veränderlichen. *Math. Ann.* **53** (1900), 461–464. 10

[Pfl 1980] P. Pflug, Ein Fortsetzungssatz für plurisubharmonische Funktionen über reel 2-kodimensionale Flächen. *Arch. Math.* **33** (1980), 559–563. 253

[Pfl 2003] P. Pflug, Extension of separately holomorphic functions – a survey 1899–2001. Proc. Conf. Complex Analysis (Bielsko-Biała, 2001), *Ann. Polon Math.* **80** (2003), 21–36. 4

[Pfl-NVA 2003] P. Pflug and Nguyên Viêt Ahn, Extension theorems of Sakai type for separately holomorphic and meromorphic functions. *Ann. Polon. Math.* **82** (2003), 171–187. 5, 180, 265

[Pfl-NVA 2004] P. Pflug and Nguyên Viêt-Anh, A boundary cross theorem for separately holomorphic functions. *Ann. Polon. Math.* **84** (2004), 237–271. 4, 81, 121, 180

[Pfl-NVA 2007] P. Pflug and Nguyên Viêt-Ahn, Boundary cross theorem in dimension 1. *Ann. Polon. Math.* **90** (2007), 149–192. 4, 180

[Pio 1985-86] Z. Piotrowski, Separate and joint continuity. *Real Anal. Exchange* **11** (1985–86), 293–322. 1

[Pio 1996] Z. Piotrowski, The genesis of separate versus joint continuity. *Tatra Mountains Math. Publ.* **8** (1996), 113–126. 1

[Pio 2000] Z. Piotrowski, Separate versus joint continuity – an update. In *Proc. 29th Spring Conference Union Bulg. Math.*, Lovetch 2000, 93–106. 1

[Por 2002] E. Porten, On the Hartogs-phenomenon and extension of analytic hypersurfaces in non-separated Riemann domains. *Complex Variables* **47** (2002), 325–332. 213

[Ran 1995] T. Ransford, *Potential theory in the complex plane*. London Math. Soc. Students Texts 28, Cambridge University Press, Cambridge 1995. 82, 166, 188, 194, 195, 196, 197, 199, 200, 203, 216

[Rii 1989] J. Riihentaus, On a theorem of Avanissian–Arsove. *Exposition. Math.* **7** (1989), 69–72. 129

[Rii 2007] J. Riihentaus, On separately harmonic and subharmonic functions. *Int. J. Pure Appl. Math.* **35** (2007), 435–446. 130

[Roc 1972] R. T. Rockafellar, *Convex analysis*. Princeton Math. Ser. 28, Princeton University Press, Princeton, NJ, 1972. 60

[Ron 1977] L. I. Ronkin, *Elements of the theory of analytic functions of several variables*. Izdat. "Naukova Dumka", Kiev 1977 (in Russian). 108, 160

[Ros 2003a] J. P. Rosay, Poletsky theory of discs on holomorphic manifolds. *Indiana Univ. Math. J.* **52** (2003), 157–169. 132

[Ros 2003b] J. P. Rosay, Approximation of non-holomorphic maps, and Poletsky theory of discs. *J. Korean Math. Soc.* **40** (2003), 423–434. 132

[Rose 1955] A. Rosenthal, On the continuity of functions of several variables. *Math. Z.* **63** (1955), 31–38. 1

[Rot 1950] W. Rothstein, Ein neuer Beweis des Hartogsschen Hauptsatzes und seine Ausdehnung auf meromorphe Funktionen. *Math. Z.* **53** (1950), 84–95. 23, 255, 258

[Roy 1974] H. L. Royden, The extension of regular holomorphic maps. *Proc. Amer. Math. Soc.* **43** (1974), 306–310. 132

[Rud 1970] W. Rudin, *Lectures on the edge-of-the-wedge theorem*. CBMS Regional Conference Series in Mathematics 6, Amer. Math. Soc., Providence, RI, 1971. 160

[Rud 1974] W. Rudin, *Real and complex analysis*. Second edition, McGraw-Hill Book Company, New York 1974. 171

[Rud 1980] W. Rudin, *Function theory in the unit ball of* $\mathbb{C}^n$, Grundlehren Math. Wiss. 241, Springer-Verlag, New York 1980. 170, 171

[Rud 1981] W. Rudin, Lebesgue's first theorem. In *Mathematical analysis and applications*, Part B, Adv. Math. Suppl. Stud. 7b, Academic Press, New York, London 1981, 741–747. 1, 2

[Sad 1982] A. Sadullaev, Rational approximation and pluripolar sets. *Mat. Sb.* (*N.S.*) **119** (161) (1982), 96–118; English transl. *Math. USSR-Sb.* **47** (1984), 91–113. 208

[Sad 1984] A. Sadullaev, A criterion for rapid rational approximation in $\mathbb{C}^n$. *Mat. Sb.* (*N.S.*) **125** (167) (1984), 269–279; English transl. *Math. USSR-Sb.* **53** (1986), 271–281. 205, 209

[Sad 2005] A. Sadullaev, Pluriharmonic continuation in some fixed direction. *Mat. Sb.* **196** (2005), 145–156; English transl. *Sb. Math.* **196** (2005), 765–775. 24, 211, 212, 243, 252

[Sad-Imo 2006a] A. S. Sadullaev and S. A. Imomkulov, Continuation of separately analytic functions defined on a part of the domain boundary. *Mat. Zametki* **79** (2006), 234–243; English transl. *Math. Notes* **79** (2006), 215–223. 121

[Sad-Imo 2006b] A. S. Sadullaev and S. A. Imomkulov, Extension of holomorphic and pluriharmonic functions with thin singularities on parallel sections. *Proc. Steklov Inst. Math.* **253** (2006), 144–159. 121, 211, 243

[Sad-Tui 2009] A. S. Sadullaev and T. T. Tuichiev, On continuation of Hartogs series that admit holomorphic extention to parallel sections. *Uzbek. Mat. Zh.* **1** (2009), 148–157 (in Russian). 21, 22

[StR 1990] J. Saint Raymond, Fonctions séparément analytiques. *Ann. Inst. Fourier* **40** (1990), 79–101. 3, 129

[Sak 1957] E. Sakai, A note on meromorphic functions in several complex variables. *Mem. Fac. Sci. Kyusyu Univ. Ser. A. Math.* **11** (1957), 75–80. 259, 264, 265

[Sha 1976] B. V. Shabat, *An introduction to complex analysis*. Vol. I, II, Izdat. "Nauka", Moscow 1976 (in Russian). 28

[Shi 1968] B. Shiffman, On the removal of singularities of analytic sets. *Michigan Math. J.* **15** (1968), 111–120. 253

[Shi 1986] B. Shiffman, Complete characterization of holomorphic chains of codimension one. *Math. Ann.* **274** (1986), 233–256. 259

[Shi 1989] B. Shiffman, On separate analyticity and Hartogs theorem. *Indiana Univ. Math. J.* **38** (1989), 943–957. 4, 5, 104, 259, 264

[Shi 1990] B. Shiffman, Hartogs theorems for separately holomorphic mappings into complex spaces. *C. R. Acad. Sci. Paris Ser. I Math.* **310** (1990), 89–94. 6

[Shi 1991] B. Shiffman, Separately meromorphic functions and separately holomorphic mappings. In *Several complex variables and complex geometry* (Santa Cruz, CA, 1989), Proc. Sympos. Pure Math. 52, Part 1, Amer. Math. Soc., Providence, RI, 1991, 191–198. 5, 6

[Shim 1957] I. Shimoda, Notes on the functions of two complex variables. *J. Gakugei Tokushima Univ.* **8** (1957), 1–3. 27

[Sic 1962] J. Siciak, A note on rational functions of several complex variables. *Ann. Polon. Math.* **12** (1962), 139–142. 104

[Sic 1968] J. Siciak, Separately analytic functions and envelopes of holomorphy of some lower dimensional subsets of $\mathbb{C}^n$. In *Séminaire Pierre Lelong (Analyse)*, Lecture Notes in Math. 71, Springer-Verlag, Berlin 1968, 21–32. 4, 104

[Sic 1969a] J. Siciak, Analyticity and separate analyticity of functions defined on lower dimensional subsets of $\mathbb{C}^n$. *Zeszyty Nauk. Uniw. Jagiello. Prace Mat. Zeszyt* **13** (1969), 53–70. 4, 93, 104, 108, 126

[Sic 1969b] J. Siciak, Separately analytic functions and envelopes of holomorphy of some lower dimensional subsets of $\mathbb{C}^n$. *Ann. Polon. Math.* **22** (1969–1970), 147–171. 4, 93, 104, 108, 126

[Sic 1974] J. Siciak, Holomorphic continuation of harmonic functions. *Ann. Polon. Math.* **29** (1974), 67–73. 91

[Sic 1981a] J. Siciak, Extremal plurisubharmonic functions in $\mathbb{C}^N$. *Ann. Polon. Math.* **39** (1981), 175–211. 4, 70, 104

[Sic 1981b] J. Siciak, A modified edge of the wedge theorem. *Zeszyty Nauk. Uniw. Jagiellon. Prace Mat.* **22** (1981), 15–17. 124

[Sic 1983] J. Siciak, On the equivalence between locally polar and globally polar sets in $\mathbb{C}^n$. In *Complex analysis*, Banach Center Publ. 11, PWN–Polish Scientific Publishers, Warsaw 1983, 299–309. 44

[Sic 1990] J. Siciak, Singular sets of separately analytic functions. *Colloq. Math.* **60/61** (1990), 181–290. 3, 129

[Sic 1995] J. Siciak, Polynomial extensions of functions defined on subsets of $\mathbb{C}^n$. *Univ. Iagel. Acta Math.* **32** (1995), 7–16. 99, 101

[Sic 2001] J. Siciak, Holomorphic functions with singularities on algebraic sets. *Univ. Iagel. Acta Math.* **39** (2001), 9–16. 5, 223

[STW 1990] K. Spallek, P. Tworzewski, and T. Winiarski, Osgood-Hartogs-theorems of mixed type. *Math. Ann.* **288** (1990), 75–88. 104

[Sto 1977] E. L. Stout, The boundary values of holomorphic functions of several complex variables. *Duke Math. J.* **44** (1977), 105–108. 92

[Sto 1991] E. L. Stout, A note on removable singularities. *Boll. Un. Mat. Ital. A* (7) **5** (1991), 237–243. 253

[Ter 1967] T. Terada, Sur une certaine condition sous laquelle une fonction de plusieurs variables complexes est holomorphes. *Publ. Res. Inst. Math. Sci. Ser. A* **2** (1967), 383–396. 3, 86

[Ter 1972] T. Terada, Analyticitès relatives à chaque variable. *J. Math. Kyoto Univ.* **12** (1972), 263–296. 3, 26, 27, 88

[Tho 1870] J. Thomae, *Abriss einer Theorie der complexen Functionen und der Thetafunctionen einer Veränderlichen*. L. Nebert, Halle 1870. 1

[Tho 1873] J. Thomae, *Abriss einer Theorie der complexen Functionen und der Thetafunctionen einer Veränderlichen*. Second edition, L. Nebert, Halle 1873. 1

[Thor 1989] J. Thorbiörnson, Construction of extremal plurisubharmonic functions for Reinhardt domains. Preprint 1989, University of Umeå, Sweden. 73

[Tol 1949] G. P. Tolstov, On partial derivatives. *Izvestiya Akad. Nauk SSSR. Ser. Mat.* **13** (1949), 425–446; English transl. *Amer. Math. Soc. Translation* **1952** (1952), 55–83. 2

[Tui 1985] T. T. Tuichiev, An analogue of Hartogs' theorem. *Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk* **6** (1985), 26–29, 68–69 (in Russian). 21, 22

[Vla 1966] V. S. Vladimirov, *Methods of the theory of functions of many complex variables*. The M.I.T. Press, Cambridge, Mass., London 1966. 72, 160

[Wie 1988] J. Wiegerinck and R. Zeinstra, Separately subharmonic functions need not be subharmonic. *Proc. Amer. Math. Soc.* **104** (1988), 770–771. 129

[Wie-Zei 1991] J. Wiegerinck and R. Zeinstra, Separately subharmonic functions. When are they subharmonic. *Proc. Symp. Pure Math.* **52** (1991), 245–249. 129

[Wil 1969] G. K. Williams, On continuity in two variables. *Proc. Amer. Math. Soc.* **23** (1969), 580–582. 23

[Zah 1976] V. P. Zahariuta, Separately analytic functions, generalizations of Hartogs theorem, and envelopes of holomorphy. *Math. USSR-Sb.* **30** (1976), 51–67. 4, 104, 117

[Zer 1982] A. Zeriahi, Bases communes dans certains espaces de fonctions harmoniques et fonctions séparement harmoniques sur certains ensembles de $\mathbb{C}^n$. *Ann. Fac. Sci. Toulouse V. Ser. Math.* **4** (1982), 75–102. 117

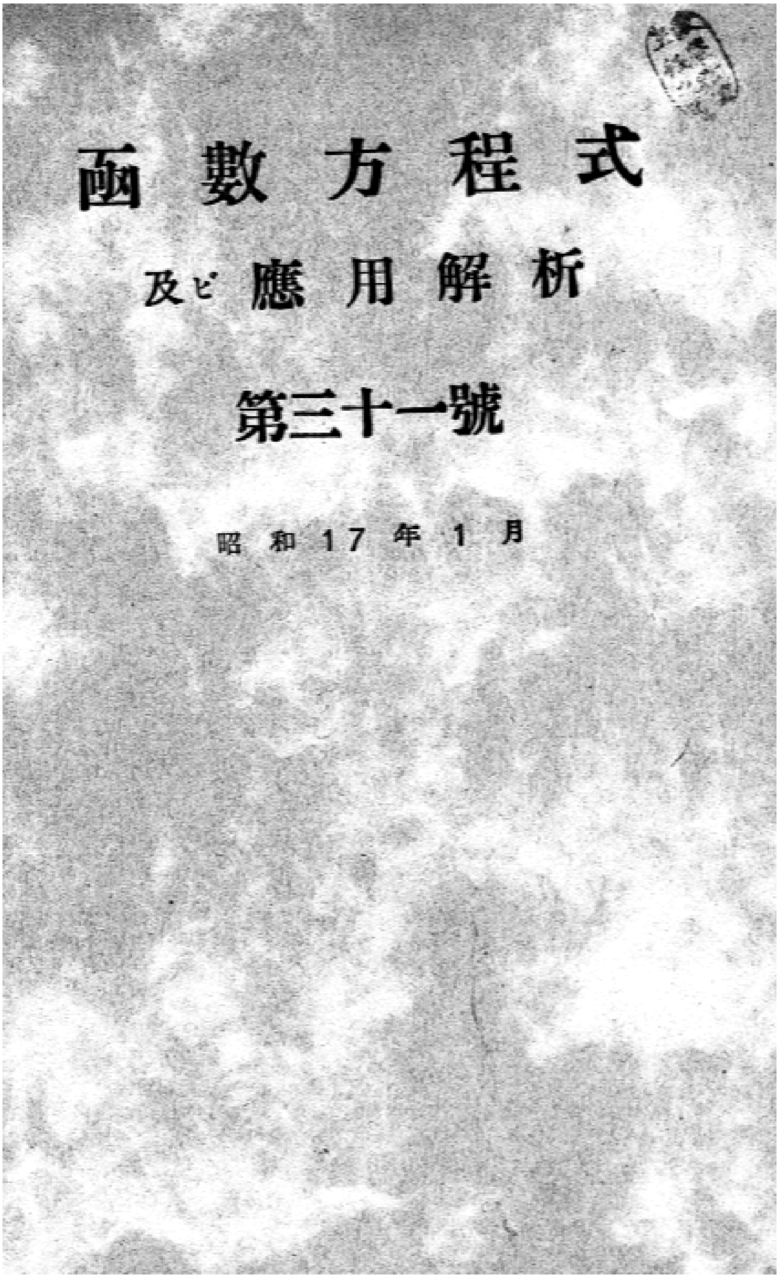
[Zer 1986] A. Zeriahi, Fonctions plurisousharmoniques extrémales. Approximation et croissance des fonctions holomorphes sur des ensembles algébriques. Theèse de Doctorat d'État, Sciences, U.P.S. Toulouse, 1986. 71, 117

[Zer 1991] A. Zeriahi, Fonction de Green pluricomplexe à pôle à infinie sur un espace de Stein parabolique et applications. *Math. Scand.* **69** (1991), 89–126. 117

[Zer 2002] A. Zeriahi, Comportement asymptotique des systemes doublement orthogonaux de Bergman: une approche élémentaire. *Vietnam J. Math.* **30** (2002), 177–188. 4, 104, 117

[Zern 1961] M. Zerner, *Quelques résultats sur le prolongement analytique des fonctions de variables complexes*. Mimeographed notes of a seminar given in Marseille, 1961. 158, 160

It is rather difficult to find [Huk 1942]; the more we thank Professor T. Ohsawa for his strong help in getting a copy of the paper presented below:



函數方程式

及ビ應用解析

第三十一號

昭和17年1月

Osgood 及ビ Hartogs ノ定理擴張

福原　満洲雄（九大）

多変数ノ複素函数ニツイテハ全ク素人ナノデ，ドウイフ結果ガ既ニ得ラレテヰルカヨク知ラナイ。コノ方面ニツイテ讀者ノ御教示ヲ得レバ幸デアル。

Osgood ノ定理ハ普通次ノヤウニ述ベラレル.（D, Δ ハ常ニ複素数平面ノ領域トスル）

「$f(z, w)$ ガ $D \times \Delta$ ニ於テ有界，且ツ z, w ノ各々ニ關シテ正則ナラバ，$f(z, w)$ ハ $D \times \Delta$ ニ於テ (z, w) ノ函数トシテ正則デアル。」

併シソノ証明ヲ見レバ直グニ分ルヤウニコノ定理ハ次ノヤウニ拡張サレル。

「$E (\subset \Delta)$ ハ Δ ノ内部ニ集積点ヲ持ツ集合トスル。$f(z, w)$ ハ $D \times \Delta$ デ有界デ，w ニ関シテハ正則，$w \in E$ ナルトキ z ニ関シテ D ニ於テ正則ナラバ $f(z, w)$ ハ (z, w) ノ函数トシテ $D \times \Delta$ ニ於テ正則デアル。」

Osgood ノ定理ニ於テ有界ノ假設ヲ除イタモノガ Hartogs ノ定理デアル，即チ

「$f(z, w)$ ガ $D \times \Delta$ ニ於テ z, w ノ各々ニ関シテ正則ナラバ，$f(z, w)$ ハ (z, w) ノ函数トシテ正則デアル。」

所ガコノ定理モソノ証明カラ分ルヤウニ次ノヤウニ擴

張サレル。

「Eハ Δ ニ含マレル閉集合トスル。$f(z, w)$ ハ $z \in D$ ナルトキ w ニ関シテ Δ ニ於テ正則，$w \in E$ ナル時 z ニ関シテ D ニ於テ正則ナラバ，$f(z, w)$ ハ $D \times \Delta$ ニ於テ (z, w) ノ函数トシテ正則デアル。」

以上ノ拡張ハ大シテ困難デハナイガ，Osgood ノ定理ノ拡張ノ場合ニハ E ハ閉集合デナクテモ Δ ノ内部ニ集積点ヲ持テバヨイ（従ッテ可附番集合デモヨイ）ノデアルカラ，Hartogs ノ定理モソノヤウナ形ニ拡張サレナイデアラウカトイフ問題ガ考ヘラレル．此ノ種ノ定理ハ應用モ廣イノデアルカラ明快ナ解決ガ望マシイ。

尚特ニ二変数ノ場合ニツイテ述ベタガ，一般ニ n 変数ノ場合モ同様デアルコトハ言フ迄モナイ。

Symbols

General symbols

$\mathbb{N}$:= the set of natural numbers, $0 \notin \mathbb{N}$; $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$; $\mathbb{N}_k := \{n \in \mathbb{N} : n \geq k\}$;
$\mathbb{Z}$:= the ring of integer numbers;
$\mathbb{Q}$:= the field of rational numbers;
$\mathbb{R}$:= the field of real numbers;
$\mathbb{C}$:= the field of complex numbers;
$\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ = the Riemann sphere;
$\operatorname{Re} z$:= the real part of $z \in \mathbb{C}$, $\operatorname{Im} z$:= the imaginary part of $z \in \mathbb{C}$;
$\bar{z} := x - iy$ = the conjugate of $z = x + iy$;
$|z| := \sqrt{x^2 + y^2}$ = the modulus of a complex number $z = x + iy$;
$\arg z := \{\varphi \in \mathbb{R} : z = |z|e^{i\varphi}\}$;
$\mathbb{C} \setminus \{0\} \ni z \mapsto \operatorname{Arg} z \in (-\pi, +\pi]$ the main argument; $\operatorname{Arg} 0 := 0$;
A^n := the Cartesian product of n copies of the set A, e.g. $\mathbb{C}^n$;
$x \leq y :\iff x_j \leq y_j,\ j = 1, \dots, n$, where $x = (x_1, \dots, x_n),\ y = (y_1, \dots, y_n) \in \mathbb{R}^n$;
$A_* := A \setminus \{0\}$, e.g. $\mathbb{C}_*, (\mathbb{C}^n)_*$; $A_*^n := (A_*)^n$, eg. $\mathbb{C}_*^n$;
$A_+ := \{a \in A : a \geq 0\}$, e.g. $\mathbb{Z}_+, \mathbb{R}_+$; $A_+^n := (A_+)^n$, e.g. $\mathbb{Z}_+^n, \mathbb{R}_+^n$;
$A_- := \{a \in A : a \leq 0\}$;
$A_{>0} := \{a \in A : a > 0\}$, e.g. $\mathbb{R}_{>0}$; $A_{>0}^n := (A_{>0})^n$, e.g. $\mathbb{R}_{>0}^n$;
$A_{<0} := \{a \in A : a < 0\}$;
$\mathbb{R}_{-\infty} := \{-\infty\} \cup \mathbb{R}$, $\mathbb{R}_{+\infty} := \mathbb{R} \cup \{+\infty\}$;
$A + B := \{a + b : a \in A,\ b \in B\}$, $a + B := \{a\} + B$, where $A, B \subset \boldsymbol{X}$, $a \in \boldsymbol{X}$;
$A \cdot B := \{a \cdot b : a \in A,\ b \in B\}$, where $A \subset \mathbb{C}$, $B \subset \mathbb{C}^n$;
$\delta_{j,k} := \begin{cases} 0, & \text{if } j \neq k \\ 1, & \text{if } j = k \end{cases}$ = the Kronecker symbol;
$\boldsymbol{e} = (e_1, \dots, e_n)$:= the canonical basis in $\mathbb{C}^n$, $e_j := (\delta_{j,1}, \dots, \delta_{j,n})$, $j = 1, \dots, n$;
$\mathbb{1} = \mathbb{1}_n := (1, \dots, 1) \in \mathbb{N}^n$; $\mathbb{2} := 2 \cdot \mathbb{1} = (2, \dots, 2) \in \mathbb{N}^n$;

$\langle z, w\rangle := \sum_{j=1}^n z_j \overline{w}_j$ = the Hermitian scalar product in $\mathbb{C}^n$;
$\overline{w} := (\overline{w}_1, \dots, \overline{w}_n)$, $w = (w_1, \dots, w_n) \in \mathbb{C}^n$;
$z \cdot w := (z_1 w_1, \dots, z_n w_n)$, $z = (z_1, \dots, z_n)$, $w = (w_1, \dots, w_n) \in \mathbb{C}^n$;
$e^z := (e^{z_1}, \dots, e^{z_n})$, $z = (z_1, \dots, z_n) \in \mathbb{C}^n$;
$\|z\| := \langle z, z\rangle^{1/2} = \left(\sum_{j=1}^n |z_j|^2 \right)^{1/2}$ = the Euclidean norm in $\mathbb{C}^n$;
$\|z\|_\infty := \max\{|z_1|, \dots, |z_n|\}$ = the maximum norm in $\mathbb{C}^n$;
$\|z\|_1 := |z_1| + \cdots + |z_n|$ = the ℓ^1-norm in $\mathbb{C}^n$;

$\#A$:= the number of elements of A;
$\operatorname{diam} A$:= the diameter of the set $A \subset \mathbb{C}^n$ with respect to the Euclidean distance;

χ_A := the characteristic function of A;
$\operatorname{conv} A = \operatorname{conv}(A)$:= the convex hull of the set A;
$A \subset\subset X \;:\Longleftrightarrow\; A$ is relatively compact in X;
$\operatorname{pr}_X\colon X \times Y \to X$, $\operatorname{pr}_X(x, y) := x$, or $\operatorname{pr}_X\colon X \oplus Y \to X$, $\operatorname{pr}_X(x + y) := x$.

Euclidean balls
$\mathbb{B}(a,r) = \mathbb{B}_n(a,r) := \{z \in \mathbb{C}^n : \|z - a\| < r\}$ = the open Euclidean ball in $\mathbb{C}^n$ with center $a \in \mathbb{C}^n$ and radius $r > 0$; $\mathbb{B}(a, +\infty) := \mathbb{C}^n$;
$\overline{\mathbb{B}}(a,r) = \overline{\mathbb{B}}_n(a,r) := \overline{\mathbb{B}_n(a,r)} = \{z \in \mathbb{C}^n : \|z - a\| \le r\}$ = the closed Euclidean ball in $\mathbb{C}^n$ with center $a \in \mathbb{C}^n$ and radius $r > 0$;
$\mathbb{B}(r) = \mathbb{B}_n(r) := \mathbb{B}_n(0,r)$; $\overline{\mathbb{B}}(r) = \overline{\mathbb{B}}_n(r) := \overline{\mathbb{B}}_n(0,r)$;
$\mathbb{B} = \mathbb{B}_n := \mathbb{B}_n(1)$ = the unit Euclidean ball in $\mathbb{C}^n$;
$\mathbb{D}(a,r) := \mathbb{B}_1(a,r)$; $\mathbb{D}(r) := \mathbb{D}(0,r)$;
$\overline{\mathbb{D}}(a,r) := \overline{\mathbb{B}}_1(a,r)$; $\overline{\mathbb{D}}(r) := \overline{\mathbb{D}}(0,r)$;
$\mathbb{D}_*(a,r) := \mathbb{D}(a,r) \setminus \{a\}$; $\mathbb{D}_*(r) := \mathbb{D}_*(0,r)$;
$\mathbb{D} := \mathbb{D}(1) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$ = the unit disc;
$\mathbb{T} := \partial\mathbb{D}$.

Polydiscs
$\mathbb{P}(a,r) = \mathbb{P}_n(a,r) := \{z \in \mathbb{C}^n : \|z - a\|_\infty < r\}$ = the polydisc with center $a \in \mathbb{C}^n$ and radius $r > 0$; $\mathbb{P}_n(a, +\infty) := \mathbb{C}^n$;
$\overline{\mathbb{P}}(a,r) = \overline{\mathbb{P}}_n(a,r) := \overline{\mathbb{P}_n(a,r)}$; $\overline{\mathbb{P}}_n(a,0) := \{a\}$;
$\mathbb{P}(r) = \mathbb{P}_n(r) := \mathbb{P}_n(0,r)$;

$\mathbb{P}(a,\boldsymbol{r}) = \mathbb{P}_n(a,\boldsymbol{r}) := \mathbb{D}(a_1,r_1) \times \cdots \times \mathbb{D}(a_n,r_n)$ = the polydisc with center $a \in \mathbb{C}^n$ and multiradius (polyradius) $\boldsymbol{r} = (r_1, \dots, r_n) \in \mathbb{R}^n_{>0}$; notice that $\mathbb{P}(a,r) = \mathbb{P}(a, r\cdot\boldsymbol{1})$;
$\mathbb{P}(\boldsymbol{r}) = \mathbb{P}_n(\boldsymbol{r}) := \mathbb{P}_n(0,\boldsymbol{r})$;
$\partial_0\mathbb{P}(a,\boldsymbol{r}) := \partial\mathbb{D}(a_1,r_1) \times \cdots \times \partial\mathbb{D}(a_n,r_n)$ = the distinguished boundary of $\mathbb{P}(a,\boldsymbol{r})$.

Annuli
$\mathbb{A}(a,r^-,r^+) := \{z \in \mathbb{C} : r^- < |z - a| < r^+\}$, $a \in \mathbb{C}$, $-\infty \le r^- < r^+ \le +\infty$, $r^+ > 0$; if $r^- < 0$, then $\mathbb{A}(a,r^-,r^+) = \mathbb{D}(a,r^+)$; $\mathbb{A}(a,0,r^+) = \mathbb{D}(a,r^+) \setminus \{a\}$;
$\mathbb{A}(r^-,r^+) := \mathbb{A}(0,r^-,r^+)$.

Laurent series
$z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$ $(0^0 := 1)$;
$\alpha! := \alpha_1! \cdots \alpha_n!$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_+$;
$|\alpha| := |\alpha_1| + \cdots + |\alpha_n|$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$;
$\binom{\alpha}{\beta} := \frac{\alpha(\alpha-1)\cdots(\alpha-\beta+1)}{\beta!}$, $\alpha \in \mathbb{Z}$, $\beta \in \mathbb{Z}_+$;
$\binom{\alpha}{\beta} := \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n}$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$, $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n_+$.

Functions
$\|f\|_A := \sup\{|f(a)| : a \in A\}$, $f\colon A \to \mathbb{C}$;
$\operatorname{supp} f := \overline{\{x : f(x) \neq 0\}}$ = the support of f;
$\mathcal{P}(\mathbb{K}^n)$:= the space of all polynomial mappings $F\colon \mathbb{K}^n \to \mathbb{K}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$;

$\mathcal{P}_d(\mathbb{K}^n) := \{F \in \mathcal{P}(\mathbb{K}^n) : \deg F \le d\}$;
$\mathcal{C}^{\uparrow}(\Omega) :=$ the set of all upper semicontinuous functions $u\colon \Omega \to \mathbb{R}_{-\infty}$;
$\frac{\partial f}{\partial z_j}(a) := \frac{1}{2}\Big(\frac{\partial f}{\partial x_j}(a) - i\frac{\partial f}{\partial y_j}(a)\Big)$, $\frac{\partial f}{\partial \bar z_j}(a) := \frac{1}{2}\Big(\frac{\partial f}{\partial x_j}(a) + i\frac{\partial f}{\partial y_j}(a)\Big) =$ the formal partial derivatives of f at a;
$\operatorname{grad} u(a) := (\frac{\partial u}{\partial \bar z_1}(a), \dots, \frac{\partial u}{\partial \bar z_n}(a)) =$ the gradient of u at a;
$D^{\alpha,\beta} := (\frac{\partial}{\partial z_1})^{\alpha_1} \circ \dots \circ (\frac{\partial}{\partial z_n})^{\alpha_n} \circ (\frac{\partial}{\partial \bar z_1})^{\beta_1} \circ \dots \circ (\frac{\partial}{\partial \bar z_n})^{\beta_n}$;
$\mathcal{C}^k(\Omega, F) :=$ the space of all $\mathcal{C}^k$-mappings $f\colon \Omega \to F$, $k \in \mathbb{Z}_+ \cup \{\infty\} \cup \{\omega\}$ (ω stands for the real analytic case);
$\mathcal{C}^k(\Omega) := \mathcal{C}^k(\Omega, \mathbb{C})$;
$\mathcal{C}^k_0(\Omega) := \{f \in \mathcal{C}^k(\Omega) : \operatorname{supp} f \subset\subset \Omega\}$;
$\mathcal{L}^N :=$ Lebesgue measure in $\mathbb{R}^N$;
$L^p(\Omega) :=$ the space of all p-integrable functions on Ω;
$\| \ \|_{L^p(\Omega)} :=$ the norm in $L^p(\Omega)$;
$L^p_{\mathrm{loc}}(\Omega) :=$ the space of all locally p-integrable functions on Ω;
$\mathcal{O}(X, Y) :=$ the space of all holomorphic mappings $f\colon X \to Y$;
$\mathcal{O}(\Omega) := \mathcal{O}(\Omega, \mathbb{C}) =$ the space of all holomorphic functions $f\colon \Omega \to \mathbb{C}$;
$\frac{\partial f}{\partial z_j}(a) := \lim\limits_{\mathbb{C}_* \ni h \to 0} \frac{f(a+he_j)-f(a)}{h} =$ the j-th complex partial derivative of f at a;
$D^{\alpha} := (\frac{\partial}{\partial z_1})^{\alpha_1} \circ \dots \circ (\frac{\partial}{\partial z_n})^{\alpha_n} = \alpha$-th partial complex derivative;
$L^p_h(\Omega) := \mathcal{O}(\Omega) \cap L^p(\Omega) =$ the space of all p-integrable holomorphic functions on Ω;
$L^\infty_h(\Omega) :=$ the space of all bounded holomorphic functions on Ω;
$\mathcal{H}(\Omega) :=$ the space of all harmonic functions on Ω, $\Omega \subset \mathbb{C}$;
$\mathcal{SH}(\Omega) :=$ the set of all subharmonic functions on Ω, $\Omega \subset \mathbb{C}$;
$\mathcal{PSH}(X) :=$ the set of all plurisubharmonic functions on X;
$\mathcal{L}u(a;\xi) := \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar z_k}(a)\xi_j \bar\xi_k =$ the Levi form of u at a.

List of symbols

Chapter 3

Chapter 4

Chapter 5

Chapter 6

Subject index